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THE STRUCTURE OF SPECIAL ENDOMORPHISM RINGS OVER ARTIN ALGEBRAS[1]

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Introduction

For a ring Λ and a finitely generated Λ-module M we have from [2], [3] a correspondence between Mod Λ (the category of left Λ-modules) and Mod End $(M)^{\text{op}}$ (the category of left End $(M)^{\text{op}}$-modules). This correspondence is especially interesting when M is a finitely generated generator or a special type of finitely generated projective Λ-module.

In these cases we look at a pair (Λ, M) (resp. (Γ, P)) where Λ is any ring and M a finitely generated generator (resp. Γ is any ring and P a Wedderburn projective). (For definition of Wedderburn projective, see the text.) Given (Λ, M) we construct the pair (End $(M)^{\text{op}}$, (M, Λ)). (M, Λ) is then a Wedderburn projective End $(M)^{\text{op}}$-module. (Given (Γ, P) we construct the pair (End $(P)^{\text{op}}$, (P, Γ)). (P, Γ) is then a finitely generated generator.) If we now use this map between pairs as above twice, we have the identity.

By using this correspondence on Artin algebras Λ with the Loewy length of Λ, $\mathscr{L}(\Lambda) = n$ and the finitely generated generator

$$M = \Lambda \amalg \Lambda/r_\Lambda^{n-1} \amalg \cdots \amalg \Lambda/r_\Lambda,$$

M. Auslander has proved that the global dimension of End $(M)^{\text{op}}$ is less than or equal to n. This shows that any Artin algebra is isomorphic to End $(P)^{\text{op}}$ for a finitely generated projective Γ-module where Γ is an Artin algebra of finite global dimension. In [5] we give examples which show that the inequality in this result of M. Auslander is optimal. In [3], M. Auslander asked for an abstract characterization of the Artin algebras we get as End $(M)^{\text{op}}$ where M is as in the theorem referred to. This is partially done in [2] and the main purpose of this paper is to give a complete such characterization. We also give a complete characterization of the End $(M)^{\text{op}}$ when $M = \Lambda \amalg \Lambda/r_\Lambda$ for an Artin algebra Λ. We then see that if $\mathscr{L}(\Lambda) = 2$ we get End $(M)^{\text{op}}$ as a special case of both these cases.

Section 1

Here we give some general results from ring theory and some results from [2], [3] which we shall need in the following sections and give some conventions

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about notation. See [1] and [4] for general background in ring theory and homological algebra.

DEFINITION. Let Λ be a ring and A and B Λ-modules. Then A is relatively injective to B if for every Λ-module X, homomorphism $f\colon X \to A$ and monomorphism $i\colon X \to B$ there exists a $g\colon B \to A$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & B \arrow[dl, "g"] \\ A & \end{tikzcd}$$

commutes.

For the proof of the following result we refer to [1, p. 186].

PROPOSITION 1.1. *If A and B are Λ-modules for a ring Λ where $A = A' \amalg A''$ (direct sum) then A is relatively injective to B if and only if both A' and A'' are relatively injective to B.*

Now let Λ be a ring, Mod Λ the category of left Λ-modules, M a finitely generated Λ-module and $\Gamma = \operatorname{End}(M)^{\mathrm{op}}$. Then M has a natural structure as an End (M)-module by $f \cdot m = f(m)$ for $f \in \operatorname{End}(M)$ and $m \in M$. This structure will in a natural way induce a Γ-module structure on (M, X) for every X in Mod Λ given by $f \cdot g = g \circ f$ for $f \in \Gamma$ and $g \in (M, X)$.

We are now going to give some results about the functor $(M, -)$. Let **A** be the full subcategory of Mod Λ consisting of the Λ-modules X such that there exists a presentation $\coprod M \to \coprod M \to X \to 0$ with the property that

$$(M, \coprod M) \to (M, \coprod M) \to X \to 0$$

is exact. Let **B** be the full subcategory of Mod Γ consisting of the (M, X) for some X in Mod. Λ. From [2, p. 3] and results of [3, Sections 2, 3, 4] we then have the following result.

PROPOSITION 1.2. $(M, -)\colon \mathbf{A} \to \mathbf{B} \subseteq \operatorname{Mod}\Gamma$ *is a full and faithful functor, where $(M, -)$ also denotes $(M, -)$ restricted to* $\mathbf{A} \subseteq \operatorname{Mod}\Lambda$.

DEFINITION. Let add M denote the full subcategory of Mod Λ consisting of the Λ-modules which are summands of finite sums of copies of M and let Add M denote the full subcategory of Mod Λ consisting of the Λ-modules which are summands of arbitrary sums of copies of M.

PROPOSITION 1.3. $(M, -)$ *induces an equivalence between* Add M *and the full subcategory of* Mod Γ *consisting of the projective Γ-modules and between* add M *and the full subcategory of finitely generated projective Γ-modules.*

See [2, p. 5] *and* [3, Sections 2, 3, 4].

PROPOSITION 1.4. *If M is a finitely generated generator then* $\mathbf{A}$ *described above is all of* Mod Λ.
See [2, p. 5] *and* [3, p. 247].

In the following let Γ be a ring, P a finitely generated projective Γ-module and $\Lambda = \text{End}\,(P)^{\text{op}}$. We then have the following results taken from [2, p. 15] and [3, p. 218].

PROPOSITION 1.5. *The full subcategory* $\mathbf{B}$ *of* Mod Λ *consisting of the* Λ-*modules* $B = (P, X)$ *for a* Γ-*module* X *is equivalent to* Mod (Λ).

COROLLARY 1.6. $(P, -)$: Mod $\Gamma \to$ Mod Λ *will induce an equivalence between the full subcategory* $\mathbf{C}_P$ *of* Mod Γ *consisting of the* Γ-*modules* Y *with presentation* $\coprod P \to \coprod P \to Y \to 0$, *and* Mod Λ.

DEFINITION. Let Ker $(P, -)$ be the full subcategory of Mod Γ consisting of the Γ-modules A such that $(P, -)A = (P, A) = 0$.

We now give some results about Ker $(P, -)$ taken from [2, p. 17] and [3, p. 222].

PROPOSITION 1.6. (a) *If* $0 \to A' \to A \to A'' \to 0$ *is exact in* Mod Γ *then* A *lies in* Ker $(P, -)$ *if and only if* A' *and* A'' *lie in* Ker $(P, -)$.
(b) Ker $(P, -)$ *is closed under direct limits.*
(c) *For every* Γ-*module* B *there exists a unique maximal submodule* $B' \subseteq B$ *such that* B' *is in* Ker $(P, -)$.
(d) *If*

$$0 \to K' \to A' \xrightarrow{f} A \to K \to 0$$

is exact in Mod Γ, *then* (P, f) *is an isomorphism if and only if* K' *and* K *lie in* Ker $(P, -)$.

We now give some useful definitions.

DEFINITION. The unique maximal submodule B' of B in Ker $(P, -)$ is called the $(P, -)$-torsion of B.

DEFINITION. A Γ-module A is said to be $(P, -)$-torsionfree if the $(P, -)$-torsion of A is zero.

As before, let Γ be a ring, P a finitely generated projective Γ-module and $\Lambda = \text{End}\,(P)^{\text{op}}$. We will now describe a full subcategory of Mod Γ which also is equivalent to Mod (Λ) via the functor $(P, -)$. But first we will describe a functor G: Mod $\Gamma \to$ Mod Γ [2, p. 23], [3, p. 221].

Let X be a Γ-module and $X' = X/(P, -)$-torsion X. Now consider the exact sequence

$$0 \to X' \to E_0(X') \xrightarrow{\phi} V \to 0,$$

where $E_0(X')$ is the injective envelope of X'. Let $G(X) = \phi^{-1}[(P, -)\text{-torsion } V]$ and let $X \to G(X)$ be the composition $X \to X/(P, -)\text{-torsion } X \to G(X)$. Observe that if X is $(P, -)$-torsionfree then $X \to G(X)$ is a monomorphism. The operation of G on morphisms is natural and done in [2], [3].

PROPOSITION 1.7. *Let Γ be a ring and P a finitely generated projective Γ-module. For an A in* Mod Γ *the following are equivalent.*

(a) $(P, -)\colon (\Gamma, A) \to ((P, \Gamma), (P, A))$ *is an isomorphism.*
(b) $(P, -)\colon (X, A) \to ((P, X), (P, A))$ *is an isomorphism for all X in* Mod Γ.
(c) $A \to G(A)$ *is an isomorphism.*
(d) $\operatorname{Ext}^i(K, A) = 0$ $i = 0, 1$ *and K in* Ker $(P, -)$.
(e) *If $0 \to A \to E_0(A) \to E_1(A)$ is a minimal injective copresentation of A then $E_i(A)$ is $(P, -)$-torsionfree for $i = 0, 1$.*
See [2, p. 24] *and* [3, pp. 221–223].

DEFINITION. The full subcategory of Γ-modules consisting of the Γ-modules A which satisfy the equivalent properties in Proposition 1.7 will be denoted by $\mathbf{D}_P$.

PROPOSITION 1.8. $\mathbf{D}_P$ *is equivalent to* Mod Λ *via the functor* $(P, -)$.

Further, $\mathbf{D}_P$ has that following property.

PROPOSITION 1.9. *The inclusion of* $\mathbf{D}_P$ *in* Mod Γ *is left exact.*

Now let Λ be an Artin algebra and M a finitely generated generator. Then of course Λ is in add M, so (M, Λ) is a finitely generated projective Γ-module where $\Gamma = \operatorname{End}(M)^{\mathrm{op}}$.

PROPOSITION 1.10. *Let Λ be an Artin algebra, M a finitely generated generator and* $\Gamma = \operatorname{End}(M)^{\mathrm{op}}$. *Then the following hold.*

(a) *The natural ring homomorphism* $\beta\colon \Lambda \to \operatorname{End}_\Gamma(M, \Lambda)^{\mathrm{op}}$ *is an isomorphism.*
(b) $((M, \Lambda), -) \circ (M, -) = ((M, \Lambda), (M, -)) = \mathrm{id}_{\operatorname{Mod}\Lambda}$.
(c) *For every Γ-module X, $((M, \Lambda), X)$ is a finitely generated Λ-module if and only if $T_{(M,\Lambda)}(X)$ (the submodule of X generated by all* Im f *where $f\colon (M, \Lambda) \to X$) is a finitely generated Γ-module.*
(d) *The natural ring homomorphism* $(\Gamma, \Gamma) \to ((M, \Lambda), \Gamma), ((M, \Lambda), \Gamma))$ *is an isomorphism.*
See [2, p. 31] *and* [3, p. 221].

We are now going to give the definition of a Wedderburn module (see [2, p. 23] and [3, pp. 244–246].

DEFINITION. Let Γ be an Artin algebra. If P is a finitely generated projective Γ-module with the property that $(P, -)\colon (\Gamma, \Gamma) \to ((P, \Gamma), (P, \Gamma))$ is an isomorphism, then P is called a Wedderburn Γ-module.

We now see that if Λ is an Artin algebra and M is a finitely generated generator, then (M, Λ) is a Wedderburn End $(M)^{\mathrm{op}}$-module.

PROPOSITION 1.11. *Let Γ be an Artin algebra, P a Wedderburn Γ-module and $\Lambda = \mathrm{End}\ (P)^{\mathrm{op}}$. Then the following hold.*

(a) (P, Γ) *is a finitely generated generator over* Λ.
(b) $(P, -)\colon \Gamma^{\mathrm{op}} \to \mathrm{End}_\Lambda\ (P, \Gamma)$ *is an isomorphism.*
(c) *The natural homomorphism* $P \to ((P, \Gamma), \Lambda)$ *is an isomorphism.*

Now consider the map V between pairs given by $V(R, A) = (\mathrm{End}\ (A)^{\mathrm{op}}, (A, R))$ for a ring R and an R-module A. If we now let (R, A) be the pair (Λ, M) $((\Gamma, P))$ when $\Lambda(\Gamma)$ is an Artin algebra and M a finitely generated generator (P a Wedderburn Γ-module), we see that V applied twice gives the identity and that (M, Λ) is a Wedderburn End $(M)^{\mathrm{op}}$-module ((P, Γ) a finitely generated End $(P)^{\mathrm{op}}$-generator).

DEFINITION. Let Λ be a ring and A a Λ-module. Then the socle of A is defined by Soc $A = M$ where M is the maximal semisimple submodule in A.

DEFINITION. A Wedderburn Γ-module P is a minimal Wedderburn Γ-module if the only indecomposable summands in P are the projective covers of the simple modules in Soc $(E_0(\Gamma) \amalg E_1(\Gamma)$ (see [2, p. 33] and [3, pp. 255–256]).

PROPOSITION 1.12. *Let Λ be an Artin algebra and M a finitely generated generator over* Mod Λ. *Then (M, Λ) is a minimal Wedderburn Γ-module where $\Gamma = \mathrm{End}\ (M)^{\mathrm{op}}$ if and only if every indecomposable injective Λ-module occurs as a summand of $E_0(M) \amalg E_1(M)$.*

In the rest of this paper let r_Λ be the radical of the ring Λ. By the Loewy length $\mathscr{L}(\Lambda)$ of an Artin ring Λ we mean the least integer n such that $r_\Lambda^n = 0$.

Section 2

In this section we are going to give a complete characterization of End $(M)^{\mathrm{op}}$ for an Artin algebra Λ and $M = \Lambda \amalg \Lambda/r_\Lambda$, where there is no restriction on the Loewy length of Λ.

DEFINITION. Let R be a ring and A an R-module. A submodule B of A is characteristic if whenever $f \in (A, A), f(B) \subseteq B$.

The following proposition is a generalization of a result stated in [2, p. 43] without proof.

PROPOSITION 2.1. *Let Λ be an Artin algebra, $\{Q_1, \ldots, Q_s\}$ a complete set of nonisomorphic projective Λ-modules and for each of these let $Q_i \supset A_{0,i} \supset A_{1,i} \supset \cdots A_{m_i,i} = 0$ be a proper chain of submodules of Q_i with each $A_{k,i}$ a characteristic submodule of Q_i and $A_{0,i} = r_\Lambda Q_i$. Let*

$$M = \coprod_i (Q_i \amalg Q_i/A_{m_i-1,i} \amalg \cdots \amalg Q_i/A_{0,i}) \quad \text{and} \quad \Gamma = \mathrm{End}(M)^{\mathrm{op}}.$$

Then Γ has the following properties:

(1) *Every projective Γ-module P without proper projective submodules has a unique composition series with nonisomorphic composition factors.*

(2) *Let $\{P_1, \ldots, P_n\}$ be a complete set of nonisomorphic projective Γ-modules without proper projective submodules and let P' be an arbitrary indecomposable projective Γ-module. Then there exists a unique $i \in \{1, 2, \ldots, n\}$ such that $P'/r_\Gamma P'$ occurs as a composition factor in the composition series of P_i.*

Proof. We first observe that since every simple Λ-module is in add M, the projective Γ-modules without proper projective submodules are exactly the projective Γ-modules of form (M, M_i) for a simple Λ-module M_i.

If we now look at one particular of these, we have the following exact sequences in mod Λ:

$$\begin{array}{llll}
0 \to A_{0,i} & \to Q_i & \xrightarrow{p_{m_i}} & M_i \to 0 \\
0 \to A_{0,i}/A_{m_i-1,i} & \to Q_i/A_{m_i-1,i} & \xrightarrow{p_{m_i-1}} & M_i \to 0 \\
 & \vdots & & \\
0 \to A_{0,i}/A_{1,i} & \to Q_i/A_{1,i} & \xrightarrow{p_1} & M_i \to 0 \\
0 \to 0 & \to Q_i/A_{0,i} & \xrightarrow{p_0} & M_i \to 0.
\end{array}$$

Here Q_i is the projective cover of the simple Λ-module M_i and the maps are the natural ones. These sequences now give rise to the following exact sequences in mod Γ:

$$\begin{array}{lll}
0 \to (M, A_{0,i}) & \to (M, Q_i) & \to (M, M_i) \\
0 \to (M, A_{0,i}/A_{m_i-1,i}) & \to (M, Q_i/A_{m_i-1,i}) & \to (M, M_i) \\
\quad\vdots & \quad\vdots & \quad\vdots \\
0 \to (M, A_{0,i}/A_{1,i}) & \to (M, Q_i/A_{1,i}) & \to (M, M_i) \\
\quad 0 & \to (M, Q_i/A_{0,i}) & = (M, M_i).
\end{array}$$

Now let

$$\begin{aligned}
B_{k,i} &= \mathrm{Im}\,(M, p_{k,i}) \\
&= \{f\colon M \to M_i \mid \text{there exists } g\colon M \to Q_i/A_{k,i} \text{ such that } f = p_{k,i} \circ g\}.
\end{aligned}$$

It now follows easily that

$$0 \subset B_{m_{i,i}} \subset B_{m_i-1,i} \subset \cdots \subset B_{2,i} \subset B_{1,i} \subset (M, M_i) = P_i$$

is a proper chain of submodules of P_i since $(M, Q_i/A_{k,i})$, $k = 0, 1, \ldots, m_i$, is a set of nonisomorphic indecomposable projective Γ-modules. Now an easy exercise using that each $A_{k,i}$ is a characteristic submodule of the projective Λ-module Q_i gives that $\operatorname{Im}(M, g) = B_{k,i}$ for every nonzero Λ-homomorphism $g\colon Q_i/A_{k,i} \to M_i$.

To prove (1) it now only remains to show that P_i does not have other submodules than the ones constructed. This follows since $(M, -)$ is a full and faithful functor and since projective covers exist in Mod Λ when Λ is an Artin algebra.

Property (2) now follows by the construction above, from the fact that every indecomposable projective Γ-module is of the form $(M, Q_i/A_{k,i})$. Observe that $\mathscr{L}(P_i) = m_i + 1$ if P_i is as in the proof of Proposition 2.1.

DEFINITION. An Artin algebra Γ is called a *pre unipro* Artin algebra if it satisfies conditions 1 and 2 in Proposition 2.1.

We now give a definition concerning the length of an End $(P)^{\mathrm{op}}$-module B in terms of a certain length of B as a Γ-module, where P is a projective Γ-module.

DEFINITION. Let Γ be an Artin ring, P a finitely generated projective Γ-module and B any finitely generated Γ-module. Then we define $l_P(B) = n$ where n is the number of composition factors in a composition series of B isomorphic to a simple summand in $P/r_\Gamma P$.

We now have the following lemma.

LEMMA 2.2. *Let Γ be an Artin algebra, P a finitely generated projective Γ-module and B any finitely generated Γ-module. Then $l(P, B) = l_P(B)$ where $l(P, B)$ is the length of (P, B) as an* End $(P)^{\mathrm{op}}$*-module.*

Proof. Suppose first $l_P(B) = 0$. Then $(P, B) = 0$ and vice versa. We claim that (P, S) is simple as an End $(P)^{\mathrm{op}}$-module for every simple summand S in $P/r_\Lambda P$. Let $f\colon P \to S$ be a nonzero homomorphism and $g\colon P \to S$ any other homomorphism. Then there exists an $h\colon P \to P$ such that the diagram

$$\begin{tikzcd} & P \arrow[dl, "h"'] \arrow[d, "g"] \\ P \arrow[r, "f"'] & S \end{tikzcd}$$

commutes. In other words, f generates (P, S) as an End $(P)^{\mathrm{op}}$-module. f was arbitrary so (P, S) is simple as an End $(P)^{\mathrm{op}}$-module. Now the rest follows by induction.

We now give a complete characterization of the Artin algebras we get as End $(M)^{\mathrm{op}}$ for an Artin algebra Λ and $M = \Lambda \amalg \Lambda/r_\Lambda$ with no restriction on the Loewy length of Λ.

THEOREM 2.3. *An Artin algebra Γ' is Morita equivalent to an Artin algebra Γ where $\Gamma = \mathrm{End}\,(\Lambda \amalg \Lambda/r_\Lambda)^{\mathrm{op}}$ for an Artin algebra Λ if and only if:*

(a) *Γ' is pre unipro.*

(b) *$l(P) \leq 2$ for every projective Γ'-module P without proper projective submodules.*

(c) *$P/r_{\Gamma'}P \not\subseteq \Gamma'$ for every nonsimple projective Γ'-module P without proper projective submodules.*

(d) *Γ' is relatively injective to every projective Γ'-module P without proper projective submodules.*

Proof. Suppose first that $\Gamma' = \mathrm{End}\,(M)^{\mathrm{op}}$ for an Artin algebra Λ and $M = \Lambda \amalg \Lambda/r_\Lambda$. Then Γ' is Monita equivalent to the ring $\Gamma'' = \mathrm{End}\,(N)^{\mathrm{op}}$ where $N = \coprod_i M_i$ for a complete set $\{M_i\}$ of nonisomorphic indecomposable summands in M by [1, p. 268]. Now from Proposition 2.1, Γ'' is pre unipro.

Let the equivalence from Mod Γ' to Mod Γ'' be given by F and the inverse by G. Let P be a projective Γ'-module without proper projective submodules. Then $F(P)$ has the same property as Γ''-module since equivalence preserves the lattice of submodules and projective modules [1, pp. 257–258]. So $F(P)$ has a unique composition series with nonisomorphic composition factors. Now since G is exact, takes nonisomorphic modules to nonisomorphic modules, and preserves the lattice of submodules $P \simeq GF(P)$ has the same property, i.e., Γ' satisfies (1) in the definition of pre unipro Artin algebra.

Now let $P_1, \ldots, P_n$ be a complete set of nonisomorphic projective Γ'-modules without proper projective submodules. Then $F(P_1), \ldots, F(P_n)$ is a complete set of nonisomorphic Γ'-modules without proper projective submodules. Now if P is any indecomposable projective Γ'-module there exists a unique i such that $F(P)/r_{\Gamma''}F/(P)$ is a composition factor in $F(P_i)$. Then $GF(P)/r_{\Gamma'}GF(P)$ is a composition factor in $GF(P_i)$ and this is unique. So we have that pre unipro is preserved by equivalence and therefore Γ' is pre unipro. Now suppose that $P/r_{\Gamma'}P \subseteq \Gamma'$ for a nonsimple projective Γ'-module P without proper projective submodules. Then $P = (M, M_i)$ for a simple nonprojective Λ-module M_i and since $(M, -)$ is a full functor there exists a nonzero homomorphism $f\colon M_i \to M$ such that $\mathrm{Im}\,(M,f) = P/r_{\Gamma'}P \subseteq \Gamma'$. Since M_i is simple f must be mono; therefore also $(M,f)\colon (M, M_i) \to (M, M)$ is mono. This is a contradiction, since $\mathrm{Im}\,(M,f)$ was supposed to be $P/r_\Gamma P$.

To prove (d) consider the diagram

$$\begin{array}{c} 0 \to X \to P \\ \quad\quad\downarrow \\ \quad\quad\Gamma' \end{array}$$

for every X such that $X \subseteq P$, where P is a projective Γ'-module without proper projective submodules. From (b) it follows that $X = P$ or $X = r_{\Gamma'}P$ which is simple. If $X = P$, $X \to P$ must be an isomorphism since P is of finite length, so the diagram can trivially be completed to a commutative diagram. Now suppose $l(P) = 2$ and that $X = S = r_{\Gamma'} P$. From the proof of Proposition 2.1 we have that $P = (M, M_i)$ for a simple Λ-module M_i and that $X = \operatorname{Im}(M, p)$ where $p\colon Q_i \to M_i$ is a projective cover. We have then the diagram

$$\begin{tikzcd} (M, Q_i) \arrow[dr] & & \\ & S \arrow[r] \arrow[d] & P = (M, M_i) \\ & \Gamma = (M, M). & \end{tikzcd}$$

But now $(M, -)$ is an equivalence between add M and $p(\Gamma')$, the category of finitely generated projective Γ'-modules, so we get the diagram

$$\begin{tikzcd} Q_i \arrow[r, "p"] \arrow[d, "g"] & M_i \\ M_i \arrow[r] & M \end{tikzcd}$$

in mod Λ. Then since M_i is simple and Q has a unique maximal submodule, there exists an $h\colon M_i \to M_i$ such that the diagram

$$\begin{tikzcd} Q_i \arrow[r] \arrow[d] & M_i \arrow[dl, "h"] \\ M_i \arrow[r] & M \end{tikzcd}$$

commutes. Then the diagram

$$\begin{tikzcd} (M, Q_i) \arrow[d] & \\ S \arrow[r] \arrow[d] & (M, M_i) = P \arrow[dl] \\ (M, M_i) \arrow[d] & \\ (M, M) & \end{tikzcd}$$

also commutes, i.e., Γ' is relatively injective to P.

To prove the converse we first need a couple of lemmas.

LEMMA 2.4. *Suppose that Γ' satisfies conditions* (a), (b), *and* (c) *in Theorem* 2.3 *and let $P_{\Gamma'} = P_0(\operatorname{Soc} \Gamma')$ be the projective cover of* Soc Γ'. *Then an indecomposable Γ'-module K is in* Ker $(P_{\Gamma'}, -)$ *if and only if $K \cong P/r_{\Gamma'}P$ for a nonsimple projective Γ'-module P without proper projective submodules.*

Proof. Suppose $K = P/r_{\Gamma'}P$ for a nonsimple projective Γ-module P without proper projective submodules. Then $(P_{\Gamma'}, K) = 0$ by Lemma 2.2.

Now suppose K is an indecomposable Γ-module in Ker $(P_{\Gamma'}, -)$. Then the projective cover of $K, f\colon \coprod P_i \to K$, is such that all the P_i are indecomposable nonsimple projective Γ-modules without proper projective submodules. Because every simple summand in the socle of $\coprod P_i$ is a summand in $P_{\Gamma'}/r_{\Gamma'}P_{\Gamma'}$ it follows that Soc $\coprod P_i = \coprod$ Soc $P_i \subseteq \operatorname{Ker} f$, but $\coprod$ Soc $P_i = \coprod r_{\Gamma'}P_i$, so K is semisimple. Therefore K is simple and isomorphic to $P/r_{\Gamma'}P$ for a nonsimple projective Γ'-module without proper projective submodules.

LEMMA 2.5. *Let Γ' satisfy conditions* (a), (b), (c), *and* (d) *in Theorem* 2.3 *and let $P_{\Gamma'}$ be as in Lemma* 2.4. *Then $P_{\Gamma'}$ is a minimal Wedderburn Γ'-module.*

Proof. If we are able to prove that $P_{\Gamma'}$ is a Wedderburn Γ'-module it is clear from the way we select $P_{\Gamma'}$ that it is minimal. Now let K be an indecomposable Γ'-module in Ker $(P_{\Gamma'}, -)$. Then from Lemma 2.4 we have the exact sequence $0 \to r_{\Gamma'}P \to P \to K \to 0$ for a projective Γ'-module P without proper projective submodules. From this sequence we get the long exact sequence

$$0 \to (K, \Gamma') \to (P, \Gamma') \to (r_{\Gamma'}P, \Gamma') \to \operatorname{Ext}' (K, \Gamma') \to \operatorname{Ext}' (P, \Gamma') = 0.$$

Now it follows from (c) that $(K, \Gamma') = 0$ and from (d) that $(P, \Gamma') \to (r_{\Gamma'}P, \Gamma')$ is epic, so $\operatorname{Ext}^1 (K, \Gamma') = 0$, i.e., $P_{\Gamma'}$ is a Wedderburn Γ'-module.

We are now able to give the rest of the proof of Theorem 2.3. It follows from the general Wedderburn correspondence that $\Gamma' \cong \operatorname{End} (P_{\Gamma'}, \Gamma')^{\mathrm{op}}$ so it remains to show that $(P_{\Gamma'}, \Gamma')$ has as summands precisely the projective and the simple End $(P_{\Gamma'})^{\mathrm{op}}$-modules. This follows trivially because Λ is a summand and that every simple End $(P_{\Gamma'})^{\mathrm{op}}$-module is $(P_{\Gamma'}, P_i)$ for a projective Γ'-module P_i without proper projective submodules.

Section 3

In this section we are going to specialize our result from Section 2 to the case where Λ is an Artin algebra with $\mathscr{L}(\Lambda) = 2$ and $M = \Lambda \amalg \Lambda/r_\Lambda$.

THEOREM 3.1. *An Artin algebra Γ' is Morita equivalent to an Artin algebra Γ where $\Gamma = \operatorname{End} (\Lambda \amalg \Lambda/r_\Lambda)^{\mathrm{op}}$ for an Artin algebra Λ with $\mathscr{L}(\Lambda) = 2$ if and only if Γ' satisfies the following conditions.*

(a) *Γ' is a pre unipro Artin algebra.*

(b) *$l(P) \le 2$ for every projective Γ'-module without proper projective submodules.*

(c) *$r_{\Gamma'}P = \amalg P_i$ for every indecomposable projective Γ'-module P with proper projective submodule, where the P_i are projective Γ'-modules without proper projective submodules.*

(d) *Every projective Γ'-module P without proper projective submodule is relatively injective to P.*

Proof. Suppose $\Gamma' = \text{End}\,(M)^{\text{op}}$ for an Artin algebra Λ with $\mathscr{L}(\Lambda) = 2$ and $M = \Lambda \amalg \Lambda/r_\Lambda$. Then one easily sees that Γ' has the property claimed when one observes that $r_{\Gamma'}(M, Q) = (M, r_\Lambda Q)$ for every indecomposable projective Λ-module Q.

For the converse we need a lemma. Observe that property (c) in Theorem 3.1 implies property (c) in Theorem 2.3.

LEMMA 3.2. *Suppose Γ' satisfies conditions* (a), (b), (c), *and* (d) *in Theorem* 3.1. *Then Γ' is relatively injective to every projective Γ'-module P without proper projective submodules.*

Proof. The uniqueness in the definition of pre unipro and property (c) in Theorem 3.1 forces every $f\colon r_\Gamma P \to \Gamma'$ for a projective Γ'-module P without proper projective submodules to factor through a sum of copies of this P. So we have the picture

$$\begin{tikzcd} r_\Gamma P \arrow[r] \arrow[d] & P \\ \coprod P \arrow[d] & \\ \Gamma' & \end{tikzcd}$$

and by assumption this diagram can be completed to a commutative diagram

$$\begin{tikzcd} r_\Gamma P \arrow[r] \arrow[d] & P \arrow[dl] \\ \coprod P \arrow[d] & \\ \Gamma' & \end{tikzcd}$$

and therefore Γ' is relatively injective to every projective Γ'-module P without proper projective submodules.

We are now able to complete the proof of Theorem 3.1. From Lemma 3.2 and Lemma 2.5 it follows that $P_{\Gamma'}$ defined as in Lemma 2.4 is as minimal Wedderburn Γ'-module. From Theorem 2.3 it follows that Γ' is Morita equivalent to $\text{End}\,(\Lambda \amalg \Lambda/r_\Lambda)^{\text{op}}$ where $\Lambda = \text{End}\,(P_{\Gamma'})^{\text{op}}$. It now remains to show that $r_\Lambda^2 = 0$. We have $r_\Lambda = (P_{\Gamma'}, r_{\Gamma'} P_{\Gamma'}) = (P_{\Gamma'}, \coprod P_i)$ where the P_i are projective Γ'-modules without proper projective submodules, i.e., r_Λ is semisimple and therefore $r_\Lambda^2 = 0$.

Section 4

In this section we are going to give the main result of this paper which gives a complete characterization of End $(M)^{\mathrm{op}}$ where Λ is an Artin algebra with $\mathscr{L}(\Lambda)=n$ and $M=\Lambda \amalg \Lambda/r_\Lambda^{n-1} \amalg \cdots \amalg \Lambda/r_\Lambda$. This is thus a generalization of the result in Section 3.

Now let Γ be a pre unipro Artin ring. Then every projective Γ-module P without proper projective submodules will, in a unique way, define $l(P)-1$ other indecomposable projective Γ-modules, namely the projective covers of the nonisomorphic composition factors. Since such a composition series is unique, it follows that these $l(P)-1$ other indecomposable projective Γ-modules connected with P are the projective covers of the submodules of P. Now let P_i, $i=1,\dots,s$, be a basic set of projective Γ-modules without proper projective submodules. For every such P_i we get the following situation:

$$\begin{tikzcd}[column sep=small] & P_{i,l(P_i)-1} \arrow[d, "p_{i,l(P_i)-1}"] & & P_{i,2} \arrow[d, "p_{i,2}"] & P_{i,1} \arrow[d, "p_{i,1}"] & \\ 0 \subset & r_\Gamma^{l(P_i)-1}P_i \subset \cdots \subset & & r_\Gamma^2 P_i \subset & r_\Gamma P_i \subset & P_i. \end{tikzcd}$$

Since the $P_{i,j}$ are projective and the $p_{i,j-1}$ are epic, there exist Γ-homomorphisms $f_{j,j-1}^{(i)}$ such that the diagram

$$\begin{tikzcd} P_{i,j} \arrow[r, "f_{j,j-1}^{(i)}"] \arrow[d, "p_{i,j}"] & P_{i,j-1} \arrow[d, "p_{i,j-1}"] \\ r_\Gamma^j P_i \arrow[r] & r_\Gamma^{j-1} P_i \end{tikzcd}$$

commutes. Now let $f_{j,k}=f_{k+1,k}^{(i)}\circ\cdots\circ f_{j,j-1}^{(i)}$ for $j>k$ and $f_{j,j}^{(i)}=\mathrm{id}\,P_{i,j}$. We now give a useful definition concerning these homomorphisms.

DEFINITION. Let Γ be an pre unipro Artin ring and let $P_\Gamma=P_0(\mathrm{Soc}\,\Gamma)$ (the projective cover of Soc Γ). If there exist $f_{j,k}^{(i)}$ as described above such that

(1) $l_{P_\Gamma}(P_{i,j})-l_{P_\Gamma}(\mathrm{Ker}\,f_{j,k}^{(i)})=l_{P_\Gamma}(P_{i,k})$

(2) $\mathrm{Ker}\,f_{j,j-1}^{(i)}$ is maximal in $\mathrm{Ker}\,f_{j,j-2}^{(i)}$ of the form $\amalg\, P_i$ where P_i are as above and such that none of the composed homomorphisms

$$P_i \to \mathrm{Ker}\,f_{j,j-1}^{(i)} \to \mathrm{Ker}\,f_{j,j-2}^{(i)}$$

split, Γ will be called a *unipro* Artin ring.

We are now in a situation where we can state and prove the main result of this paper.

THEOREM 4.1. *An Artin algebra Γ' is Morita equivalent to an Artin algebra Γ where $\Gamma=\mathrm{End}\,(\Lambda \amalg \Lambda/r_\Lambda^{n-1} \amalg\cdots\amalg \Lambda/r_\Lambda)^{\mathrm{op}}$ for an Artin algebra Λ with $\mathscr{L}(\Lambda)=n$ if and only if the following conditions are satisfied.*

(a) *Γ' is unipro,*

(b) *$\max\{l(P)\mid P$ a projective Γ'-module without proper projective submodules$\} = n$*

(c) *Every simple summand S in* $\operatorname{Soc}(E_0(\Gamma') \amalg E_1(\Gamma'))$ *is isomorphic to* $\operatorname{Soc} P$ *for a projective Γ'-module P without proper projective submodules.*

Proof. Let Λ be an Artin algebra with $\mathscr{L}(\Lambda) = n$,

$$M = \Lambda \amalg \Lambda/r_\Lambda^{n-1} \amalg \cdots \amalg \Lambda/r_\Lambda$$

and $\Gamma = \operatorname{End}(M)^{\mathrm{op}}$. From Proposition 2.1 it follows that Γ is pre unipro and that (b) is satisfied. Since (M, Λ) is a Wedderburn Γ-module

$$\operatorname{Soc}(E_0(\Gamma) \amalg E_1(\Gamma))$$

will contain only simple summands of the form $(M, Q)/r_\Gamma(M, Q)$ for an indecomposable projective Λ-module Q. From the construction of the unique composition series of $(M, Q/r_\Lambda Q)$ for an indecomposable projective Λ-module Q in the proof of Proposition 2.1, it then follows that Γ satisfies condition (c). So it remains only to show that Γ is unipro. Consider therefore the commutative diagram

$$\begin{tikzcd}
(M, Q) \arrow[r] \arrow[d, "(M, p_{\mathscr{L}(Q)-1})"] & \cdots \arrow[r] & (M, Q/r_\Lambda^3 Q) \arrow[r] \arrow[d, "(M, p_2)"] & (M, Q/r_\Lambda^2 Q) \arrow[d, "(M, p_1)"] & \\
0 \subset r_\Gamma^{\mathscr{L}(Q)-1}(M, Q/r_\Lambda Q) & \subset \cdots \subset & r_\Gamma^2(M, Q/r_\Lambda Q) & \subset r_\Gamma(M, Q/r_\Lambda Q) & \subset (M, Q/r_\Lambda Q)
\end{tikzcd}$$

for an indecomposable projective Λ-module Q. We are now in the situation described before the definition of a unipro Artin ring. We claim that the homomorphisms $(M, p_{j,k})$, where

$$p_{j,k}\colon Q/r_\Lambda^{j+2}Q \to Q/r_\Lambda^{k+1}Q$$

are the natural epimorphisms, $j \geq k$, have the properties in the definition of a unipro Artin ring. Now as before let $P_\Gamma = P_0(\operatorname{Soc}\Gamma)$. We then have

$$\begin{aligned}
& l_{P_\Gamma}(M, Q/r_\Lambda^{i+1}Q) - l_{P_\Gamma}\operatorname{Ker}(M, P_{i,j}) \\
&= l_{(M,\Lambda)}(M, Q/r_\Lambda^{i+1}Q) - l_{(M,\Lambda)}\operatorname{Ker}(M, P_{i,j}) \\
&= l((M, \Lambda), (M, Q/r_\Lambda^{i+1}Q)) - l((M, \Lambda), (M, r_\Lambda^{j+1}/r_\Lambda^{i+1}Q)) \\
&= l(Q/r_\Lambda^{i+1}Q) - l(r_\Lambda^{j+1}Q/r_\Lambda^{i+1}Q) \\
&= l(Q/r_\Lambda^{j+1}Q) \\
&= l_{(M,\Lambda)}(M, Q/r_\Lambda^{j+1}Q) = l_{P_\Gamma}(M, Q/r_\Lambda^{j+1}Q)
\end{aligned}$$

from the general Wedderburn correspondence and Lemma 2.2.

Part (b) in the definition of unipro is satisfied since none of the composed homomorphisms $M_i \subseteq r_\Lambda^j Q/r_\Lambda^{j+1}Q \to r_\Lambda^{j-1}Q/r_\Lambda^{j+1}Q$ where M_i is a simple Λ-module splits and $r_\Lambda^j Q/r_\Lambda^{j+1}Q$ is maximal with this property.

Before we continue we need some lemmas.

LEMMA 4.2. *Let* Γ *be an unipro Artin ring and let the notation be as in the definition of unipro. Then*

$$l_{P_\Gamma}(\operatorname{Ker} f^{(i)}_{j,j-1}) = l_{P_\Gamma}(\operatorname{Ker} f^{(i)}_{j+1,j-1}) - l_{P_\Gamma}(\operatorname{Ker} f^{(i)}_{j+1,j})$$

Proof. Trivial from (a) in the definition of unipro Artin algebras.

LEMMA 4.3. *Let* Γ *be an unipro Artin algebra such that for every simple summand* S *in* $\operatorname{Soc}(E_0(\Gamma) \amalg E_1(\Gamma))$ *there exists a projective* Γ*-module* P *without proper projective submodules with* $S \cong \operatorname{Soc} P$. *If we now use the notation in the definition of unipro Artin ring we have*

$$(P_\Gamma, \operatorname{Ker} f^{(i)}_{j,j-1}) = r_\Lambda(P_\Gamma, \operatorname{Ker} f^{(i)}_{j,j-2}) \quad \textit{where } \Lambda = \operatorname{End}(P_\Gamma)^{\mathrm{op}}.$$

Proof. Observe that P_Γ now is a minimal Wedderburn Γ-module. In the rest of the proof, for simplicity, let $K^{(i)}_{j,k}$ denote $\operatorname{Ker} f^{(i)}_{j,k}$, $j \geq k$. From Lemma 4.2 it follows that

$$l_{P_\Gamma}(K^{(i)}_{j-1,j-2}) = l_{P_\Gamma}(K^{(i)}_{j,j-2}) - l_{P_\Gamma}(K^{(i)}_{j,j-1}).$$

From Lemma 2.2 it then follows that

$$l(P_\Gamma, K^{(i)}_{j,j-2}/K_{j,j-1}) = l(P_\Gamma, K^{(i)}_{j-1,j-2}).$$

Now $K^{(i)}_{j,j-2}/K^{(i)}_{j,j-1} \subseteq K^{(i)}_{j-1,j-2}$ and since $(P_\Gamma, -)$ is exact we have

$$(P_\Gamma, K^{(i)}_{j,j-2}/K^{(i)}_{j,j-1}) = (P_\Gamma, K^{(i)}_{j,j-2})/(P_\Gamma, K^{(i)}_{j,j-1}) = (P_\Gamma, K^{(i)}_{j-1,j-2}).$$

By assumption $K^{(i)}_{j-1,j-2} \cong \coprod P_n$, where the P_i are projective Γ-modules without proper projective submodules. It then follows that $(P_\Gamma, K^{(i)}_{j-1,j-2})$ is semisimple, i.e., $r_\Lambda(P_\Gamma, K^{(i)}_{j,j-2}) \subseteq (P_\Gamma, K^{(i)}_{j,j-1})$. Suppose now that this inclusion is proper. Then there exists a Λ-module B such that B is a maximal submodule of $(P_\Gamma, K^{(i)}_{j,j-2})$ and $(P_\Gamma, K^{(i)}_{j,j-1}) \nsubseteq B$. Since $(P_\Gamma, K^{(i)}_{j,j-1})$ is semisimple there exists a simple summand $S = (P_\Gamma, P_n)$ in $(P_\Gamma, K^{(i)}_{j,j-1})$ such that $S \amalg B = (P_\Gamma, K^{(i)}_{j,j-2})$. So the composed homomorphism

$$S \subset (P_\Gamma, K^{(i)}_{j,j-1}) \subset (P_\Gamma, K^{(i)}_{j,j-2})$$

splits. Now let $G\colon \operatorname{Mod} \Gamma \to \mathbf{D}_P$ be the functor described in the preliminaries. $K^{(i)}_{j,j-2}$ is $(P_\Gamma, -)$-torsionfree because it is a submodule of a projective Γ-module. So

$$K^{(i)}_{j,j-2} \subseteq G(K^{(i)}_{j,j-2}).$$

Further, we have $(P_\Gamma, K_{j,j-2}) = (P_\Gamma, G(K^{(i)}_{j,j-2}))$. From [2], [3], $\mathbf{D}_{P_\Gamma}$ is equivalent to $\operatorname{Mod}(\Lambda)$ where the equivalence is given by $(P_\Gamma, -)$. We now have that the composed homomorphism

$$(P_\Gamma, P_n) \subset (P_\Gamma, K^{(i)}_{j,j-1}) \subset (P_\Gamma, K^{(i)}_{j,j-2}) = (P_\Gamma, G(K^{(i)}_{j,j-2}))$$

splits with P_i and $G(K^{(i)}_{j,j-2})$ both in $\mathbf{D}_{P_\Gamma}$. Therefore the composed Γ-homomorphism

$$P_n \subset K^{(i)}_{j,j-1} \subset K^{(i)}_{j,j-2} \subset G(K^{(i)}_{j,j-2})$$

also splits, so $P_n \subset K^{(i)}_{j,j-1} \subset K^{(i)}_{j,j-2}$ splits, a contradiction.

LEMMA 4.4. *Let Λ be an Artin ring and $0 \to C' \to C \to C'' \to 0$ an exact sequence of Λ-modules. If $C' \subseteq r_\Lambda C$ then $f^{-1}(r_\Lambda C'') = r_\Lambda C$.*

Proof. Trivial.

We are now able to prove the rest of Theorem 4.1. Let P_i be a projective Γ'-module without proper projective submodules and $P_{\Gamma'} = P_0(\mathrm{Soc}\,\Gamma')$. We are now in the following situation:

$$\begin{array}{ccccc} P_{i,l(P_i)-1} \to \cdots \to P_{i,2} & \xrightarrow{f^{(i)}_{2,1}} & P_{i,1} & & \\ \downarrow \qquad\qquad\quad \downarrow & & \downarrow & & \\ r^{l(P_i)-1}_{\Gamma'} P_i \subset \cdots \subset r^2_{\Gamma'} P_i & \subset & r_{\Gamma'} P_i & \subset & P_i. \end{array}$$

Since

$$l_{P_{\Gamma'}}(P_{i,j}) = l_{P_{\Gamma'}}(P_{i,k}) - l_{P_{\Gamma'}}(\mathrm{Ker}\, f^{(i)}_{k,j}),\ k \geq j,$$

it follows that

$$(P_\Gamma, P_{i,l(P_i)-1}) \to (P_{\Gamma'}, P_{i,l(P_i)-2}) \to \cdots \to (P_{\Gamma'}, P_{i,2}) \to (P_{\Gamma'}, P_{i,1}) \to (P_{\Gamma'}, P_i)$$

is a proper chain of epimorphisms with $(P_{\Gamma'}, P_i)$ simple. Furthermore we have

$$(P_{\Gamma'}\ \mathrm{Ker}\, f_{i,1}) = r_\Lambda(P_\Gamma, P_{i,2}).$$

By induction we get $(P_{\Gamma'}, \mathrm{Ker}\, f^{(i)}_{j,k}) = r^{j-k}_\Lambda(P_{\Gamma'}, P_{i,j})$. From the general Wedderburn correspondence $\Gamma' \leftrightarrow \mathrm{End}\,(P_{\Gamma'}, \Gamma')^{\mathrm{op}}$ it now follows that Γ' is of the type claimed since $Q/r^j_\Lambda Q$ is a summand of $(P_{\Gamma'}, \Gamma')$ for every indecomposable projective Λ-module Q for $1 \leq j \leq \mathscr{L}(Q)$ and these are the only ones. This finishes the proof of our main result.

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