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RIEMANN-LEBESGUE CENTERS OF PLANE DOMAINS

BY

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1. Introduction

Let $G \subseteq \mathbf{C}$ be a plane domain which supports nonconstant bounded analytic functions, and let $\zeta \in G$. For $n = 0, 1, 2, \ldots$ define

$$A_n = A_n(\zeta, G) = \sup \{ |f^{(n)}(\zeta)| : f \in B_H(G), \|f\|_\infty = 1\},$$

where $B_H(G)$ is the space of functions analytic and bounded in G, and $\|f\|_\infty$ denotes the supremum norm. A point $\zeta \in G$ such that for each $f \in B_H(G)$,

$$\frac{f^{(n)}(\zeta)}{A_n} \to 0 \quad (n \to \infty),$$

is called a Riemann-Lebesgue center (R.L.C.) of G. The name "Riemann-Lebesgue center" is an allusion to the well-known Riemann-Lebesgue lemma, which implies that 0 is a R.L.C. of the unit disc $\{z: |z| < 1\}$.

In this paper we shall be concerned with certain problems about R.L.C.'s from the point of view of "hard" analysis. However, the idea of R.L.C. arose in conversation with Stephen D. Fisher with reference to different topologies on function spaces. The latter are dealt with in a paper of Rubel and Ryff [2]. A number of particular topics are considered in greater detail in the papers listed by Rubel and Ryff in their bibliography. The results of the present paper do not, unfortunately, seem to have any applicability to the problem of different topologies on function spaces, so we present them for their intrinsic interest.

2. Statement of results

THEOREM 1. *Let $D = \{z \in \mathbf{C}: |z| < 1\}$ and*

$$A_n = A_n(\zeta; D) = \sup \{|f^{(n)}(\zeta)| : f \in B_H(D), \|f\|_\infty \leq 1\},$$

where $0 < |\zeta| < 1$. If $f \in B_H(D)$, then

$$\liminf_{n \to \infty} \frac{|f^{(n)}(\zeta)|}{A_n} = 0;$$

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and there is an $f \in B_H(D)$ *such that*

$$\limsup_{n\to\infty} \frac{|f^{(n)}(\zeta)|}{A_n} > 0.$$

COROLLARY. *The origin is the only R.L.C. of D.*

THEOREM 2. *Let* β *satisfy* $0 < \beta < \pi/2$ *and G be a domain satisfying*

$$D \subseteq G \subseteq D \cup \{z \in \mathbf{C}\colon |\arg(-z)| < \beta\}.$$

Then the origin is a R.L.C. of G.

THEOREM 3. *Let* $G \subseteq \mathbf{C}$ *be a bounded domain,* $\zeta \in G, \Delta \subseteq G$, *and* $\partial G \cap \partial\Delta$ *be nonempty and finite, where* $\Delta = \{z \in \mathbf{C}\colon |z - \zeta| < 1\}$. *Suppose that from each point of* $\partial G \cap \partial \Delta$ *emanates a half-line lying in the complement of G. Then* ζ *is not a R.L.C. of G.*

THEOREM 4. *If* $n = 0, 1, 2, \ldots,$ *or* $n = \aleph_0$, *then there is a domain G which contains precisely n R.L.C.'s.*

3. Proof of Theorem 1

We shall indicate the idea of the proof of the first part of Theorem 1 after the following lemma.

LEMMA 5. *Suppose that* $0 < \beta < 1$ *and let* $A_n = A_n(\beta, D)$. *There is a constant* $\Lambda > 1$ *such that*

$$\frac{1}{\Lambda}\frac{n!}{\sqrt{n}(1-\beta)^n} \le A_n \le \Lambda \frac{n!}{\sqrt{n}(1-\beta)^n} \quad (n \ge 1).$$

Proof. The precise values of the A_n for odd n have been given by Macintyre and Rogosinski [1], but for even n the precise values of the A_n do not appear to be known.

If $f \in B_H(D)$ is extremal for A_n consider $g = (z - \beta)f$. Then

$$g^{(n+1)}(\beta) = (n+1)f^{(n)}(\beta)$$

and so

$$A_n = |f^{(n)}(\beta)| = \|g\|_\infty \frac{1}{n+1}\frac{|g^{(n+1)}(\beta)|}{\|g\|_\infty} \le (1+\beta)\frac{A_{n+1}}{n+1}.$$

If $g \in B_H(D)$ and is extremal for A_{n+1} consider

$$f = \frac{g - g(\beta)}{z - \beta}.$$

Then $g^{(n+1)}(\beta) = (n+1)f^{(n)}(\beta)$ and so

$$A_{n+1} = |g^{(n+1)}(\beta)| = \|f\|_\infty(n+1)\frac{|f^{(n)}(\beta)|}{\|f\|_\infty} \le \frac{2}{1-\beta}(n+1)A_n.$$

Hence

$$\frac{1-\beta}{2}\frac{A_{n+1}}{n+1} \le A_n \le (1+\beta)\frac{A_{n+1}}{n+1}$$

so that from the known values of the A_k for k odd one gets estimates for the A_k with k even. The lemma follows from the results of Macintyre and Rogosinski and the preceding inequalities.

Let $f \in B_H(D)$ with $\|f\|_\infty = 1$ and assume $\zeta = \beta$, where $0 < \beta < 1$. For positive integers n define

$$d = d(n) = \min\{|f^{(\nu)}(\beta)|/A_\nu : \nu = n, n+1, \ldots, 2n\}.$$

If $\zeta_0, \ldots, \zeta_n$ are suitably chosen of unit modulus, then

$$\sum_{\nu=0}^{n} \zeta_\nu \frac{f^{(n+\nu)}(\beta)}{A_{n+\nu}} = \sum_{\nu=0}^{n} \frac{|f^{(n+\nu)}(\beta)|}{A_{n+\nu}} \ge (n+1)\,d \tag{3.1}$$

In the left hand sum of (3.1) use

$$f^{(n+\nu)}(\beta) = \frac{(n+\nu)!}{2\pi i}\int_{|z|=1}\frac{f(z)}{(z-\beta)^{n+\nu+1}}\,dz$$

and the estimates of Lemma 5 and it follows that

$$\Lambda\sqrt{2n}\sum_{\nu=0}^{n}(1-\beta)^{n+\nu}\frac{\zeta_\nu}{2\pi i}\int_{|z|=1}\frac{f(z)}{(z-\beta)^{n+\nu+1}}\,dz \ge (n+1)\,d.$$

Hence, since $\|f\|_\infty = 1$,

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{\nu=0}^{n}\frac{\zeta_\nu(1-\beta)^\nu}{(e^{i\theta}-\beta)^{n+\nu+1}}\right|d\theta \ge \frac{d}{\Lambda\sqrt{2}}\frac{\sqrt{n}}{(1-\beta)^n}. \tag{3.2}$$

The first part of Theorem 1 follows by finding a suitable upper estimate for the left hand side of (3.2) and the next few lemmas contain results that are appropriate for such an estimate.

LEMMA 6. *Given $\varepsilon > 0$ there is an $\alpha = \alpha(\varepsilon) > 0$ so that*

$$\int_{\alpha/\sqrt{n}\le|\theta|\le\pi}\frac{d\theta}{|e^{i\theta}-\beta|^n} \le \frac{\varepsilon}{\sqrt{n}}\frac{1}{(1-\beta)^n}$$

for all large n.

Proof. For $|\theta| \le \pi$,

$$|e^{i\theta}-\beta|^2 = (1-\beta)^2 + 4\beta\sin^2\theta/2 \ge (1-\beta)^2 + 4\beta\theta^2/\pi^2.$$

There is a $\gamma > 0$ such that

$$\frac{1}{1+t} \le e^{-\gamma t} \quad \left(0 \le t \le \frac{4\beta}{(1-\beta)^2}\right)$$

and so, for $|\theta| \le \pi$,

$$\frac{1}{|e^{i\theta}-\beta|^2} \le \frac{1}{(1-\beta)^2} \exp\left[-\gamma \frac{4\beta}{(1-\beta)^2} \cdot \frac{\theta^2}{\pi^2}\right] = \frac{1}{(1-\beta)^2} e^{-\gamma'\theta^2}, \text{ say.}$$

Therefore, given $\xi > 0$, for all large n,

$$\int_{\xi/\sqrt{n} \le |\theta| \le \pi} \frac{d\theta}{|e^{i\theta}-\beta|^n} \le \frac{2}{(1-\beta)^n} \int_{\xi/\sqrt{n}}^{\pi} e^{-\gamma' n\theta^2/2}\, d\theta$$

and putting $\sqrt{\frac{1}{2}\gamma' n}\,\theta = g$ we obtain

$$\int_{\xi/\sqrt{n}}^{\pi} e^{-\gamma' n\theta^2/2}\, d\theta \le \sqrt{\frac{2}{\gamma' n}} \int_{\xi\sqrt{\gamma'/2}}^{\infty} e^{-\phi^2}\, d\phi.$$

Hence, given $\varepsilon > 0$, if $\alpha = \alpha(\varepsilon) > 0$ is chosen so that

$$\frac{2\sqrt{2}}{\sqrt{\gamma'}} \int_{\alpha\sqrt{\gamma'/2}}^{\infty} e^{-\phi^2}\, d\phi < \varepsilon$$

the lemma follows from the above estimates by setting $\xi = \alpha$.

LEMMA 7. *Given $\varepsilon > 0$, let $\alpha = \alpha(\varepsilon)$ be the same as in Lemma 5. For all large n, if $\zeta_0, \zeta_1, \ldots, \zeta_n$ are unimodular numbers then*

$$\int_{\alpha/\sqrt{n} \le |\theta| \le \pi} \left| \sum_{\nu=0}^{n} \frac{\zeta_\nu (1-\beta)^\nu}{(e^{i\theta}-\beta)^{n+\nu+1}} \right| d\theta \le \frac{\varepsilon\sqrt{n}}{(1-\beta)^{n+1}}.$$

Proof. The integral on the left is bounded above by

$$\sum_{\nu=0}^{n} (1-\beta)^\nu \int_{\alpha/\sqrt{n} \le |\theta| \le \pi} \frac{d\theta}{|e^{i\theta}-\beta|^{n+\nu+1}} \le \sum_{\nu=0}^{n} (1-\beta)^\nu \frac{\varepsilon}{\sqrt{n}} \frac{1}{(1-\beta)^{n+\nu+1}}$$

from Lemma 6, noting that $\alpha/\sqrt{n+\nu+1} < \alpha/\sqrt{n}$ $(\nu = 0, \ldots, n)$.

LEMMA 8. *Given β $(0 < \beta < 1)$ there is an $\eta > 0$ and a function $\psi(\theta)$ analytic for $|\theta| < \eta$ with $\psi(0) = \psi'(0) = 0$ such that*

$$\frac{1}{e^{i\theta}-\beta} = \frac{1}{1-\beta} \exp\left(\frac{-i\theta}{1-\beta} + \psi(\theta)\right) \quad (|\theta| < \eta).$$

Proof. It is easily checked that $\psi(\theta)$ defined by

$$\psi(\theta) = \log\left[\frac{1-\beta}{e^{i\theta}-\beta} e^{i\theta/(1-\beta)}\right]$$

for small θ satisfies the requirements of the lemma.

LEMMA 9. *Let $\zeta_0, \ldots, \zeta_n$ be unimodular and $\psi(\theta)$ the function of Lemma* 8 *and set*

$$F(\theta) = \sum_{\nu=0}^{n} \zeta_\nu \exp\left(\frac{-i\nu\theta}{1-\beta} + \nu\psi(\theta)\right),$$

$$G(\theta) = \sum_{\nu=0}^{n} \zeta_\nu e^{-i\nu\theta/(1-\beta)}.$$

Given $\xi > 0$, for all large n

$$\int_{|\theta| \le \xi/\sqrt{n}} |F(\theta) - G(\theta)| \, d\theta \le \text{const. } n^{1/4}.$$

where const. *denotes a constant which may depend on ξ.*

Proof. From Lemma 8 we can write $\psi(\theta) = c\theta^2 + \lambda(\theta)$ $(|\theta| < \eta)$ for some constant c, where $|\lambda(\theta)| \le \text{const. } |\theta|^3$ $(|\theta| \le \eta/2)$. Consider

$$e^{\nu\psi(\theta)} - 1 = e^{c\nu\theta^2}(e^{\nu\lambda(\theta)} - 1) + e^{c\nu\theta^2} - 1. \tag{3.3}$$

For $\xi > 0$ given and all large n,

$$|e^{c\nu\theta^2}(e^{\nu\lambda(\theta)} - 1)| \le \text{const. } \nu|\theta|^3 \quad (|\theta| \le \xi/\sqrt{n};\ \nu = 0, 1, 2, \ldots, n).$$

Hence

$$\int_{|\theta| \le \xi/\sqrt{n}} \left| \sum_{\nu=0}^{n} \zeta_\nu e^{-i\nu\theta/(1-\beta)} e^{c\nu\theta^2}(e^{\nu\lambda(\theta)} - 1) \right| d\theta \le \text{const. } n^2 \int_0^{\xi/\sqrt{n}} \theta^3 \, d\theta \le \text{const.} \tag{3.4}$$

Also,

$$\int_{|\theta| \le \xi/\sqrt{n}} \left| \sum_{\nu=0}^{n} \zeta_\nu e^{-i\nu\theta/(1-\beta)}(e^{c\nu\theta^2} - 1) \right| d\theta$$

$$\le \sum_{k=1}^{\infty} \frac{|c|^k}{k!} \int_{|\theta| \le \xi/\sqrt{n}} \left| \sum_{\nu=0}^{n} \zeta_\nu \nu^k e^{-i\nu\theta/(1-\beta)} \theta^{2k} \right| d\theta. \tag{3.5}$$

For $k = 1, 2, \ldots$ and large n,

$$\int_{|\theta| \le \xi/\sqrt{n}} \left| \sum_{\nu=0}^{n} \zeta_\nu \nu^k e^{-i\nu\theta/(1-\beta)} \right| \theta^{2k} \, d\theta$$

$$\le \left(\int_{|\theta| \le \xi/\sqrt{n}} \left| \sum_{\nu=0}^{n} \zeta_\nu \nu^k e^{-i\nu\theta/(1-\beta)} \right|^2 d\theta \right)^{1/2} \left(\int_{|\theta| \le \xi/\sqrt{n}} \theta^{4k} \, d\theta \right)^{1/2}$$

$$\le \text{const. } \frac{\xi^{2k}}{\sqrt{k}\, n^{k+1/4}} \left(\int_{-\xi/(1-\beta)\sqrt{n}}^{\xi/(1-\beta)\sqrt{n}} \left| \sum_{\nu=0}^{n} \zeta_\nu \nu^k e^{-i\nu\phi} \right|^2 d\phi \right)^{1/2}$$

$$\le \text{const. } \frac{\xi^{2k}}{\sqrt{k}\, n^{k+1/4}} \left(\sum_{\nu=0}^{n} \nu^{2k} \right)^{1/2}$$

$$\le \text{const. } \frac{\xi^{2k}}{k} n^{1/4}, \tag{3.6}$$

where we have assumed n large enough to ensure that $\xi/(1-\beta)\sqrt{n} < \pi$ and used Parseval's relation.

Noting (3.3), the lemma follows from (3.4), (3.5), and (3.6).

LEMMA 10. *Suppose $\zeta_0, \zeta_1, \ldots, \zeta_n$ are unimodular and $\varepsilon > 0$. For all large n,*

$$\int_0^{2\pi} \left| \sum_{\nu=0}^{n} \frac{\zeta_\nu(1-\beta)^\nu}{(e^{i\theta}-\beta)^{n+\nu+1}} \right| d\theta \leq \frac{\varepsilon\sqrt{n}}{(1-\beta)^{n+1}} + \frac{\text{const. } n^{1/4}}{(1-\beta)^{n+1}}.$$

Proof. Choose $\alpha = \alpha(\varepsilon)$ so that Lemma 5 holds and assume n is large enough to ensure that $\alpha/\sqrt{n} < \eta$, where η is the number of Lemma 7. Using Lemma 8 we find, in the notation of Lemma 9, that

$$\begin{aligned}\int_{|\theta|\leq\alpha/\sqrt{n}} \left| \sum_{\nu=0}^{n} \frac{\zeta_\nu(1-\beta)^\nu}{(e^{i\theta}-\beta)^{n+\nu+1}} \right| d\theta &= \frac{1}{(1-\beta)^{n+1}} \int_{|\theta|\leq\alpha/\sqrt{n}} |F(\theta)|\, d\theta \\ &\leq \frac{\text{const. } n^{1/4}}{(1-\beta)^{n+1}} + \frac{1}{(1-\beta)^{n+1}} \int_{|\theta|\leq\alpha/\sqrt{n}} |G(\theta)|\, d\theta,\end{aligned}$$

by the result of Lemma 9.

Using Schwarz's inequality and then setting $-\theta/(1-\beta) = \phi$ we find that

$$\begin{aligned}\int_{|\theta|\leq\alpha/\sqrt{n}} |G(\theta)|\, d\theta &\leq \text{const. } n^{-1/4} \left(\int_{-\alpha/(1-\beta)\sqrt{n}}^{\alpha/(1-\beta)\sqrt{n}} \left| \sum_{\nu=0}^{n} \zeta_\nu e^{i\nu\phi} \right|^2 d\phi \right)^{1/2} \\ &\leq \text{const. } n^{1/4},\end{aligned}$$

provided n is large enough to ensure that $\alpha/(1-\beta)\sqrt{n} < \pi$. This estimate together with the preceding shows that

$$\int_{|\theta|\leq\alpha/\sqrt{n}} \left| \sum_{\nu=0}^{n} \frac{\zeta_\nu(1-\beta)^\nu}{(e^{i\theta}-\beta)^{n+\nu+1}} \right| d\theta \leq \text{const. } \frac{n^{1/4}}{(1-\beta)^{n+1}} \tag{3.7}$$

for all large n.

The lemma follows from Lemma 6 and (3.7).

From Lemma 10 and (3.2) it follows at once that $d = d(n) \to 0$ $(n \to \infty)$ and this gives the first part of Theorem 1 (and a little more).

In dealing with the second part of Theorem 1 again assume that $\zeta = \beta$, where $0 < \beta < 1$. If one observes from Lemma 6 that when $f \in B_H(D)$ the integral for $f^{(n)}(\beta)$ round $|z| = 1$ is dominated by the contribution from an arc about $z = 1$ that shrinks to $z = 1$ as $n \to \infty$, then it is not surprising that a "gliding hump" construction leads to a function of the kind sought. We shall not go into details of the construction apart from giving in the next lemma the basic result required. Note that this result and its proof make it clear that one can ensure that $f \in B_H(D)$ with $\|f\|_\infty = 1$ and

$$\limsup_{n\to\infty} \frac{|f^{(n)}(\beta)|}{A_n} = 1.$$

LEMMA 11. *Given $\sigma > 0$, ε $(0 < \varepsilon < \pi)$, ε' $(0 < \varepsilon' < 1)$ and $N \in \mathbf{N}$ there is a function $f \in B_H(D)$ and δ $(0 < \delta < \varepsilon)$ such that*

(i) $\|f\|_\infty = 1$,
(ii) $|f^{(n)}(\beta)| > (1 - \varepsilon')A_n$ *for some* $n > N$,
(iii) $|f(e^{i\theta})| < \sigma$ *for* $\varepsilon \leq |\theta| \leq \pi$ *and for* $|\theta| \leq \delta$.

Proof. We shall give a descriptive account of the construction of $f(z)$.

If $\gamma > 0$ is small then $1 - (1 - z)^\gamma$ will be small on $|z| = 1$ outside some arc about $z = 1$ and near to 1 on another such arc. Now

$$h(z) = z^m \frac{(z - \beta)^{m+1}}{(1 - \beta z)^{m+1}}$$

is extremal for A_{2m+1} and the dominant arc of the integral for $h^{(2m+1)}(\beta)$ round $|z| = 1$ will be within that where $1 - (1 - z)^\gamma$ is near to 1 for all large m. Hence the $(2m + 1)$st derivative of $(1 - (1 - z)^\gamma)h(z)$ at β will be near A_{2m+1} for γ "small" and m "large." This leads to a function satisfying (i) and (ii).

If $\gamma' > 0$ the function $(1 - (1 - z)^\gamma)(1 - z)^{\gamma'}h(z)$ will have the preceding properties and in addition will be small on $|z| = 1$ when z is "very near" to 1 provided γ' is sufficiently small. Hence one sees that f satisfying (i), (ii), and (iii) can be constructed.

4. Proof of Theorem 2

A number of lemmas are required. In what follows we assume that $0 < \beta < \pi/2$, $D = \{z\colon |z| < 1\}$ and that

$$O = D \cup \{z \in \mathbf{C}\colon |\arg(-z)| < \beta\}.$$

LEMMA 12. *If $n \in \mathbf{N}$, $k \in \mathbf{Z}$ such that $2\pi|k|/n \leq \pi/2 - \beta$ and $\zeta = e^{2\pi i k/n}$, then*

$$|e^{\zeta z - 1}| < 1/e \quad \textit{for } z \in O \cap \{z\colon |z| > 1\}.$$

Proof. For $z = re^{i\theta}$ with $\pi - \beta < \theta < \pi + \beta$,

$$\operatorname{Re} \zeta z = r \cos(\theta + 2\pi k/n)$$

and

$$\theta + 2\pi k/n < \pi + \beta + 2\pi|k|/n \leq \pi + \beta + \pi/2 - \beta = 3\pi/2,$$

and

$$\theta + 2\pi k/n > \pi - \beta - 2\pi|k|/n \geq \pi - \beta - (\pi/2 - \beta) = \pi/2.$$

Hence $\operatorname{Re} \zeta z < 0$ in $G \cap \{z\colon |z| > 1\}$, and so the result follows.

LEMMA 13. *If $n \in \mathbf{N}$, $N = [\sqrt{n}]$, $M \leq [N/2]$, and $\zeta_\nu = e^{2\pi i \nu N/n}$ for $\nu = -M, \ldots, M$, then for $|z| \leq 1$,*

$$\left| \sum_{\nu=-M}^{M} e^{n(\zeta_\nu z - 1)} \right| < \text{const.} \left(= 2 \sum_0^\infty e^{-2\nu^2} \right).$$

Proof. For $-\pi \le \phi \le \pi$ it is easy to see that $|e^{e^{i\phi}-1}| = e^{-2\sin^2(\phi/2)}$ has a maximum at $\phi = 0$ and decreases as $|\phi|$ increases from 0 to π. Therefore for $|z| \le 1$,

$$\left|\sum_{\nu=-M}^{M} e^{n(\zeta_\nu z-1)}\right| \le 2\sum_{\nu=0}^{M} |e^{n(e^{2\pi i\nu N/n}-1)}|$$
$$= 2\sum_{\nu=0}^{M} \exp\left(-2n\sin^2\frac{\pi\nu N}{n}\right).$$

For $\nu = 0, \dots, M$,

$$\frac{\nu N}{n} \le \frac{MN}{n} = \frac{[[\sqrt{n}]/2][\sqrt{n}]}{n} \le \frac{1}{2}$$

and hence, using

$$\sin\phi \ge \frac{2}{\pi}\phi \quad \left(0 \le \phi \le \frac{\pi}{2}\right),$$

we find that for $|z| \le 1$,

$$\left|\sum_{\nu=-M}^{M} e^{n(\zeta_\nu z-1)}\right| \le 2\sum_{\nu=0}^{M} \exp\left(-2n\left(\frac{2}{\pi}\frac{\pi\nu N}{n}\right)^2\right)$$
$$= 2\sum_{\nu=0}^{M} e^{-8\nu^2N^2/n}.$$

Since $N^2/n = [\sqrt{n}]^2/n \ge 1/4$, the lemma follows.

We now give the proof of Theorem 2.

Let $n \in \mathbf{N}$ and $N = [\sqrt{n}]$. Choose $M \in \mathbf{N}$ as large as possible with

$$\frac{2\pi M}{\sqrt{n}} \le \frac{\pi}{2} - \beta.$$

Then $M \ge c\sqrt{n}$ for all large n, where $c = c(\beta) > 0$ and $M \le [N/2]$. From Lemmas 12 and 13 it follows that

$$\left|\sum_{\nu=-M}^{M} e^{n(\zeta_\nu z-1)}\right| < \text{const.} \quad (z \in G).$$

If, for z in D, $\sum_{\nu=-M}^{M} e^{n(\zeta_\nu z-1)} = \sum_{\nu=0}^{\infty} a_\nu z^\nu$, then

$$a_n = \frac{e^{-n}}{n!}\sum_{\nu=-M}^{M} (n\zeta_\nu)^n = \frac{(2M+1)n^n e^{-n}}{n!} > \text{const.} > 0,$$

from Stirling's formula. Hence, for some $\alpha > 0$, $B_n \ge \alpha$, where

$$B_n = \sup\{|f^{(n)}(0)|/n!: f \in B_H(G), \|f\|_\infty \le 1\}.$$

However, by the Riemann-Lebesgue lemma, for any $f \in B_H(G)$, $f^{(n)}(0)/n! \to 0$ $(n \to \infty)$ and so the origin is a R.L.C. of G.

5. Proof of Theorem 3

The form of this theorem is to a large extent due to its use in the Proof of Theorem 4. It will be clear from our arguments that much greater generality is in fact possible.

LEMMA 14. *Assume the hypotheses of Theorem* 3 *are satisfied and define*

$$B_n = \sup \{|f^{(n)}(\zeta)|/n! : f \in B_H(G),\ \|f\|_\infty \leq 1\}.$$

There are constants $c > 0$, $d > 0$ *such that* $B_n \geq c/n^d$ $(n = 1, 2, \ldots)$.

Proof. Suppose that $0 \in \partial G \cap \partial\Delta$ and the negative real axis lies in the complement of G and that $\zeta = e^{i\phi}$, where $-\pi < \phi < \pi$. Choose $\alpha > 0$ to satisfy $\alpha(\pi + |\phi|) \leq \pi/2$ and then $g(z) = \exp\{-(ze^{-i\phi})^\alpha\}$ with $g(e^{i\phi}) = e^{-1}$ is in $B_H(G)$ and $\|g\|_\infty = 1$. We shall establish the lemma by showing that for some constants $c > 0$, $d > 0$.

$$\frac{|g^{(n)}(e^{i\phi})|}{n!} \geq \frac{c}{n^d} \quad (n = 1, 2, \ldots).$$

Consider for $n \in \mathbf{N}$,

$$\frac{g^{(n)}(e^{i\phi})}{n!} = \frac{1}{2\pi i}\int_{\partial\Delta} \frac{g(z)}{(z - e^{i\phi})^{n+1}}\, dz.$$

To estimate this integral we consider g restricted to Δ and cut the plane along the half-ray $\arg z = \pi + \phi$. We next continue the restriction of g to Δ into this cut plane and distort the path of integration, $\partial\Delta$, so that it wraps itself along both sides of $\arg z = \phi + \pi$. Hence we find that

$$\frac{g^{(n)}(e^{i\phi})}{n!} = \frac{(-1)^n e^{-in\phi}}{\pi}\int_0^\infty \frac{e^{-r^\alpha \cos \alpha\pi} \sin(r^\alpha \sin \alpha\pi)}{(r+1)^{n+1}}\, dr$$

By considering $\int_0^\infty$ above as $\int_0^1 + \int_1^\infty$ it follows easily that for some $c > 0, d > 0$,

$$\frac{|g^{(n)}(e^{i\phi})|}{n!} \geq \frac{c}{n^d} \quad (n = 1, 2, \ldots).$$

The following is an outline of the proof of Theorem 3.

There is a rectifiable contour Γ in $\bar{G}$ about ζ through the points of $\partial G \cap \partial\Delta$ such that any point of Γ apart from these latter points is at a distance greater than 1 from ζ. Suppose $f_n \in B_H(G)$, $\|f_n\|_\infty = 1$ is extremal for B_n of Lemma 14 and consider

$$\frac{f_n^{(n)}(\zeta)}{n!} = \frac{1}{2\pi i}\int_\Gamma \frac{f_n(z)}{(z - \zeta)^{n+1}}\, dz.$$

From Lemma 14 it follows that the integral on the right is dominated by the contribution from arcs containing the points of $\partial G \cap \partial\Delta$ which shrink to these points as $n \to \infty$.

Hence one can use a "gliding hump" argument based on the result for the present situation which is analogous to that of Lemma 11, with $h(z)$ of that lemma replaced by $f_m(z)$ for some m. From the hypotheses that G is bounded and complementary to half-lines emanating from $\partial G \cap \partial\Delta$ it follows that suitable factors similar to those in Lemma 11 exist in the present case.

6. Proof of Theorem 4

If $n = 0$, take G to be the inside of a triangle and use Theorem 3.

If $n \in \mathbf{N}$ let $N = \{P_1, \ldots, P_n\}$ be a regular n-gon of side 1 and for $k = 1, \ldots, n$ let

$$\Delta_k = \{z \in \mathbf{C}\colon |z - P_k| < 1/4\}$$

and define $G = (\bigcup_{k=1}^{n} \Delta_k) \cup N$. From Theorem 2 it follows that $P_1, \ldots, P_n$ are R.L.C.'s of G and from Theorem 3 it follows that any other point of G is not a R.L.C.

If $n = \aleph_0$ let $D_0 = \{z\colon |z| < 1\}$ and let $D_1, D_2, \ldots,$ be small discs centered on ∂D_0 such that if $G = \bigcup_{k=0}^{\infty} D_k$ then for any point of G apart from the centers of $D_0, D_1, \ldots$ the conditions of Theorem 3 are satisfied. Since the center of each D_k is a R.L.C. of G, from Theorem 2, it follows that G has precisely $\aleph_0$ R.L.C.'s.

7. Concluding remarks

The following result is perhaps true. Let G be a domain supporting nonconstant bounded analytic functions and suppose that no point of ∂G is a removable singularity for all functions in $B_H(G)$. Let $\zeta \in G$ and Δ be the largest disc centered on ζ lying in G. If the linear measure of $\partial G \cap \partial\Delta$ is 0, then ζ is not a R.L.C. of G.

In the above notation one might also consider whether or not ζ is a R.L.C. of G when the linear measure of $\partial G \cap \partial\Delta$ is positive. It seems somewhat unlikely that this condition would be sufficient to ensure that ζ is always a R.L.C. of G. But it might be the case that ζ is a R.L.C. of G if $\partial G \cap \partial\Delta$ contains an arc of $\partial\Delta$ of positive length which separates G from its complement. Perhaps the simplest case to consider first of all would be the one similar to that of Theorem 2 when β satisfies $\pi/2 < \beta < \pi$.

Finally there is the question of how many R.L.C.'s a domain can possess. Is Theorem 4 best possible? If the answer is no, then what conditions does the set of R.L.C.'s satisfy?

Added in proof. A natural question that arises is whether, for every domain G, every point $\zeta \in G$ must be a weak R.L.C. in the sense that

$$|f^n(\zeta)|/A_n = 0.$$

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