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ORIENTATION REVERSING MAPS OF SURFACES

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1. Introduction

In this paper we characterize the conjugacy classes (in the diffeomorphism group) of orientation reversing maps whose squares have prime order $p>2$ on a compact surface. These results extend the author's previous work [7], in which the case $p=2$ was considered, and are analogous to the results of Nielsen [3] and Gilman [2], where orientation preserving maps were considered. Our main theorem is the following.

THEOREM 1.1. *Let X be a smooth compact surface of genus n and let $g_i: X \to X$, $i=1,2$ be two orientation reversing maps with the property that g_1^2 and g_2^2 both have prime order p. Then g_1 and g_2 are conjugate in the group of diffeomorphisms of X if and only if* (1) *g_1^2 and g_2^2 are conjugate,* (2) *$X\langle g_1\rangle$ and $X\langle g_2\rangle$ are diffeomorphic, and* (3) *g_1^p is conjugate to g_2^p.*

We remark that necessity is trivial. To prove sufficiency we may replace condition (1) by (1)$'$: $g_1^2=g_2^2$. The map g_i^p is an orientation reversing map of order two, and the conjugacy class of such a map is determined by the topological type of the quotient space, (see e.g., [1, pp. 57–58]), so that (3) is equivalent to (3)$'$:

$$X/\langle g_1^p\rangle \cong X/\langle g_2^p\rangle.$$

Here $\cong$ means homeomorphic. Also, if $g_1^2=g_2^2=f$, then each g_i induces an orientation reversing map g_i' of order two on $X'=X/\langle f\rangle$. It is clear that $X'/\langle g_i'\rangle \cong X/\langle g_i\rangle$ and that (2) may be replaced by (2)$'$: g_1' is conjugate to g_2'. Finally, it is not difficult to construct examples which show that conditions (1), (2) and (3) are independent.

The notation which we use is much the same as that used in [7]. For completeness we review it here. Let X be a smooth compact surface of genus n and let $g: X \to X$ be an orientation reversing map such that $f=g^2$ has prime order $p>2$. Let $\pi: X \to X'=X/\langle f\rangle$ denote the (possibly branched) covering. The surface X' has genus m and g induces an orientation reversing involution g' on X'.

If Y is a Riemann surface with an automorphism h then we say that h is embeddable if there is a conformal map $d: Y \to \mathbf{R}^3$ such that dhd^{-1} is the restriction of a rotation. Rüedy [4] has given necessary and sufficient

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conditions for an automorphism to be embeddable, and using his result we prove in [6] that if X is given a conformal structure so that f and g are, respectively, conformal and anticonformal, then f is embeddable. Thus by [4] f has an even number $2a$ of fixed points. We may now calculate m, the genus of X', from the Riemann–Hurwitz formula, and we get $n-1=p(m-1)+a(p-1)$. Let $\alpha(f)$ denote the angle of the rotation of f. If we normalize by requiring that $0<\alpha(f)<2\pi$ then we must have $\alpha(f)=2\pi j/p$, $j=1,2,\ldots,p-1$. It is not hard to show, using the Chinese remainder theorem, that in proving 1.1 it suffices to consider only $\alpha(f)=2\pi/p$.

We prove 1.1 by considering separately three cases. Case one is g' has fixed points and $X'/\langle g'\rangle$ is orientable; case two is g' has fixed points and $X'/\langle g'\rangle$ is not orientable; and case three is g' has no fixed points. We remark that in the third case $X'/\langle g'\rangle$ must be non-orientable [5].

2. Case 1: $X'/\langle g'\rangle$ is orientable and g' has fixed points

In addition to proving Theorem 1.1 in this case, we prove several results which indicate what g is like geometrically.

LEMMA 2.1. *$X'/\langle g'\rangle$ is orientable if and only if $X/\langle g^p\rangle$ is.*

Proof. We know that f induces an mapping f' on $X/\langle g^p\rangle$. Also

$$X'/\langle g'\rangle\cong(X/\langle g^p\rangle)/\langle f'\rangle.$$

The lemma now follows easily.

PROPOSITION 2.2. *Let X be a Riemann surface and suppose that $H\colon X\to X$ is an embeddable automorphism of odd order and $K\colon X\to X$ is an anti-automorphism of order two. Assume further that $X/\langle K\rangle$ is orientable and that H and K commute. Then there exists a conformal embedding of X in $\mathbf{R}^3$ so that H and K become, respectively, the restrictions of a rotation about the z-axis and a reflection in the x-y plane.*

Proof. We will first show that X may be topologically embedded in $\mathbf{R}^3$ so that the above conditions hold. Since $X/\langle K\rangle$ is orientable, the fixed point set of K is a collection of curves which divides X. Thus there exist two surfaces with boundary, X_1 and X_2 such that $X=X_1\cup X_2$, $K(X_1)=X_2$, $K(X_2)=X_1$ and $X_1\cap X_2$ is the fixed point set of K. Now clearly

$$H(X_1\cap X_2)=H(K(X_1\cap X_2))=K(H(X_1\cap X_2)).$$

Thus $H(X_1\cap X_2)\subset X_1\cap X_2$. Similarly $H^{-1}(X_1\cap X_2)\subset X_1\cap X_2$, and hence

$$H(X_1\cap X_2)=X_1\cap X_2.$$

Now let $r,s\in X_1-X_1\cap X_2$ and let $\gamma\subset X_1-X_1\cap X_2$ be a path with end-points r and s. If $H(r)\in X_1$ and $H(s)\in X_2$ then $H(\gamma)$ intersects $X_1\cap X_2$, which contradicts the fact that

$$H(X_1 \cap X_2) = X_1 \cap X_2.$$

Thus either $H: X_1 \to X_1$ and $H: X_2 \to X_2$ or $H: X_1 \to X_2$ and $H: X_2 \to X_1$. But from the fact that H has odd order the latter is impossible.

Since H is embeddable we may embed X_1 in $\mathbf{R}^3$ so that H becomes the restriction of a rotation about the z-axis. Now change this embedding, if necessary, so that the boundary curves are in the x-y plane and so that the rest of X_1 lies below the x-y plane, and so that H still remains the restriction of a rotation. Clearly $X_i \cong X/\langle K \rangle$, $i = 1, 2$, so if we double across the boundary curves we obtain (topologically) our result. To finish one need only deform the embedded surface slightly so that it is still fixed by the rotation H about the z-axis and reflection in the x-y plane and so that the embedding is conformal. One can do this by the same argument used by Rüedy in 4.2 of [4, pp. 416–417].

By 2.1, $X/\langle g^p \rangle$ is orientable, so we may apply 2.2 to the case in which $K = g^p$, $H = f$. Thus we now assume that X is embedded in $\mathbf{R}^3$ so that g^p and f are induced by, respectively, reflection in the x-y plane and rotation about the z-axis.

The intersection of X with the x-y plane contains c components, each of which is mapped onto itself by f, and pd components, which are permuted by f.

PROPOSITION 2.3. *The number $a + c$ is even, where $2a$ is the number of fixed points of f.*

Proof. We first cut X along each of the $c + pd$ curves. This divides X into two surfaces X_1 and X_2. Thus f induces two maps $f_i: X_i \to X_i$, $i = 1, 2$, which are both the restrictions of a rotation about the z-axis. Now we may glue discs to the boundary components of, say, X_1, and f_1 may be extended to a map on the resulting surface. This map has $a + c$ fixed points and is embeddable. Thus $a + c$ is even.

Proof of 1.1 (*case* 1). Let g_1 and g_2 be two orientation reversing maps of order $2p$ with the property that $g_1^2 = g_2^2$ and let c_i (resp. pd_i), $i = 1, 2$, be the number of boundary components of $X/\langle g_i^p \rangle$ which are fixed by (resp. are not fixed by) f_i, the action induced by f on $X/\langle g_i^p \rangle$. Thus if $X/\langle g_1^p \rangle \cong X/\langle g_2^p \rangle$, then

$$c_1 + pd_1 = c_2 + pd_2.$$

Also, $(X/\langle g_i^p \rangle)/\langle f_i \rangle$ has $c_i + d_i$ boundary components. It is clear that

$$X/\langle g_i \rangle \cong (X/\langle g_i^p \rangle)/\langle f_i \rangle$$

so that if $X/\langle g_1 \rangle \cong X/\langle g_2 \rangle$ then $c_1 + d_1 = c_2 + d_2$. Thus we must have $c_1 = c_2$ and $d_1 = d_2$.

By previous remarks we may identify $X/\langle g_i^p \rangle$, $i = 1, 2$, with a surface embedded in $\mathbf{R}^3$ beneath the x-y plane so that the boundary components lie in the x-y plane and so that the f_i are restrictions of rotations about the z-axis. Since we assumed that $\alpha(f)=2\pi/p$, it is easy to see that the boundary curves of $X/\langle g_1^p\rangle$ and $X/\langle g_2^p\rangle$ and the fixed points of f_1 and f_2 may be paired so that the valence of corresponding curves and points are the same. Thus by [3, p. 53] or [2] there is a map $h: X/\langle g_1^p\rangle \to X/\langle g_2^p\rangle$, $f_2 = hf_1h^{-1}$. Since g_i^p has order two, h lifts to a map k so that the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{k} & X \\ \downarrow & & \downarrow \\ X/\langle g_1^p\rangle & \xrightarrow{h} & X/\langle g_2^p\rangle \end{array}$$

Thus $g_2^p = kg_1^pk^{-1}$ and either (a) $f=kfk^{-1}$ or (b) $f=kfk^{-1}g_1^p$. A short calculation using the order of f shows that (a) holds.

We now write $g_2 = f^{-j}g_2^p$, $j=(p-1)/2$, so that

$$g_2 = f^{-j}kg_1^pk^{-1} = kf^{-j}g_1^pk^{-1} = kg_1k^{-1}.$$

This completes the proof.

3. Case 2: X'/g' is not orientable and g' has fixed points

We prove here several results which are also used in §4. We start by first looking at the behavior of orientation reversing involutions. If Y is a compact surface, possibly with boundary, and G is an orientation reversing involution, then by an annular region for G we mean an open set $A \subset Y$, homeomorphic to an annulus, which is fixed by G and with the property that $A/\langle G\rangle$ is a moebius strip.

LEMMA 3.1. *Let Y and G be as above. Assume also that $Y/\langle G\rangle$ is not orientable. Then there exist either one or two annular regions A on Y with the property that if we remove the interiors of these regions from Y, the quotient of the remaining surface modulo G is orientable. If G has fixed points, the number of annular regions needed is determined by the topological type of $Y/\langle G\rangle$.*

Proof. We first remark that a torus has an orientation reversing involution with the property that the quotient surface is a moebius strip. To see this think of the torus as the quotient of the plane by the lattice generated by τ and 1, where $\operatorname{Im}(\tau)>0$, $\operatorname{Re}(\tau)=1/2$. Then the map $z\to\bar z$ covers a map of the torus with the desired property.

We consider the case in which Y has boundary components and G has fixed points. It is shown in [5, pp. 224–225] that G may be obtained by glueing to a double one or two tori, each with an involution as described above. The glueing is accomplished by removing two small discs intersecting the fixed point sets on the torus and the double and glueing along the boundary. It is not hard to see that a regular neighborhood about the loop corresponding to the line $y = 1/2$ may be found on each adjoined torus which satisfies the required properties for our annular region. We adjoin one (resp. two) tori iff $Y/\langle G\rangle$ has an odd (resp. even) number of cross caps.

We now consider the case in which G has no fixed points. If we refer to [5, pp. 225–226] it is easy to find exactly one annular region with the desired property. If m is odd then this annular region does not divide Y and if m is even then it does divide Y.

Now assume that Y has a finite number of boundary components. In this case $\operatorname{Int}(Y)$ is homeomorphic to a compact surface with finitely many punctures. Thus assume

$$\operatorname{Int}(Y) = Z - \{z_1, \ldots, z_N\},$$

where Z is compact. We may extend g to Z, and by what we have just shown there exists either one or two annular regions for g on Z which satisfy the conclusion of the theorem.

We now claim that these regions may be chosen so that they avoid the points z_i. We took as our annular region a regular neighborhood of the loop corresponding to the line $y = 1/2$. If this loop contains any points z_i, then let γ be a simple smooth arc contained in a fundamental domain for $\langle \tau, 1\rangle$ which connects $x + i/2$ and $x + 1/2 + i/2$, for some x, which does not pass through any z_i, and whose image under the map $z \to \bar{z}$ also does not pass through any z_i. The arc γ and its image under the map $z \to \bar{z}$ correspond to a loop on the adjoined torus, and a regular neighborhood of this loop has the desired property.

The annular regions which we have constructed are contained in $\operatorname{Int}(Y)$, and if we remove them from Y and take the quotient of the remaining surface modulo G we obtain an orientable surface. This completes the proof.

LEMMA 3.2. *If Y is a compact surface of genus zero, possibly with boundary, and G is an orientation reversing involution with the property that $Y/\langle G\rangle$ is not orientable, then there exists an annular region A on Y so that $(Y - A)/\langle G\rangle$ is orientable. Also it is impossible to have two disjoint annular regions for G.*

Proof. We consider first the case in which $Y = S^2$. In this case G is conjugate to the antipodal map [1, p. 59], so that a neighborhood, which is fixed by G, of a great circle may serve as an annular region which satisfies the conclusion of the lemma. If A is any annular region for G on Y, then A divides Y into two components which are interchanged by G. Hence it is impossible to have two disjoint annular regions for G. The case in which Y has boundary components follows by an argument similar to that used in proving 3.1.

We remark now that g' has fixed points and $X'/\langle g'\rangle$ is not orientable iff g^p also has fixed points and $X/\langle g^p\rangle$ is not orientable. This follows from 2.1 and from the following lemma.

LEMMA 3.3. *If $x \in X$ is not a fixed point of f, then $g^p(x)=x$ if and only if $g'(x')=x'$, where $x'=\pi(x)$. Also g' has fixed points only if p is odd.*

Proof. We first remark that if g' has fixed points then this set consists of the points along a set of loops. If $g'(x')=x'$, then we must have $g(x)=f^j(x)$ for some j, where $0\le j\le p$. Thus $x=g^{-1}f^j(x)=g^{2j-1}(x)$, hence $x=f^{2j-1}(x)$. Since this equation holds for infinitely many $x\in X$, we must have that $2j-1=p$. Therefore $g^p(x)=x$. The converse is trivial.

LEMMA 3.4. *We assume that $X'/\langle g'\rangle$ is not orientable. Then we may find either one or two annular regions for g^p on X such that each of these regions is fixed by f and each projects to an annular region for g' on X'. If we remove the interiors of these regions from X and X', then the quotients of the remaining surfaces by g^p and g', respectively, are orientable.*

Proof. Let Σ be the union of either one or two annular regions for g^p on X with the property that $(X-\Sigma)/\langle g^p\rangle$ is orientable. By 3.1 such a Σ exists. Also by an argument similar to that used in 3.1 we may assume that Σ contains no fixed points of f. We first consider the case in which Σ consists of one annular region A. The map g' acts on $\pi(A)$ and since $A/\langle g^p\rangle$ is not orientable, $\pi(A)/\langle g'\rangle$ is also not orientable. Thus by 3.1 or 3.2 there are one or two annular regions on $\pi(A)$ with the property that if we remove them and take the quotient modulo g', we obtain an orientable surface.

We claim that there is only one such annular region on $\pi(A)$. Assume that there are two, α' and β'. These lift to two disjoint annular regions α and β on A. But A has genus zero so by 3.2 we get a contradiction. Thus there is only one annular region α' for g' on $\pi(A)$, and this lifts to an annular region α for g^p on A. Since $(\pi(A)-\alpha')/\langle g'\rangle$ is orientable, $(A-\alpha)/\langle g^p\rangle$ is orientable. Also by the choice of A, $(X-A)/\langle g^p\rangle$ is orientable. Thus $(X-\alpha)/\langle g^p\rangle$ is orientable. If α' lifted to p regions on X then each would be an annular region for g^p and $(X-\alpha)/\langle g^p\rangle$ would not be orientable, contradiction. Thus α is fixed by f and hence satisfies the conclusion of the theorem.

We now consider the case in which Σ consists of two annular regions A and B. If $\pi(A)$ and $\pi(B)$ are disjoint, then by an argument similar to that used in the case in which Σ has one component, we may find annular regions α' and β' on $\pi(A)$ and $\pi(B)$ which lift to annular regions α and β which satisfy the conclusion of the theorem.

If $\pi(A)\cap\pi(B)\neq\emptyset$, then let $C=\pi(B)-\pi(A)$. Since $g'(\pi(B))=\pi(B)$ and $g'(\pi(A))=\pi(A)$, $g'(C)=C$. The set C need not be connected but we claim that C must have one component C', which is fixed by g', with the property that $C'/\langle g'\rangle$ is not orientable. If this is not the case then $C/\langle g'\rangle$ is orientable. We may thus find an annular region α' for g' on $\pi(A)$, which lifts to an annular region α for g^p on A, with the property that $(\pi(A)-\alpha')/\langle g'\rangle$ is orientable. Thus $(A-\alpha)/\langle g^p\rangle$ is also orientable. Since $C/\langle g'\rangle$ is orientable, $(\pi^{-1}(C)\cap B)/\langle g^p\rangle$ is orientable as well. Now

$$A\cup B-\alpha=(A-\alpha)\cup(\pi^{-1}(C)\cap B).$$

What we have shown implies that $((A-\alpha)\cup(\pi^{-1}(C)\cap B))/\langle g^p\rangle$ is orientable so that $(A\cup B-\alpha)/\langle g^p\rangle$ is orientable. This implies that $(X-\alpha)/\langle g^p\rangle$ is orientable. This contradicts the fact that we needed two annular regions for g^p on X. Thus there is a component C' of C with the property that $C'/\langle g'\rangle$ is not orientable.

Now we find disjoint annular regions on $\pi(A)\cup\pi(B)$ which lift to annular regions on X which are fixed by f. We let β' be an annular region for g' on C' with the property that $(C'-\beta')/\langle g'\rangle$ is orientable. Then β' lifts to an annular region β for g^p on B. Since B has genus zero g^p interchanges the two components of $B-\beta$, so that $(B-\beta)/\langle g^p\rangle$ is orientable. Hence $(C-\beta')/\langle g'\rangle$ is orientable. Now let α' be an annular region for g' on $\pi(A)$ with the property that $(\pi(A)-\alpha')/\langle g'\rangle$ is orientable. By an argument similar to that used before α' lifts to an annular region α on A with the property that $(A-\alpha)/\langle g^p\rangle$ is orientable. Thus $(A\cup B-(\alpha\cup\beta))/\langle g^p\rangle$ is orientable, so that $(X-\alpha\cup\beta)/\langle g^p\rangle$ is orientable. If α' or β' lifts to p annular regions on X, then $(X-\alpha\cup\beta)/\langle g^p\rangle$ is not orientable. Thus α' and β' each lift to one component on X, and each lift is fixed by f. This completes the proof.

LEMMA 3.5. *Suppose A and A' are two annular regions with maps*

$$H\colon A\to A,\quad H'\colon A'\to A',$$

both fixed point free of order p, and

$$K\colon A\to A,\quad K'\colon A'\to A',$$

orientation reversing of order two with the property that the quotient spaces are both moebius strips. Also H commutes with K and H' commutes with K'. Assume further that there exists a map $h\colon \partial A\to\partial A'$ with the property that (1) *$h\circ H=H'\circ h$ and* (2) *$h\circ K=K'\circ h$. Then h may be extended to a map $h\colon A\to A'$ so that* (1) *and* (2) *hold.*

Proof. The maps H and H' induce fixed point free actions L and L', both of order p, on $A/\langle K\rangle$ and $A'/\langle K'\rangle$. We may find fundamental domains D and D' for L and L', respectively, both of which are homeomorphic to a square, and bounded by four arcs. Two arcs α_1,α_2 (α_1',α_2') are subsets of $\partial A/\langle K\rangle$ $(\partial A'/\langle K\rangle)$ and the other two join α_1 and α_2 $(\alpha_1'$ and $\alpha_2')$. The map h induces a map

$$h_1\colon \partial A/\langle K\rangle\to A'/\langle K'\rangle$$

and the two fundamental domains may be chosen so that $h_1(\alpha_i)=\alpha_i'$, $i=1,2$. Now h_1 may clearly be extended to D, and by means of the formula

$h_1(L(x)) = L'(h_1(x))$, to all of $A/\langle K\rangle$. Since A and A' are two sheeted coverings of $A/\langle K\rangle$ and $A'/\langle K'\rangle$ we may lift h_1 to a map with the desired properties.

Proof of Theorem 1.1 (*in case* 2). Let g_1 and g_2 be two orientation reversing maps satisfying the hypothesis of 1.1. Thus we may assume that $g_1^2 = g_2^2 = f$ and that the maps g_1' and g_2', which are induced on X' by g_1 and g_2, respectively, are conjugate.

By 3.4 there are one or two annular regions on X for each of the maps g_1^p and g_2^p, which project to annular regions on X' for the maps g_1' and g_2'. If we remove these annular regions we obtain surfaces X_i, $i = 1, 2$, which are fixed by g_i^p and which have the property that $X_i/\langle g_i^p\rangle$ is orientable. Since the annular regions of g_i^p are fixed by f, X_i is also fixed by f. Also, since g_1^p and g_2^p are conjugate, $X_1 \cong X_2$.

Now let c_i (resp. d_i), $i = 1, 2$, be the number of loops on X which are fixed pointwise by g_i' and which lift to one (resp. p) loops on X. Since g_1' is conjugate to g_2' we must have $c_1 + d_1 = c_2 + d_2$. Also by 3.3 the loops which are fixed pointwise by g_i' lift to loops which are fixed pointwise by g_i^p. Since there are $c_i + pd_i$ such loops and since g_1^p and g_2^p are conjugate, we must have $c_1 + pd_1 = c_2 + pd_2$. Thus $c_1 = c_2$ and $d_1 = d_2$.

The map f induces a map f_i on $X_i/\langle g_i^p\rangle$. Also f is embeddable with $2a$ fixed points and $\alpha(f) = 2\pi/p$. By mimicking the argument used in 2.2 we may embed X_i in $\mathbf{R}^3$ so that g_i^p becomes reflection in the x-y plane and f becomes a rotation about the z-axis through an angle of $2\pi/p$. We may thus identify $X_i/\langle g_i^p\rangle$ with that part of X_i which lies below and in the x-y plane, and in this case f_i becomes a rotation about the z-axis through an angle of $2\pi/p$. Also $X_i/\langle g_1^p\rangle$ and $X_2/\langle g_2^p\rangle$ are both homeomorphic and the maps f_1 and f_2 both fix the same number of boundary components and both have the same number of fixed points. Thus by [3, p. 53] or [2], there is a map $k\colon X_1/\langle g_1^p\rangle \to X_2/\langle g_2^p\rangle$ such that $f_2 = kf_1k^{-1}$. By doubling across the $c_i + pd_i$ boundary components which come from loops which are fixed pointwise by g_i^p, we obtain a map $h\colon X_1 \to X_2$ so that (1) $hfh^{-1} = f$ and (2) $hg_1^ph^{-1} = g_2^p$. By repeated application of 3.5 we may extend h so that $h\colon X \to X$ and so that (1) and (2) still hold. It is now trivial to show that $hg_1h^{-1} = g_2$. This completes the proof.

4. Case 3: X'/g' is not orientable and g' has no fixed points

The proof of 1.1 in this case depends on our previous results. If m is even then we use the results of §2 and if m is odd we use the results of §3.

We first consider the case in which m is even. By [5, pp. 225–226], there is a dividing cycle A on X', which is fixed by g', with the property that $X' - A$ consists of two components which are interchanged by g'. We may move A slightly if necessary so that it still has the above properties and so that it does not intersect the branch point set. The lifts of A divide X into two components. We now cut X' along A to produce two homeomorphic surfaces with boundary X_1' and X_2'. Similarly we cut X along the lifts of A to produce two homeomorphic surfaces with boundary X_1 and X_2 on which the map f induces actions f_1 and f_2, respectively. Also assume that these surfaces are numbered so the map π_i, $i = 1, 2$, induced by π on X_i, is such that $\pi: X_i \to X_i'$, and $X_i/\langle f_i \rangle \cong X_i'$.

LEMMA 4.1. *If m is even then the loop A described above lifts to one loop if a is odd, and lifts to p loops if a is even.*

Proof. Assume first that a is odd and A lifts to p loops. Then the surface X_1 has p boundary components which are permuted by f_1. Now glue discs to the boundary components of X_1 and extend f_1 to this surface. We obtain a map which is embeddable and which contains an odd number, a, of fixed points, a contradiction. Thus A lifts to one loop.

Similarly, if a is even and A lifts to one loop, then we glue a disc to the one boundary component of X_1 and extend f_1 to the resulting surface to obtain an embeddable map with an odd number, $a+1$, of fixed points, a contradiction. Thus A lifts to one loop.

We now construct a surface on which g^p and f both induce mappings. If A lifts to p loops then none of these loops are fixed by g, since none are fixed by f. Since p is prime, g^p must fix each of these loops. Also g^p induces a map K' of order two on the disjoint union of X_1 and X_2. This map K' has the property that $K'f_1 = f_2K'$ and $K'f_2 = f_1K'$. Now glue ∂X_1 to ∂X_2 by identifying $x \in \partial X_1$ to $K'(x) \in \partial X_2$. We produce a surface Z which is homeomorphic to X and on which K' induces a map K which is orientation reversing of order two. The maps f_1 and f_2 together induce a map F on Z, which commutes with K. Also K fixes pointwise each loop which came from a boundary component of X_i, $i = 1, 2$. We also have $X/\langle K \rangle \cong X_i$ and is thus orientable. Let c be the number of loops which are fixed pointwise by K and fixed by F, and let pd be the number of loops which are fixed pointwise by K and permuted by F. We then have the following.

LEMMA 4.2. *We may embed Z in $\mathbf{R}^3$ so that F and K become, respectively, the restriction of a rotation about the z-axis through an angle of $2\pi/p$ and a reflection in the x-y plane. If a is odd then $c = 1$ and $d = 0$. If a is even, then $c = 0$ and $d = 1$.*

Proof. This follows immediately from 2.2 and 4.1.

We now consider the case in which m is odd. We first prove the following.

PROPOSITION 4.3. *If m is odd then a is even.*

Proof. By [5, pp. 225–226] there are two loops A and B on X' which divide X' into two components X_1 and X_2. The map g' interchanges X_1 and

X_2 and A and B. Also A and B may be adjusted slightly, if necessary, so that neither loop contains a branch point. Assume that $a>0$. Then X_1 contains branch points so that $\pi^{-1}(X_1)$ is connected.

We now use an argument similar to that used in 4.1. Thus from $\pi^{-1}(X_1)$ we construct a surface on which f induces an embeddable map with either a or $a+2$ fixed points, depending on whether A and B both lift to one or p loops. Hence a is even.

We remark that by 3.4 and 3.1 there exists an annular region α for g^p on X which projects to an annular region for g' on X. The annular region α does not divide X as is seen from the proofs of 3.1 and 3.4. The region α is a neighborhood of a loop A which is fixed by g^p and f and which projects to a loop on X' which is fixed by g'.

We now construct a surface Y on which g^p and f both induce mappings. First we cut X along A and we obtain a surface X_1 with two boundary components on which g^p and f induce maps. Let K' denote the map induced by g^p. The boundary components of X_1 are interchanged by K', and we identify $x\in\partial X_1$ with $K'(x)\in\partial X_1$ to produce a surface Y on which g^p and f induce maps K and F, respectively. Clearly $Y\cong X$. Let B denote the loop on Y which came from A. Clearly K fixes B pointwise. Since B is the only loop which K fixes pointwise and since B does not divide Y, we must have that $Y/\langle K\rangle$ is not orientable. Also the Euler characteristic of $Y/\langle K\rangle$ is $(2n-2)/2=n-1$.

Proof of 1.1. We first consider the case in which m is even. We assume $g_1^2=g_2^2=f$. For each of the maps g_i, $i=1,2$, we construct a surface Z_i, on which each of the maps g_i^p and f induce maps K_i and F_i. By Lemma 4.2 Z_1 and Z_2 are both embeddable in $\mathbf{R}^3$ so that F_i becomes a rotation through an angle of $2\pi/p$ and K_i becomes a reflection in the x-y plane. Also there are c loops fixed pointwise by K_i and fixed by F_i, and pd loops which are fixed pointwise by K_i and permuted by F_i. The numbers c and d are the same for both $i=1$ and $i=2$. Thus by mimicking the argument used in §2, we may construct a map $k\colon Z_1\to Z_2$ such that $kK_1=K_2k$ and $kF_1=F_2k$. Clearly k maps the fixed point set of K_1 onto the fixed point set of K_2.

We now use the map k to construct a map h which will conjugate g_1 and g_2. First cut Z_i along the loops which are fixed pointwise by K_i. We then obtain two surfaces from Z_i, X_{i1} and X_{i2}. The map k induces two maps k_i, $i=1$, 2, where $k_i\colon X_{i1}\to X_{i2}$, after possibly renumbering. Now reglue X_{i1} and X_{i2} to recover X. By observing carefully k_i on ∂X_{i1}, we see that, after reglueing, k_1 and k_2 induce a map $h\colon X\to X$, with the property that $hg_1^p=g_2^ph$ and $hf=fh$. Thus if $j=(p+1)/2$, then

$$hg_1h^{-1}=hg_1^pf^jh^{-1}=hg_1^ph^{-1}hf^jh^{-1}=g_2^pf^j=g_2.$$

This completes the proof if m is even.

We now consider the case in which m is odd. We construct surfaces

Y_i, $i=1,2$, on which g_i^p and f induce maps K_i and F_i, respectively. Let $G_i = K_iF_i^j$, $2j=p+1$, so that $G_i^p = K_i$. The surface $Y_i/\langle K_i\rangle$ is nonorientable with one boundary component and has Euler characteristic $n-1$. Thus

$$Y_1/\langle K_1\rangle \cong Y_2/\langle K_2\rangle.$$

Also G_i^2 is embeddable with $2a$ fixed points and $\alpha(G_i^2)=\alpha(f_i)=2\pi/p$. Also if G_i' denotes the map induced by G_i on $Y_i/\langle F_i\rangle = Y_i'$, then $Y_i'/\langle G_i'\rangle$ is not orientable by 2.1, has exactly one boundary component, and has Euler characteristic $m-1$. Thus

$$Y_1'/\langle G_1'\rangle \cong Y_2'/\langle G_2'\rangle.$$

It is easy to see that the results of §3 imply that there is a map $k\colon Y_1 \to Y_2$ such that $kG_1 = G_2k$. Let B_i denote the curve fixed pointwise by K_i on Y_i. Then $k(B_1)=B_2$. If we cut Y_i along B_i and reglue to recover X, then it is easy to check that k induces a map $h\colon X\to X$. Also $hg_1 = g_2h$. This completes the proof.

References

1. N. L. Alling and N. Greenleaf, *Foundations of the theory of Klein surfaces*, Lecture Notes in Math., vol. 219, Springer-Verlag, New York, 1971.
2. J. Gilman, *On conjugacy classes in the Teichmüller modular group*, Michigan Math. J., vol. 23 (1976), pp. 53–64.
3. J. Nielsen, *Die Struktur periodischer Transformationen von Flächen*, Danske Vid. Selsk. Mat.-Fys. Medd., no. 1 (1937), pp. 1–77.
4. R. Rüedy, *Symmetric embeddings of Riemann surfaces, discontinuous groups and Riemann surfaces*, Ann. of Math. Studies 79, Princeton University Press, 1974.
5. R. Zarrow, *A canonical form for symmetric and skew-symmetric extended symplectic modular matrices with applications to Riemann surface theory*, Trans. Amer. Math. Soc., vol. 204 (1975), pp. 207–227.
6. ———, *Anti-conformal automorphisms of compact Riemann surfaces*, Proc. Amer. Math. Soc., vol. 54 (1976), pp. 162–164.
7. ———, *Orientation reversing square roots of involutions*, Illinois J. Math., vol. 23 (1979), pp. 71–80 (this issue).

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