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SECTIONAL REPRESENTATION OF MULTITOPOLOGICAL SPACES RELATIVE TO A FAMILY OF SMOOTHNESS CATEGORIES

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Introduction

The purpose of this paper is to study those multitopological spaces (a set with a family of topologies on it) that can be represented, relative to a family of differentiability classes, by an embedding of the set in a differentiable manifold. For example, if the family of differentiability classes is $\{C^r\}$, $r = 1, 2, \ldots$, then an embedding $A \subset E$ of a set A in a smooth manifold E is said to represent a sequence of topologies $\{\tau^r\}$, $r = 1, 2, \ldots$, on A if τ^r is coinduced by the family of all C^r-maps from the reals to E with image in A, for $r = 1, 2, \ldots$.

After this notion of representation is discussed in some detail, the problem of representing multitopological structures on a manifold by sections of a differentiable vector bundle is then studied. In particular, it is shown that those structures that are induced by a locally finite, decreasing sequence of regular, local kernels can be so represented. From this follows a generalized Whitney embedding theorem: namely, any decreasing sequence of foliation topologies on an n-manifold can be represented by an embedding in Euclidean $2n$-space. The case in which the foliations are trivial (leaf = manifold) reduces to the classical Whitney embedding theorem, while the case in which the foliations have points for leaves reduces to a generalized form of the construction of a continuous, nowhere differentiable function. The paper concludes with a discussion of some further problems.

The general procedure is to construct global representations by "pasting together" local ones. However, the usual technique of forming global sections from local ones by using a partition of unity does not work since sectional representations, in general, are not closed under addition and scalar multiplication. The key assumption is regularity (§5) since it allows one to build global representations from local ones by other means (5.1 and 5.2).

See [6], [7], and [9] for closely related topics.

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1. Germ topologies and smoothness categories

Recall [11] that a smoothness category C is a category whose objects are the open sets U in finite dimensional real vector spaces and whose morphisms $C(U_1, U_2)$ consist of certain C^1 (continuously differentiable) functions from U_1 to U_2 subject to the following conditions:

(1) If $U \in C$ and V is a finite-dimensional real vector space then $C(U, V)$ is a linear subspace of the real vector space $C^1(U, V)$ of all C^1 maps of U into V and contains all constant maps of U into V.

(2) If $V_1, \ldots, V_m$ and W are finite-dimensional real vector spaces then $C(V_1 \oplus \cdots \oplus V_m, W)$ contains all multilinear maps.

(3) Let U_1 and U_2 be open sets in finite dimensional real vector spaces V_1 and V_2 respectively. A function $f\colon U_1 \to U_2$ is in $C(U_1, U_2)$ if, for each $x \in U_1$, there is an open set V_x of U_1 containing x such that $f \mid V_x \in C(V_x, V_2)$.

(4) If $f_1 \in C(U, V_1)$ and $f \in C(U, V_2)$ then $x \mapsto (f_1(x), f_2(x))$ is in $C(U, V_1 \oplus V_2)$.

(5) If $f \in C(U_1, U_2)$ is a bijection then $f^{-1} \in C(U_2, U_1)$ if f^{-1} is C^1.

The categories C^r, $r = 1, 2, \ldots, \infty$ and C^ω defined by C^r and real analytic functions respectively together with the Hölder $C^{k+\alpha}$ and the Lipschitz C^{k-} categories are well known examples of smoothness categories. By §5 [11], one can show that the collection of all smoothness categories can be assumed to be a set, S, that has the structure of a complete lattice under inclusion with C^1 as maximal element and C^Ω (Nash category) as minimal element. As usual, an element C of S will also denote the category of finite-dimensional C-manifolds that it determines. For C-manifolds X, Z, and a subset A of Z let $B(A)$ be the set of all C-maps β from all open sets U_β in X to Z that have image in A. Let $B_*\colon \coprod_{B(A)} U_\beta \to A$ be the map from the disjoint union (coproduct) of all the U_β's to A, whose βth component $B_\beta\colon U_\beta \to A$ is β. The set A with the coinduced topology from B_* (U is open in A iff $B_*^{-1}(U)$ is open in $\coprod U_\beta$) will be denoted by $GC(X, (Z, A))$ or $GC(X, Z)$ if $A = Z$. In sheaf theoretic terms, $GC(X, (Z, A))$ is the "C-germ" topology on A in the sense that if T is the total space of the sheaf of germs of C maps of X to Z with image in A then there is a factorization

$$\coprod U_\beta \longrightarrow T \xrightarrow{\phi} A$$

of B_* with ϕ and B_* coinducing the same topology on A (cf. [3, p. 5]).

1.1 Proposition. (a) *If* $\dim X = n$ *then* $GC(X, (Z, A)) = GC(R^n, (Z, A))$. *Let* $GC(R, R^n) = R^n$ *for* $n = 1, 2, \ldots$. *Then:*

(b) $GC(R^n, (Z, A)) = GC(R, (Z, A))$ *for* $n = 1, 2, \ldots$.

(c) $GC(R, (Z, A))$ *is universal for C-maps into A; i.e., each* ($f\colon Y \to A \subset Z$) $\in C$ *has a unique factorization through* $GC(R, (Z, A)) \leq A$ ($\leq$ *denotes the identity map on A when it is continuous*).

(d) $GC^r(R, R^n) = R^n$ *for* $1 \le r \le \infty$ *and* $n = 1, 2, \ldots$ ($C^r = C^r$ – *differentiable maps*).

Proof. (a) This is trivial since X and R^n are locally C-diffeomorphic.

(b) Clearly $GC(R, R^n) = R^n$ implies $GC(R, U_\beta) = U_\beta$ for any open set U_β in R^n. Hence U_β has the coinduced topology from the C-maps from open sets in R and the result then follows from the "transitivity" of coinduced topologies since the topology of $GC(R^n, (Z, A))$ is coinduced by $B_*: \coprod U_\beta \to A$.

(c) Since any $(f: Y \to A \subset Z) \in C$ has an obvious factorization through $GC(Y, (Z, A)) \le A$, the result follows from (a) and (b).

(d) Since $GC^\infty(R, R^n) \le GC^r(R, R^n) \le R^n$ it is enough to show that $R^n \le GC^\infty(R, R^n)$; i.e., that any closed set in $GC^\infty(R, R^n)$ is closed in R^n or equivalently that any set not closed in R^n is not closed in $GC^\infty(R, R^n)$. The latter condition is obviously satisfied if for each convergent sequence $\{y_m\} \to y$ in R^n there are a convergent sequence $\{x_m\} \to x$ in R and a C^∞-map $V: R \to R^n$ so that $V(x_m) = y_m$, and $V(x) = y$. Suppose $y_m = (y_m^1, \ldots, y_m^n) \in R^n$ defines a sequence that converges to 0. Without loss of generality it may be assumed that $\{y_m\}$ lies in the first quadrant; i.e., $y_m^i > 0$ for $m = 1, 2, \ldots,$ and $i = 1, \ldots, n$. Let $U_p^i: R \to R$ be a C^∞-map such that

$$(U_p^i)^{-1}(0) = R - \left(\frac{3}{2^{p+1}}, \frac{3}{2^p}\right),$$

$$p = 1, 2, \ldots, \|D^k(U_p^i)\| \le M_p,\ k = 0, \ldots, p \quad \text{where} \quad \sum_{p=1}^{\infty} M_p < \infty.$$

Such a map exists by standard methods [4, p. 24]. Since $\{y_m\} \to 0$, there is a sufficiently large integer $m(p)$ so that $0 < y_{m(p)}^i \le U_p^i(1/2^{p-1}) = \alpha_p^i$ for $i = 1, \ldots, n$. Then

$$V = (V^1, \ldots, V^n): R \to R^n,$$

where

$$V^i = \sum_{p=1}^{\infty} \left(\frac{y_{m(p)}^i}{\alpha_p^i}\right) U_p^i$$

is a C^∞ map with $V(1/2^{p-1}) = y_{m(p)}$ and $V(0) = 0$. This implies the result.

Although it is always the case that $GC(X, (Z, A)) \le A$, it is not generally true that $GC(X, (Z, A)) \ge A$. In particular it may turn out that $GC(X, (Z, A))$ is discrete. If, for example, A is the graph of a continuous, nowhere differentiable function $R \to R$ then it is easy to see that $GC^1(R, (R^2, A))$ is discrete. The existence of certain C^r-discrete subsets A of R^n (i.e. $GC^r(R, (R^n, A))$ is discrete) is studied in §4 and they, or the functions defining them, are essential in the proof of the main results.

2. The general problem

Each subset A of Z, for $Z \in C^{\Omega}$, determines a family $\{GC(R^n, (Z, A))\}$, $C \in S$, $n \in N = \{1, 2, \ldots\}$ of topologies on A. (1.1(a) shows nothing more general than R^n need be used.) A general problem is to characterize those $S \times N$-indexed families, F, of topologies on A that can be *represented* in the sense that F can be induced from $\{GC(R^n, (Z, A))\}$ by an embedding $A \subset Z$, for some $Z \in C^{\Omega}$. Many simplified versions of this general question are possible. For example, one may consider only the S-indexed families $\{GC(R, (Z, A))\}$. For $C = C^r$ this is no loss of generality in view of 1.1(b) and (d). The remainder of this paper is concerned with the following general problem: Describe those sequences $\{\tau^r\}$ of topologies on a C^∞-manifold A that can be *represented* by a map i of A into some C^∞-manifold Z; i.e., so that $\{\tau^r\}$ is induced by i from $\{GC^r(R, (Z, i(A)))\}$. For those sequences that can be so represented, determine the "simplest" such Z. By taking A to be a C^∞-manifold it is possible to phrase questions on representing families of topologies on A in terms of realizing certain families of subsets (R-spaces) of the tangent bundle of A. The details of this translation are given in §3 with Corollary 3.5 as the main result.

3. R-Spaces

Let T be a subset of the total space $T(A)$ of the tangent bundle of A. T is an *R-space* if it is closed under scalar multiplication and contains the 0-section. For $Y \subset A$ and $r > 0$, the map $B_*\colon \coprod_{B(Y)} U_\beta \to Y$ defining $GC^r(R, (A, Y))$ induces a map $DB_*\colon \coprod T(U_\beta) \to T(A)$ where DB_β is the differential $D\beta\colon T(U_\beta) \to T(A)$ of $\beta\colon U_\beta \to Y \subset A$. Let BT_* be the restriction of B_* to those components U_β for which (image $D\beta) \subset T$ (subsequently shortened to $D\beta \subset T$) and let DBT_* be the map induced by passing to differentials. If DBT_* maps onto $T \mid Y$ we say that T is *r-proper* on Y. Let $R(A)$ be the category of R-spaces on A and $R^r(A)$ the full subcategory consisting of those R-spaces that are r-proper on A. Clearly $R(A)$ is isomorphic to $P(P_1(A))$, the category of subsets of the total space $P_1(A)$ of the associated projective space bundle to $T(A)$, and $R^r(A)$ is a coreflective subcategory (in the sense of [8]) of $R(A)$ with coreflector $M_r\colon R(A) \to R^r(A)$ given by $M_r(T) =$ (image DBT_*); i.e., $M_r(T)$ is the maximum subset of T that is r-proper on A.

If $A \downarrow C^\infty$ is the category whose objects are the functions i (not necessarily continuous) on A with codomain a C^∞-manifold and whose morphisms $i_1 \to i_2$ are those C^∞-maps f satisfying $fi_1 = i_2$, then $T^r(i) =$ image $DB_*(i)$ defines a functor $T^r\colon A \downarrow C^\infty \to R^r(A)$, where $DB_*(i)$ is the restriction of DB_* (with $Y = A$) to those components $T(U_\beta)$ for which $i\beta \in C^r$. Further, $T \to \tau^r(T) = A$ with coinduced topology from BT_* (with $Y = A$), defines a functor from $R^r(A)$ to the category of topologies on A with "$\leq$" morphisms.

3.1 Definition. An R-space T on A is said to be (*strongly*) *r-realized* by a function $i: A \to Z$ if, for all open sets U in R and all functions $\beta: U \to A$,

(a) $\beta \in C^r$ and $D\beta \subset T$ imply $i\beta \in C^r$ ((a)$'$ $\beta \in C^s$ and $D\beta \subset T$ imply $i\beta \in C^s$, for $s = 1, 2, \ldots$) and

(b) $i\beta \in C^r$ implies $\beta \in C^r$ and $D\beta \subset T$.

3.2 *Remark.* If T can be r-realized by a C^r-map i then $T = T(A)$, for then $\beta \in C^r$ implies $i\beta \in C^r$ which gives, by (b), $D\beta \subset T$.

3.3 Lemma. *If T is r-realized by an injection i, then i represents $\tau^r(T)$.*

Proof. By definition, i represents $\tau^r(T)$ if the topology of $\tau^r(T)$ is induced by

$$i = i_r: \tau^r(T) \to GC^r(R, (Z, i(A)));$$

i.e., since i_r is a bijection if i_r is a homeomorphism. If BT_* is as above and B'_* is the representation of $GC^r(R, (Z, i(A)))$ then 3.1(a) and (b) imply that the components of iBT_*, $i^{-1}B'_*$ are among those of B'_*, BT_* respectively. Hence i_r and i_r^{-1} are continuous and the result follows.

A sequence $\{T_j\}$ of R-spaces on A is said to be *locally finite* if there is a locally finite open cover $\{U_j\}$ of A so that $D_j = \{x \mid x \in A, (T_j)_x \neq (T_{j-1})_x\} \subset U_j$, where $(T_j)_x$ is the fiber of T_j at x. Note that $D_1 = A$.

3.4 Proposition. *Let $\{T_j\}$ be a locally finite, decreasing sequence of R-spaces on A and suppose that T_j is strongly r_j-realized by a C^{s_j}-section i_j of a C^∞-vector bundle $p: E \to A, j = 1, 2, \ldots.$ If $0 \leq s_j \leq r_j \leq s_{j+1}, j = 1, 2, \ldots,$ then there is a section i of p that r_j-realizes $T_j, j = 1, 2, \ldots,$*

Proof. By [4, page 24] there is a C^∞-map

$$\sigma_j: A \to R \quad \text{with} \quad \sigma_j^{-1}(0) = A - U_j \subset A - D_j.$$

Since $\{U_j\}$ is locally finite, $i = \sum_{j=1}^\infty \sigma_j i_j$ is a well defined section of p. If N is an open set that intersects only $U_{j(k)}$, for $k = 1, 2, \ldots, n$ then $i = \sum_{k=1}^n \sigma_{j(k)} i_{j(k)}$ on N since $N \cap U_j = \phi$ implies $N \subset \sigma_j^{-1}(0)$. If $j(1) < j(2) < \cdots < j(n)$ then $j(1) = 1$ since $D_1 = A$. Also, $T_j = T_{j(k)}$ on N, where j satisfies $j(k) < j < j(k+1)$ or $j(k) < j$ if $k = n$ since $N \cap D_j = \phi$ implies $T_j = T_{j-1}$ on N. If $T_{j(k)}$ is $r_{j(k)}$-realized by i on N, then i r_j-realizes T_j on N for j as above. Indeed, if $\beta \in C^{r_j}$ and $D\beta \subset T_j$ then

$$\sum_{m=1}^{k} \sigma_{j(m)}(\beta) i_{j(m)}(\beta) \in C^{r_j}$$

by 3.1(a)$'$ since $T_j \subset T_{j(m)}$ and $\sigma_{j(m)} \in C^\infty$ for $j(m) < j$; and

$$\sum_{m=k+1}^{n} \sigma_{j(m)}(\beta) i_{j(m)}(\beta) \in C^{r_j}$$

since $r_j \leq s_{j(m)}$, $i_{j(m)} \in C^{s_{j(m)}}$ and $\sigma_{j(m)} \in C^\infty$ for $k<m$. Hence 3.1(a) is proved. On the other hand, if $i\beta \in C^{r_j}$ then $\beta = pi\beta \in C^{r_j}$ and, since $r_{j(k)} \leq r_j$, $i\beta \in C^{r_{j(k)}}$ and so $D\beta \subset T_{j(k)} = T_j$ and 3.1(b) follows. Since "realization" is a local property, it is then enough to prove the proposition where $\{T_j\}$ is a finite set, $j = 1, 2, \ldots, n$. The verification of 3.1(a) is as above. Condition 3.1(b) will be proved by induction. If $i\beta \in C^{r_1}$ then $\beta = pi\beta \in C^{r_1}$ and, as above,

$$\sum_{j=2}^{n} (\sigma_j\beta)(i_j\beta) \in C^{r_1}.$$

Hence $(\sigma_1\beta)(i_1\beta) \in C^{r_1}$. Since $\sigma_1\beta \in C^{r_1}$ and is nowhere 0 ($D_1 = A$), $i_1\beta \in C^{r_1}$. Therefore $D\beta \subset T_1$ and 3.1(b) is shown for $j=1$. Suppose now that condition 3.1(b) holds for $j<k$ and that $i\beta \in C^{r_k}$. Then $D\beta \subset T_j$ for $1 \leq j \leq k-1$ by the induction hypothesis since $r_k > r_j$. Since $\beta = pi\beta \in C^{r_k}$ it follows, as above, that

$$\sum_{j=1}^{k-1} (\sigma_j\beta)(i_j\beta) \in C^{r_k} \quad \text{and} \quad \sum_{j=k+1}^{n} (\sigma_j\beta)(i_j\beta) \in C^{r_k}.$$

Consequently $(\sigma_k\beta)(i_k\beta) \in C^{r_k}$. On $\beta^{-1}(A - \sigma_k^{-1}(0))$, $\sigma_k(\beta)$ is nowhere 0 and so $i_k(\beta) \in C^{r_k}$ and $D\beta \subset T_k$ there. From above, $D\beta \subset T_{k-1}$ and, since $\sigma_k^{-1}(0) \cap D_k = \phi$, it follows that $T_k = T_{k-1}$ on $\sigma_k^{-1}(0)$ and that

$$D\beta \subset T_k \quad \text{on} \quad \beta^{-1}(A - \sigma_k^{-1}(0)) \cup \beta^{-1}(\sigma_k^{-1}(0)) = \beta^{-1}(A).$$

This completes the induction.

Note that if $s_j = r_j$ in 3.4, then, by 3.2, T_j, and consequently T_k for $k \leq j$, is $T(A)$.

3.5 Corollary. *If $\{T_r\}$ is a locally finite, decreasing sequence of R-spaces on A with T_r strongly r-realizable by a C^{r-1} section of a C^∞-vector bundle $p: E \to A$, $r = 1, 2, \ldots$, then the sequence of topologies $\{\tau^r(T_r)\}$ can be represented by a section of p.*

Proof. Since the section i of 3.4 is an injection, 3.3 implies that i represents $\tau^{r_j}(T_j)$, $j = 1, 2, \ldots$. The special case with $s_j = j-1$, $r_j = j$, and $j = r$ is the corollary.

In view of 3.4 it is desirable to find strongly realizable R-spaces. To this end define an R-space T to be *r-closed* if $T = M_r(K)$, for some K that is closed in $T(A)$. If $DB_* \subset T$, where B_* represents $GC^r(R, (A, Y))$, say that T is C^r on Y. Note that if Y is a C^r-submanifold in A then T is C^r on Y iff $T(Y) \subset T$.

3.6 Lemma. *Let T be an r-closed, R-space on A that is C^r on a closed subset Y of A. If a section i of a C^∞-map $p: Z \to A$ satisfies* 3.1(a) (3.1(a)′) *and if $i\beta \in C^r$ implies $D\beta \subset T$ for all β of the form $\beta: U \to (A - Y) \subset A$, $U \in \tau(R)$, then i (strongly) r-realizes T on A.*

Proof. It is enough to show i satisfies 3.1(b) and, since $i\beta \in C^r$ implies $\beta = pi\beta \in C^r$, it is enough to prove $i\beta \in C^r$ implies $D\beta \subset T$ for $\beta: U \to A$. The assumptions on i and T imply $D(\beta \mid U_1 \cup U_2) \subset T = M_r(K)$, where $U_1 = \beta^{-1}(A - Y)$ and $U_2 = \text{interior}\,(U - U_1)$. Since $U_1 \cup U_2$ is dense in U and K is closed it follows that $D\beta \subset K$. But then, by definition, $D\beta \subset M_r(K) = T$ since $\beta \in C^r$, and the result follows.

3.7 Lemma. *If T is r-closed and C^r on each Y_α, where $\{Y_\alpha\}$, $\alpha \in D$, is a locally finite family of closed sets of A, then T is C^r on $Y = \bigcup_D Y_\alpha$.*

Proof. Since $\{Y_\alpha\}$ is locally finite it is sufficient to prove the lemma when D is finite, and in fact when D consists of two points. Suppose

$$(\beta: U \to Y \subset A) \in C^r \quad \text{and} \quad Y = Y_1 \cup Y_2.$$

Let $U_1 = \beta^{-1}((Y - Y_1) \cup (Y - Y_2))$ and $U_2 = \text{interior}\ (U - U_1)$. Since

$$Y - Y_1 \subset Y_2,\ Y - Y_2 \subset Y_1, \quad (Y - Y_1) \cap (Y - Y_2) = \phi,$$

and

$$\beta(U_2) \subset Y_1 \cap Y_2 \subset Y_1,$$

the assumptions imply $D(\beta(U_1 \cup U_2) \subset T$. The result now follows as in the proof of 3.6.

In order to construct R-spaces and sections to which 3.6 applies, it is necessary to introduce a special class of maps.

4. (r, s)-maps

A C^s-map $f: X \to Y$ between C^∞-manifolds is an *(r, s)-map* if, for all $(\beta: U \to X) \in C^r$ with U an open connected subset of R, $f\beta \in C^{s+1}$ implies $\beta \in C$, the class of constant maps, where r, s are non-negative integers.

4.1 Lemma. *Let $f: R \to R$.*

(a) *f is an $(s+1, s)$-map ($a(0,0)$-map) iff f is C^s (C^0) and is not C^{s+1} (injective or constant) on any open subset.*

(b) *f is not an (r, s)-map for $r \le s \ge 1$.*

Proof. (a) If f is an $(s+1, s)$-map (a $(0,0)$-map) and f is C^{s+1} (constant) on an open subset V, then the inclusion $\beta: V \subset R$ is C^{s+1} (C^0), $f\beta \in C^{s+1}$ (C^1) and β is not constant on connected components, a contradiction. In case f is injective on V, a local C^0-section β of f with connected domain can be found. Then $f\beta = \text{id} \in C^1$, and $\beta \notin C$, a contradiction. Conversely suppose f is C^s (C^0) but not C^{s+1} (injective or constant) on any open subset and $\beta: U \to R$ is C^{s+1} (C^0), U a connected subset of R, and $f\beta \in C^{s+1}$ (C^1). If $\beta \notin C$ then $f\beta$ has a local C^{s+1} (C^1) section σ since $D(f\beta) \neq 0$ at some point. (If $D(f\beta) = 0$ on U, then $f\beta \in C$. Thus f is constant, and consequently C^{s+1}, on interior $(\beta(U))$, which is not empty since $\beta \notin C$, a contradiction.)

Thus $\beta\sigma$ is a local C^{s+1} (C^0) section of f and consequently f is C^{s+1} (injective) on some open set, a contradiction.

(b) If f is an (r, s)-map for $s \geq 1$, then f has a local C^s-section β since $f \notin C$. Hence $\beta \in C^r$, $f\beta = \mathrm{id} \in C^{s+1}$ and β is not constant on connected components, a contradiction.

4.2 LEMMA. *If $f: X \to Y$ is an (r_1, s_1)-map, $g: Y \to Z$ is an (r_2, s_2)-map with $0 \leq s_2 \leq s_1 < r_1$, $r_2 \leq s_1$ and H is a Lie group with an effective ($h \cdot y = y$ for some y implies $h = \mathrm{id}$) C^∞-action on Y, then $\bar{g} = g(\pi_1 \cdot f\pi_2): H \times X \to Z$ is an (r_1, s_2)-map, where π_i denotes the ith projection, for $i = 1, 2$.*

Proof. Let

$$\beta = (\beta_1, \beta_2) \in C^{r_1} \quad \text{and} \quad \bar{g}\beta = g(\beta_1 \cdot f\beta_2) \in C^{s_2+1}.$$

Then $\beta_1, \beta_2 \in C^{r_2}$, $f \in C^{s_1} \subset C^{r_2}$ and consequently $\beta_1 \cdot f\beta_2 \in C^{r_2}$. Since g is an (r_2, s_2)-map, $\beta_1 \cdot f\beta_2 \in C$ and thus

$$f\beta_2 = \beta_1^{-1} \cdot (\beta_1 \cdot f\beta_2) \in C^{r_1} \subset C^{s_1+1}.$$

Since f is an (r_1, s_1)-map it follows that $\beta_2 \in C$. Since H is effective and both $\beta_1 \cdot f\beta_2$ and $f\beta_2$ are constant, $\beta_1 \in C$ and thus $\beta \in C$. Clearly $\bar{g} \in C^{s_2}$ and the result follows.

4.3 PROPOSITION. *For $n, m \geq 1$ there exists an (r, s)-map $f: R^n \to (R^m - \{0\})$ for $s = 0$ or for $s \geq 1$ if $r \geq s + 1 + \{\log_2 (n/m)\}$, where $\{x\}$ is the least integer $\geq x$ if $x \geq 0$, and 0 otherwise.*

Proof. By [5, p. 150] or [10, p. 1963] there is a non-negative continuous map $g_0: R \to R$ that is neither injective nor constant, hence not C^1, on any open subset. The C^s-map g_s, where $g_s(x) = \int_{-1}^{x} g_{s-1}(t)\, dt$, for $s = 1, 2, \ldots$, is not C^{s+1} on any open subset of $(-1, \infty)$. Consequently

$$f_s^0(x) = \sigma(x) g_s(x) + \sigma(-x) g_s(-x) + 1,$$

where $\sigma: R \to [0, \infty)$ is a C^∞-map with $\sigma^{-1}(0) = (-\infty, 0]$, is a positive C^s-map that is not C^{s+1} on any open set. By 4.1, then, $f_s^0: R \to (0, \infty) \subset R$ is an $(s+1, s)$ map, $s = 0, 1, \ldots$. Also f_0^0 is easily seen to be a $(0, 0)$-map. If there exists an $(s+p+1, s)$-map $f_s^p: R^{2^p} \to (R - \{0\})$ for $s = 0, 1, \ldots$, then by 4.2, with $X = Y = R^{2^p}$, $Z = R$, $f = \prod_{2^p} f_{s+p+1}^0$, $g = f_s^p$, $H = R^{2^p}$ with action on Y by left translation, there exists an $(s+p+2, s)$-map $f_s^{p+1}: R^{2^{p+1}} \to (R - \{0\}) \subset R$. By induction, then, there is an $(s+p+1, s)$-map $f_s^p: R^{2^p} \to (R - \{0\})$ for $s = 0, 1, 2, \ldots, p = 0, 1, 2, \ldots$. Given non-negative integers n, m, either $1 \leq n \leq m$ or $2^{p-1}m \leq n \leq 2^p m$ for some $p \geq 1$. If $n \leq m$ then

$$R^n \xrightarrow{i} R^n \times R^{m-n} = R^m \xrightarrow{f} (R^m - \{0\})$$

is clearly an $(s+1, s)$-map where $i(x) = (x, 0)$ and $f = \prod_m f_s^0$. If $2^{p-1}m < n \leq 2^p m$ then

$$R^n \xrightarrow{i} R^n \times R^{2^p m - n} = R^{2^p m} \xrightarrow{f} (R^m - \{0\})$$

is an $(s+p+1, s)$-map where $i(x)=(x, 0)$ and $f=\prod_m f_s^p$. Since

$$p=0 \quad \text{or} \quad p-1<\log_2(n/m)\le p \quad \text{for} \quad p\ge 1$$

it follows that $p=\{\log_2(n/m)\}$. Since an (r', s)-map is obviously an (r, s)-map for $r\ge r'$, the result follows for the $s\ge 1$ case. For $s=0$ proceed as follows: Using the methods and results of [1] one can show the existence of a continuous function $f\colon R^2\to R$ with totally path disconnected fibers. (Such maps are called *a-light* in [12].) Then

$$f(f\times 1)\colon R^3\to R^2\to R$$

also has totally path disconnected fibers and thus, by induction, there exists such a function $g\colon R^n\to R$. Then $f_0^0 g\colon R^n\to R\to(R-\{0\})$ is a $(0,0)$-map, since if $f_0^0 g\beta\in C^1$, then $g\beta\in C$ and so $\beta\in C$ on connected components. Hence there exists $(0,0)$-, and consequently $(r,0)$- maps $R^n\to(R^m-\{0\})$.

5. Regular kernels

An R-space T is a *local r-kernel* if $T=M_r(K)$ where K is a local kernel, i.e., for each $x\in A$ there is a C^∞-map $k_x\colon W_x\to R^{n(x)}$ on an open neighborhood W_x of x such that $K\mid W_x=\ker(Dk_x)$. Let $n(T)$ be the minimum of all bounds of all local kernels K for which $M_r(K)=T$, where a *bound* for K is an integer m so that the functions k_x can be chosen with $n(x)\le m$. T is *unbounded* if $n(T)=\infty$. An R-space T is *r-regular* if for each $x\in A$ and each open neighborhood W_x of x there is an open neighborhood V_x of x such that $\bar V_x\subset W_x$ and T is C^r (§3) on $\mathrm{Bd}(V_x)=\bar V_x-V_x$.

5.1 PROPOSITION. *Let $p\colon E\to A$ be an m-dimensional C^∞-vector bundle on A and let T be an r-regular, local r-kernel on A. If $m\ge 1$, then T can be strongly r-realized by a continuous, bounded section i of p. Further, i can be taken to be C^s for*

$$1\le s\le r-1-\{\log_2(n(T)/m)\}.$$

Proof. The proof consists of constructing a Y and i so that T, Y, i and p satisfy the conditions of 3.6. Clearly there is a family $\{k_\alpha\colon W_\alpha\to R^{n(\alpha)}\}$, $\alpha\in\mathscr{A}$, of C^∞-functions for which $n(\alpha)\le n(T)$, $\ker(Dk_\alpha)=K\mid W_\alpha$, $M_r(K)=T$, $E\mid W_\alpha$ is trivial, $\bar W_\alpha$ is compact and $\{W_\alpha\}$ is a locally finite open cover of A. Since T is r-regular, it is not difficult to construct an open cover $\{V_d\}$, $d\in D$, of A and a function $\alpha\colon D\to\mathscr{A}$ such that $\bar V_d\subset W_{\alpha(d)}$, $\{\bar V_d\}_D$ is locally finite, and T is C^r on $\mathrm{Bd}(V_d)$. Since, for a well ordering of D, the set $L_d=\bigcup_{d'<d}\bar V_{d'}$ is closed, there is a C^∞-map $\sigma_d\colon A\to R$ with $\sigma_d^{-1}(0)=(A-V_d)\cup L_d$. By 4.3 there is an (r,s)-map

$$f_{\alpha(d)}\colon R^{n(\alpha(d))}\to(R^m-\{0\}) \quad \text{for } s=0 \text{ or } 1\le s\le r-1-\{\log_2(n(T)/m)\}$$

since $n(\alpha(d))\le n(T)$. The C^s-map $\sigma_d\cdot(f_{\alpha(d)}k_{\alpha(d)})\colon A\to R^m$ can be viewed as a C^s-section i_d of p since $A-\sigma_d^{-1}(0)\subset V_d\subset W_{\alpha(d)}$ and $E\mid W_{\alpha(d)}$ is trivial.

Since $\{V_d\}$ is locally finite, $i=\sum_D i_d$ is a well defined C^s-section of p. The conditions of 3.6 will now be verified for T, i, p and $Y=\bigcup_D (\mathrm{Bd}(V_d))$. T is r-closed since $T=M_r(K)$ and K, being a local kernel, is closed in $T(A)$. By 3.7, then, T is C^r on the closed set Y since $\{\bar{V}_d\}$, and hence $\{\mathrm{Bd}(V_d)\}$, is locally finite and T is C^r on each $\mathrm{Bd}(V_d)$. To see 3.1(a)′ let $(\beta\colon U\to A)\in C^s$ with $D\beta\subset T$ and set

$$U_1=\beta^{-1}(W_{\alpha(d)}) \quad\text{and}\quad U_2=\beta^{-1}(A-\bar{V}_d).$$

Then $k_{\alpha(d)}\beta$, and consequently $f_{\alpha(d)}k_{\alpha(d)}\beta$, is constant on the connected components of U_1. Since $\sigma_d\beta$ is C^s on U and 0 on U_2, $[\sigma_d\cdot f_{\alpha(d)}k_{\alpha(d)}]\beta$ is C^s on both U_1 and U_2, and consequently on $U_1\cup U_2=U$, i.e., $i_d\beta\in C^s$. Since $i\beta=\sum i_d\beta$ is locally a finite sum, $i\beta\in C^s$ and 3.1(a)′ follows. To check the last condition of 3.6 note that if $d_1<d_2$ then $L_{d_1}\subset L_{d_2}$ and $V_{d_1}\cap(A-L_{d_2})=\phi$. Hence

$$A-\sigma_{d_1}^{-1}(0)\cup\sigma_{d_2}^{-1}(0)=V_{d_1}\cap V_{d_2}\cap(A-L_{d_2})=\phi,$$

i.e., if $\sigma_{d_1}(x)\neq 0\neq\sigma_{d_2}(x)$ then $d_1=d_2$. Thus if $\sigma_d(x)\neq 0$ then $i(x)=i_d(x)$ and, since $f_{\alpha(d)}$ is nowhere 0, $i(x)\neq 0$. Suppose $i(x)=0$ and $x\notin Y$. If d_1 is minimal with $x\in V_{d_1}$ then $x\notin A-V_{d_1}$ and $x\notin\bar{V}_d$ for $d<d_1$. Hence $x\notin\sigma_{d_1}^{-1}(0)$ and $i(x)\neq 0$, a contradiction. Therefore $i^{-1}(0)\subset Y$ and, consequently, if

$$\beta\colon U\to(A-Y)\subset A$$

then $i\beta$ is nowhere 0. For any $t\in U$, then, $i(\beta(t))\neq 0$ and $\sigma_d(\beta(t))\neq 0$ for some d. By continuity, there is a connected neighborhood U_t of t so that $\sigma_d\beta$ does not vanish on U_t; i.e., $i\beta=i_d\beta$ on U_t. But then on U_t, $i\beta\in C^r$ implies $i_d\beta\in C^r$ and thus $\sigma_d(\beta)\cdot(f_{\alpha(d)}k_{\alpha(d)}\beta)\in C^r$. Since $\sigma_d(\beta)\in C^r$ and is nowhere 0 on U_t, $f_{\alpha(d)}[k_{\alpha(d)}\beta]\in C^r$. Further, since $k_{\alpha(d)}\beta\in C^r$, $s+1\leq r$, and $f_{\alpha(d)}$ is an (r,s)-map, it follows that $k_{\alpha(d)}\beta\in C$; i.e., $D\beta\subset T$ on U_t and hence on U since t was arbitrary. It now follows from 3.6 that i strongly r-realizes T. Finally, if $i(x)\neq 0$ then $i(x)=i_d(x)$ and since $A-i_d^{-1}(0)\subset\bar{V}_d$, a compact set, i_d is bounded. By multiplying by a non-zero constant (this does not destroy the essential properties of i_d used above) one can then assume $|i_d|\leq 1$ and so a bounded i exists. This completes the proof.

5.2 Corollary. *Let $p\colon E\to A$ be an m-dimensional C^∞-vector bundle on A with $m\geq 1$. If $\{T_r\}$ is a locally finite, decreasing sequence of R-spaces where T_r is an r-regular, local r-kernel on A with $n(T_r)\leq m$, $r=1,2,\ldots$, then the sequence of topologies $\{\tau^r(T_r)\}$ can be represented by a section of p.*

Proof. Apply 5.1 to each T_r and use 3.5.

The following result is useful in showing certain R-spaces are r-regular.

5.3 Lemma. *Let $p\colon N\to M$ be a C^∞-vector bundle on M and $f\colon N\to N_1$ a C^∞-map such that*

(a) *for all $v,w\in N$, the equality $f(tv)=f(tw)$ holds for all $t\in[0,1]$ iff it holds for $t=0$ and 1.*

If there is a nowhere 0, C^∞*-section* s *of* p *for which*

(b) *the equality* $f(ts(x)) = f(ts(y))$ *holds for* $t = 1$ *if it holds for* $t = 0$, *all* x, $y \in M$,

then there is a section s_1 *of* p *such that* $T = M_r$ (ker Df) *is* C^r *on* (image s_1) $\subset N$.

Proof. Let $s_1 = (gfs_0)s$, where $g: N_1 \to (0, 1)$ is a $(0, 0)$-map, and s_0 is the 0-section of p. It is sufficient to show that if $(\beta: U \to (\text{image } s_1) \subset N) \in C^r$, where U is connected, then $f\beta \in C$. From (a) and (b) it follows that s_1 satisfies (b). Condition (b) clearly implies that if $fs_0p\beta \in C$ then $fs_1p\beta$ (and consequently $f\beta_1$ since $\beta = s_1p\beta$) is constant. Thus it is enough to show $fs_0p\beta \in C$. By assumption,

$$\beta = s_1p\beta = (gfs_0p\beta) \cdot (sp\beta) \in C^r$$

and so $g(fs_0p\beta) \in C^r$. Since g is a $(0, 0)$-map, $fs_0p\beta \in C$ and the result follows.

An easy application of 5.3 gives the following result on distributions.

5.4 Lemma. *Any* m*-dimensional integrable* C^∞*-distribution* T *on* A *is an* r*-regular, local* r*-kernel with* $n(T) = n - m$, $n = \dim(A)$.

Proof. By [2, p. 24], locally T can be identified with $\ker(D\pi_2)$ where

$$\pi_2: R^n = R^m \times R^{n-m} \to R^{n-m}$$

is projection on the second factor. Thus T is a local r-kernel since

$$\ker(D\pi_2) = M_r\,(\ker(D\pi_2)).$$

If N_1, N_2 is a tubular neighborhood of the unit sphere S^{n-m-1}, S^{m-1} in R^{n-m}, R^m respectively then $N_1' = R^m \times N_1$ and $N_2' = N_2 \times R^{n-m}$ are tubular neighborhoods of the C^∞-submanifolds (of R^n) $M_1 = R^m \times S^{n-m-1}$ and $M_2 = S^{m-1} \times R^{n-m}$ respectively. As usual, N_i' can be smoothly identified with the total space of the normal bundle of M_i in R^n where the projection $p_i: N_i' \to M_i$ is "radial", $i = 1, 2$. The conditions of 5.3 with f and p replaced by π_2 and p_i respectively are easily verified. Hence M_i (viewed as 0-section of p_i) can be replaced by a manifold M_i' ($=$ image s_1 of 5.3) arbitrarily near M_i with $T\,C^r$ on M_i'. Since kernels are r-closed, 3.7 implies that T is C^r on $M_1' \cup M_2'$. Thus T is C^r on Bd $V \subset M_1' \cup M_2'$, where V is the component of $R^n - (M_1' \cup M_2')$ containing $0 \in R^n$. By reducing the radii of S^{n-m-1} and S^{m-1} and using the homogeneity of R^n one can then obtain an arbitrarily small neighborhood V about any point of R^n with $T\,C^r$ on Bd V. This implies the result.

Each integrable C^∞-distribution T on A defines a topology $\tau^r(T)$ on A which is the foliation or leaf topology on A in the sense of [9].

5.5 Theorem. *Any decreasing sequence of foliation topologies* $\{\tau^r\}$ *on* A *can be represented by an embedding of* A *in* R^{2n}, *where* $n = \dim A$.

Proof. By definition $\tau^r = \tau^r(T_r)$, where $\{T_r\}$ is a decreasing sequence of integrable C^∞-distributions on A. Since $\{T_r\}$ is a decreasing sequence of subvector bundles of $T(A)$, it is locally finite. By 5.4 and 5.2, then, $\{\tau^r\}$ can be represented by a section i of any m-dimensional C^∞-vector bundle on A with $m \geq n$. By Whitney [13], A can be embedded in R^{2n} with n-dimensional normal bundle. Then $i(A)$ defines an embedding of A in R^{2n} that represents $\{\tau^r\}$.

6. Comments and questions

The following questions arise naturally in an attempt to extend the methods of this paper to solve the general problem of §2.

(1) The topology of A is greatly suppressed by assuming A is a C^∞-manifold. For a general subspace $A \subset R^n$, how is the topology of A reflected in the R-space $T = (\text{image } DB_*)$, where B_* represents $GC^r(R, (R^n, A))$?

(2) For a smoothness category C, the topology of $GC(R, R^n)$ is sequentially determined; a convergent sequence in R^n converges in $GC(R, R^n)$ if it has a convergent subsequence lying on a C-curve in R^n (see proof of 1.1). An intrinsic characterization of such "C-sequences" would be useful. For example, one could determine those C_1, C_2 for which $GC_1(R, R^n) = GC_2(R, R^n)$. (The case $C_1 = C^r$, $C_2 = C^s$ is 1.1(d)).

(3) Proposition 4.3 shows the existence of certain (r, s)-maps $R^n \to R$. Can the condition on r and s be relaxed? In particular, do there exist $(s+1, s)$-maps $R^n \to R$? The existence of such a map implies that of a C^s-map $R^n \to R$ with totally C^{s+1}-path disconnected fibers. Also, if such maps exist, Theorem 5.5 can be improved since the restriction on the dimension of the normal bundle could be relaxed.

(4) For which smoothness categories C_1, C_2, and C_3 do there exist C_1-maps $f\colon R^n \to R$ such that if $\beta \in C_2$ and $f\beta \in C_3$ then β is constant on connected components? ($C_1 = C^s$, $C_2 = C^r$, $C_3 = C^{s+1}$ is the (r, s)-map case.)

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