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ON THE DENSITY OF SEQUENCE $\{n_k\xi\}$

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Introduction

In his paper *Problems and results in Diophantine approximations* II which appeared in [2] Erdös asked the following:

Given a sequence of integers $n_1 < n_2 < n_2 \cdots$ satisfying $n_{k+1}/n_k \geq \alpha > 1$, $k = 1, 2, \ldots$, is it true that there always exists an irrational ξ for which the sequence $\{n_k\xi\}$ is not everywhere dense?

Here $\{x\}$ denotes the fractional part of x.

Strzelecki [5] has shown that if $\alpha \geq (5)^{1/3}$, and (t_k) is a sequence of positive real numbers, not necessarily integers, with $t_{k+1}/t_k > \alpha$ then there is a ξ such that $\{t_k\xi\} \in [\beta, 1-\beta]$, $k = 1, 2, \ldots$, for some $\beta > 0$.

It is the purpose of this paper to provide a complete answer to the question of Erdös by providing the following.

THEOREM. *Let (t_n) be a sequence of positive numbers such that*

$$q_n = t_{n+1}/t_n \geq \alpha > 1 \quad \textit{for} \quad n = 1, 2, \ldots \tag{1}$$

and let s_0 be a real number $0 < s_0 < 1$ then there exists a real number $\beta = \beta(\alpha, s_0) > 0$ and a set T of Hausdorff dimension at least s_0 such that if $\xi \in T$ then

$$\{t_k\xi\} \in [\beta, 1-\beta] \quad \textit{for} \quad k = 1, 2, \ldots. \tag{2}$$

We have the following immediate corollary.

COROLLARY. *The set of numbers ξ such that $\{t_k\xi\}$ is not dense in the unit interval has Hausdorff dimension* 1.

A similar result has recently been obtained independently by B. de Mathan [3], [4].

Proof of the Theorem. We note that it is sufficient to prove the theorem under the additional restriction that $q_n \leq \alpha^2$, for we can form a new sequence (t'_n) from (t_n) by introducing new terms between t_k and t_{k+1} if $t_{k+1}/t_k > \alpha^2$, so that $\alpha \leq t'_{n+1}/t'_n \leq \alpha^2$, $n = 1, 2, \ldots$. Obviously if the assertion of the theorem holds for some sequence (t'_n) it holds for any sub-sequence (t_n) of (t'_n).

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Choose $r \in \mathbf{N}$ so large that

$$\alpha^r - (r+2) > \alpha^{rs_0} \tag{3}$$

and put

$$N = \alpha^2 \quad \text{and} \quad \varepsilon = N^{-r}(r+1)^{-1}. \tag{4}$$

We will show that (2) holds with

$$\beta = \tfrac{1}{2}N^{-r}\varepsilon \tag{4a}$$

and with ξ belonging to the intersection of a sequence of certain closed intervals. We will construct these intervals using the following lemma.

LEMMA. *There is a sequence of pairs* (a_k, b_k) *of real numbers satisfying*:

(A_{k+1}) $a_k q_k \le a_{k+1} < b_{k+1} \le b_k q_k$;
(B) $[a_k, b_k]$ *has no integer interior points*;
(C) $l([a_{rj+1}, b_{rj+1}]) = b_{rj+1} - a_{rj+1} = N^{-r}$, $j = 0, 1, 2, \ldots$;
(D_m) *if* $r(j-1)+1 \le m \le rj$

then

$$b_{rj+1} \le \frac{t_{rj+1}}{t_m}(b_m - \beta) \quad \textit{and} \quad a_{rj+1} \ge \frac{t_{rj+1}}{t_m}(a_m + \beta).$$

Proof. Choose $[a_1, b_1]$ to have no integer interior points and length N^{-r}.
Suppose that $a_1, b_1, \ldots, a_{rj+1}, b_{rj+1}$ have been constructed to satisfy the conditions (A_{i+1}), (B), (C), (D_i), $1 \le i \le rj$.
Put $k = rj+1$. We will construct $[a_{k+1}, b_{k+1}], \ldots, [a_{k+r}, b_{k+r}]$ so that (A_{k+1}), …, (A_{k+r}), (B), (C), and (D_k), …, (D_{k+r-1}) are satisfied.
Put

$$\begin{aligned}
&\Delta = [a_k, b_k]\\
&\Delta(1) = q_k\Delta = [a_k q_k, b_k q_k]\\
&\Delta(2) = q_{k+1}\Delta(1)\\
&\quad\vdots\\
&\Delta(r) = q_{k+r-1}\Delta(r-1)
\end{aligned}$$

Now $l(\Delta(1)) < l(\Delta(2)) < \cdots < l(\Delta(r)) \le 1$ since

$$l(\Delta(r)) = N^{-r}q_k \cdots q_{k+r-1} \le N^{-r} \cdot N^r = 1.$$

Hence each of the intervals $\Delta(i)$ contains at most one integer interior point, N_i say. (If there is no integer in $\Delta(i)$ choose N_i arbitrarily in $\Delta(i)$.)
In $\Delta(i)$ order the points

$$a_k q_k \cdots q_{k+i-1},\ N_1 q_{k+1} \cdots q_{k+i-1},$$
$$N_2 q_{k+2} \cdots q_{k+i-1}, \ldots, N_i,\ b_k q_k \cdots q_{k+i-1}$$

and relabel them $P_0^{(i)} \le P_1^{(i)} \le \cdots \le P_{i+1}^{(i)}$, $1 \le i \le r$. Clearly $[P_j^{(i)}, P_{j+1}^{(i)}]$ has no integer interior points and for all i, j there is an $l = l(i, j)$ such that

(5) $$[P_j^{(i)}, P_{j+1}^{(i)}] \subset q_{k+i-1}[P_l^{(i-1)}, P_{l+1}^{(i-1)}].$$

Put

$$J(r) = [P_0^{(r)} + \varepsilon/2, P_1^{(r)} - \varepsilon/2] \cup \cdots \cup [P_r^{(r)} + \varepsilon/2, P_{r+1}^{(r)} - \varepsilon/2]$$

where we take $[a, b] = \emptyset$ if $a > b$. Then $J(r)$ is the union of at most $r+1$ intervals and has measure $m(J(r)) \ge l(\Delta(r)) - (r+1)\varepsilon$. Let

$$l[P_i^{(r)} + \varepsilon/2, P_{i+1}^{(r)} - \varepsilon/2] = l_i N^{-r}.$$

Then $m(J(r)) = \sum_{i=0}^r l_i N^{-r}$ and so

$$\sum_{i=0}^{r} [l_i] > \frac{l(\Delta(r))}{N^{-r}} - \frac{(r+1)\varepsilon}{N^{-r}} - (r+1)$$
$$= \frac{l(\Delta(r))}{N^{-r}} - (r+2) \quad \text{by (4).}$$

Hence we can find at least $l(\Delta(r))/N^{-r} - (r+2)$ disjoint sub-intervals of $\Delta(r)$ of length N^{-r} whose distance from any point $P_i^{(r)}$ is at least $\varepsilon/2$.

Choose one of these arbitrarily to be $[a_{k+r}, b_{k+r}]$ then

(6) $$[a_{k+r}, b_{k+r}] \subset [P_i^{(r)}, P_{i+1}^{(r)}] \quad \text{for some } i.$$

Now suppose that $[a_{k+j}, b_{k+j}] \subset [P_i^{(j)}, P_{i+1}^{(j)}]$; then, by (5),

(7) $$[a_{k+j}, b_{k+j}] \subset q_{k+j-1}[P_l^{(j-1)}, P_{l+1}^{(j-1)}], \quad l = l(i, j).$$

Put

(8) $$a_{k+g-1} = P_l^{(g-1)} \quad \text{and} \quad b_{k+g-1} = P_{l+1}^{(g-1)}.$$

Thus starting with $[a_{k+r}, b_{k+r}]$ define $[a_{k+r-1}, b_{k+r-1}], \ldots, [a_{k+1}, b_{k+1}]$. Clearly (A_m), (B) and (C) are satisfied for $[a_m, b_m]$, $1 \le m \le k+r = (j+1)r+1$. We now have to show that (D_m) is satisfied for $rj+1 \le m \le r(j+1)$. Now by (6), (7) and (8), $b_{k+r} + \frac{1}{2}\varepsilon \le b_{k+r-1}q_{k+r-1}$ and $b_{k+j} \le b_{k+j-1}q_{k+j-1}$, $1 \le j < r$. Thus by (1), (4) and (4a),

$$b_{k+r} \le b_m \frac{t_{k+r}}{t_m} - \frac{\varepsilon}{2}$$
$$\le \frac{t_{k+r}}{t_m}\left(b_m - \frac{\varepsilon}{2q_m \cdots q_{k+r-1}}\right)$$
$$\le \frac{t_{k+r}}{t_m}(b_m - \beta).$$

Similarly $a_{k+r} \geq (t_{k+r}/t_m)(a_m + \beta)$. Hence $(A_{k+1}), \ldots, (A_{k+r})$, (B), (C), $(D_k), \ldots, (D_{k+r-1})$ are satisfied as required.

We have constructed a sequence of intervals $([a_n, b_n])$ satisfying $q_n a_n \leq a_{n+1} < b_{n+1} \leq q_n b_n$. Thus by (1),

$$\frac{a_n}{t_n} \leq \frac{a_{n+1}}{t_{n+1}} < \frac{b_{n+1}}{t_{n+1}} \leq \frac{b_n}{t_n}.$$

So $([a_n/t_n, b_n/t_n])$ forms a sequence of closed nested intervals. Consequently there is a number ξ belonging to all the intervals of this sequence. We now have to verify that $\{\xi t_m\} \in [\beta, 1-\beta]$, $m = 1, 2, \ldots$. By the condition (D_m), $rj + 1 \leq m < r(j+1)$,

$$\frac{1}{t_{rj+1}} \frac{t_{rj+1}}{t_m}(a_m + \beta) < \frac{a_{rj+1}}{t_{rj+1}} < \xi < \frac{1}{t_{rj+1}} \frac{t_{rj+1}}{t_m}(b_m - \beta),$$

thus $a_m + \beta \leq t_m \xi \leq b_m - \beta$. But by (B), $[a_m, b_m]$ has no integer interior points. Hence $\{t_m \xi\} \in [\beta, 1-\beta]$, $m = 1, 2, \ldots$.

To show that there are uncountably many such ξ we only have to note that at each stage in the construction there are two disjoint choices for $[a_{rj+1}, b_{rj+1}]$, $j = 0, 1, 2, \ldots$, and consequently for $[a_{rj+1}/t_{rj+1}, b_{rj+1}/t_{rj+1}]$, $j = 0, 1, 2, \ldots$.

We will now use a result due to H. G. Eggleston [1] to show that the set of ξ satisfying the above conditions has Hausdorff dimension, at least s_0.

THEOREM (Eggleston). *Let A_k be a set of intervals, N_k in number, each of length δ_k. Let each interval contain $n_{k+1} > 0$ disjoint intervals of length δ_{k+1} (A_{k+1}). Suppose that $0 < s_0 \leq 1$ and that for all $s < s_0$ the sum*

$$\sum_k \frac{\delta_{k-1}}{\delta_k} (N_k (\delta_k)^s)^{-1}$$

converges. Then $P = \bigcap_{k=1}^{\infty} A_k$ has dimension greater than or equal to s_0.

We apply this theorem with

$$A_k = \left\{ \text{set of possible intervals } \left[\frac{a_{rk+1}}{t_{rk+1}}, \frac{b_{rk+1}}{t_{rk+1}}\right] \text{ after } [a_1, b_1] \text{ has been selected} \right\}.$$

Then $P \subset T$, $N_k \geq \prod_{i=1}^{k} (q_{r(i-1)i} \cdots q_{ri} - (r+2))$, and

$$\delta_k = N^{-r}(q_1 \cdots q_{rk+1})^{-1}.$$

Now $\alpha^r - (r+2) > \alpha^{rs_0}$ by (3) and so since $q_{(i-1)r+1} \cdots q_{ir} > \alpha^r$ by (1) then

$$q_{(i-1)r+1} \cdots q_{ir} - (r+2) > q_{(i-1)r+1} \cdots q_{ir}$$

and so $N_k > (q_1 q_2 \cdots q_{rk})^{s_0}$.

Let $0 < s < s_0$. Then

$$\sum_k \frac{\delta_{k-1}}{\delta_k}(N_k(\delta_k)^s)^{-1} \leq \alpha^{2r+2rs} \sum_k (q_1 \cdots q_{rk})^{s-s_0}$$
$$\leq \alpha^{2r+2rs} \sum_k (\alpha^{r(s-s_0)})^k$$

which converges and so T has dimension at least s_0.

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