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ENDO-TRIVIAL MODULES OVER (p, p)-GROUPS

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1. Introduction

Let K be a field of characteristic $p > 0$ and let G be a finite p-group. If M and N are KG-modules, then the space $\mathrm{Hom}_K(M, N)$ is a KG-module under the action $(gf)(m) = g \cdot f(g^{-1}m)$ for all $g \in G$, $f \in \mathrm{Hom}_K(M, N)$ and $m \in M$. A finitely generated KG-module M is said to be an endo-trivial module if the ring $\mathrm{Hom}_K(M, M)$, of K-endomorphism of M, is isomorphic to the direct sum of the one-dimensional trivial KG-module and a free module. These modules were called invertible modules by Alperin in [2].

In [6], E. C. Dade has proved that if G is an abelian p-group then every endo-trivial module is the direct sum of a syzygy of the trivial module and a free module. The author has independently proved this theorem using somewhat different techniques. Presented in this paper is that portion of the author's proof which differs most significantly from that of Dade. Specifically we consider the case in which G is elementary abelian of order p^2. This is an important step since the proof of the larger theorem involves an induction argument which begins with this case. The proof presented here consists of characterizing the syzygies of the trivial module in terms of their restrictions to proper subgroups of G and a mapping property.

2. Notation and preliminaries

Throughout this paper K is a field of characteristic p and G is a finite p-group. All KG-modules are assumed to be finitely generated. The radical of KG is denoted Rad KG. If M is a KG-module, then $\mathrm{Rad}\, M = \mathrm{Rad}\, KG \cdot M$ is the set of nongenerators of M. We define the rank of M to be $\mathrm{Rk}\, M = \mathrm{Dim}_K\, M/\mathrm{Rad}\, M$. The socle of M is the set $\mathrm{Soc}\, M = \{m \in M \mid gm = m \text{ for all } g \in G\}$. If H is a subgroup of G, then $\tilde{H} = \sum_{h \in H} h \in KG$, $1(H)$ is the one-dimensional trivial KH-module, and M_H is the restriction of M to a KH-module.

If F is a free KG-module and if $\phi\colon F \to M$ is an epimorphism, then the kernel of φ is isomorphic to the direct sum of a free module and a module $\Omega(M)$ which has no projective components. Similarly we define $\Omega^{-1}(M)$ to be the sum of the nonprojective components of the cokernel of a monomorphism of M into a free (and hence injective) module. It is well known that $\Omega(M)$, $\Omega^{-1}(M)$ are indepen-

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dent of the choices of free modules and the homomorphisms (see [8]). Inductively we define $\Omega^n(M) = \Omega(\Omega^{n-1}(M))$ and $\Omega^{-n}(M) = \Omega^{-1}(\Omega^{-n+1}(M))$ for all $n > 0$. Let $\Omega^0(M)$ be the sum of the nonprojective components of M. Now $\Omega^n(M)$ is the *n-th* syzygy of module M.

Let M be an endo-trivial KG-module. Then $\mathrm{Hom}_K(M, M) \cong 1(G) \oplus F$ where F is a free KG-module. Clearly $\mathrm{Hom}_{KG}(M, M) = \mathrm{Soc}(\mathrm{Hom}_K(M, M))$. Hence $\tilde{G} \cdot F = \tilde{G} \cdot \mathrm{Hom}_K(M, M)$ is a submodule of $\mathrm{Hom}_{KG}(M, M)$ with co-dimension one.

Suppose P is a projective cover of M. We have an exact sequence

$$0 \to \Omega(M) \xrightarrow{i} P \xrightarrow{\psi} M \to 0. \tag{2.1}$$

This yields a commutative diagram

$$\begin{array}{ccccccc}
0 \to \mathrm{Hom}_K(M, \Omega(M)) & \to & \mathrm{Hom}_K(M, P) & \to & \mathrm{Hom}_K(M, M) \to 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 \to \mathrm{Hom}_{KG}(M, \Omega(M)) & \to & \mathrm{Hom}_{KG}(M, P) & \xrightarrow{\psi^*} & \mathrm{Hom}_{KG}(M, M)
\end{array}$$
$$\to \mathrm{Ext}^1_{KG}(M, \Omega(M)) \to 0$$

where the vertical maps are multiplication by $\tilde{G}$ (see [12, chapter III]). The image of ψ^* contains $\tilde{G} \cdot \mathrm{Hom}_K(M, M)$ and hence the dimension of $\mathrm{Ext}^1_{KG}(M, \Omega(M))$ is at most 1. This dimension cannot be zero since otherwise (2.1) would split. Since Ext^1_{KG} distributes over direct sums we have the following.

LEMMA 2.2. *Let M be an endo-trivial KG-module. Then the dimension of $\mathrm{Ext}^1_{KG}(M, \Omega(M))$ is 1 and M is the direct sum of an indecomposable endotrivial module and a free module.*

Suppose $p = 2$ and G is noncyclic of order 4. The modules over KG have been classified by Basev in [3] and Heller and Reiner in [9]. Every indecomposable KG-module of odd dimension is of the form $\Omega^n(1(G))$ for some n (see also [10]).

Suppose G is cyclic of order p. If M is an endo-trivial KG-module then

$$\mathrm{Dim}\ \mathrm{Hom}_K(M, M) = (\mathrm{Dim}\ M)^2 \equiv 1 \mod p.$$

Therefore $\mathrm{Dim}\ M \equiv \pm 1 \bmod p$.

LEMMA 2.3. *Let M be an indecomposable endo-trivial KG-module. If G is cyclic of order p, then the dimension of M is either 1 or $p - 1$ and $M \cong \Omega^k(1(G))$ for either $k = 0$ or $k = 1$. If G is a noncyclic group of order 4 then $M \cong \Omega^n(1(G))$ for some n.*

Suppose that G is abelian and M is an endo-trivial KG-module. Let $\operatorname{End}(M) = \operatorname{Hom}_{KG}(M, M)$. Now $\operatorname{Ext}^1_{KG}(M, \Omega(M))$ is an $\operatorname{End}(M)$ – $\operatorname{End}(\Omega(M))$-bimodule (see [12]). There exist homomorphisms $\sigma\colon KG \to \operatorname{End}(\Omega(M))$ and $\tau\colon KG \to \operatorname{End}(M)$ where for each $\alpha \in KG$, $\sigma(\alpha)$ and $\tau(\alpha)$ are multiplication by α. Hence $\operatorname{Ext}^1_{KG}(M, \Omega(M))$ is a KG-KG-bimodule of dimension 1. If α is in $\operatorname{Rad} KG$, then $\sigma(\alpha)$, $\tau(\alpha)$ must annihilate $\operatorname{Ext}^1_{KG}(M, \Omega(M))$. More specifically we get a diagram

$$\begin{array}{rccccccccc} E\colon & 0 & \to & \Omega(M) & \xrightarrow{i} & P & \xrightarrow{\psi} & M & \to & 0 \\ & & & \downarrow{\scriptstyle \sigma(\alpha)} & & \downarrow & & \| & & \\ E\sigma(\alpha)\colon & 0 & \to & \Omega(M) & \to & B & \to & M & \to & 0 \end{array}$$

where B is the pushout. If $\alpha \in \operatorname{Rad} KG$, then $E\sigma(\alpha)$ splits and the splitting gives a homomorphism $\theta\colon P \to \Omega(M)$ such that $\theta i = \sigma(\alpha)$ is multiplication by α. We get a similar diagram using τ and the pullback.

PROPOSITION 2.4. *Let G be an abelian p-group and let M be an endo-trivial KG-module. For any $\alpha \in \operatorname{Rad} KG$, there exist homomorphisms $\theta\colon P \to \Omega(M)$, $\mu\colon M \to P$ (see (2.1)) such that θi and $\psi\mu$ are multiplication by α on $\Omega(M)$ and M respectively.*

Actually μ can be taken to be $\alpha - i\theta$ applied to the cosets of P modulo $\Omega(M)$.

3. The main results

Assume throughout this section, that K is a field of characteristic p and $G = \langle x, y\rangle$ is a noncyclic group of order p^2. We shall prove the following.

THEOREM 3.1. *Suppose $p > 2$. Let M be a KG-module satisfying the following conditions.*

(1) $\operatorname{Rk} M > \operatorname{Rk} \Omega^{-1}(M)$.
(2) *For any element $g \in G$ with $g \neq 1$, $M_{\langle g\rangle} \cong \Omega^k(1(\langle g\rangle)) \oplus F_g$ where $k = 0$, 1 and F_g is a free $K\langle g\rangle$-module.*
(3) *Let*

$$0 \to \Omega(M) \xrightarrow{i} P \xrightarrow{\psi} M \to 0$$

be exact, where F is a free KG-module. For any element $g \in G$ there exists homomorphisms $\sigma_g\colon P \to \Omega(M)$ and $\tau_g\colon M \to P$ such that $\sigma_g i$ and $\psi\tau_g$ are multiplication by $g - 1$ on $\Omega(M)$ and M respectively.

Then $M \cong \Omega^t(1(G)) \oplus L$ for some $t > 0$ and some KG-module L whose restriction to any proper subgroup of G is free.

Before beginning the proof of this theorem, let us consider an endo-trivial KG-module M. The results of the last section show that M satisfies both conditions (2) and (3). Let $M^* = \mathrm{Hom}_K(M, 1(G))$ be the K-dual of M. It is well known that $\mathrm{Hom}_K(M, M) \cong M \otimes_K M^*$. If

$$0 \to \Omega(M) \to P \to M \to 0$$

is exact with P a free module, then

$$0 \to \Omega(M) \otimes M^* \to P \otimes M^* \to M \otimes M^* \to 0$$

is also exact, and $P \otimes M^*$ is a free module by Frobenius reciprocity (see [5]). Hence $\Omega(M) \otimes_K M^* \cong \Omega(M \otimes M^*) \oplus F_1 \cong \Omega(1(G)) \oplus F_1$ for some free KG-module F_1. Similarly the reader may check that if n, m are any integers, then

$$\Omega^n(M) \otimes \Omega^m(M^*) \cong \Omega^{n+m}(l(G)) \oplus F_2 \tag{3.2}$$

for some free module F_2. By duality $(\Omega^n(M))^* \cong \Omega^{-n}(M^*)$. By setting $m = -n$ we see that $\Omega^n(M)$ is also an endo-trivial module. Now the dimensions of $\Omega^n(1(G))$ become arbitrarily large as n increases (see [11]). From equation (3.2) it follows that the dimensions of $\Omega^n(M)$ must increase as n becomes large. Consequently for some n, $\Omega^n(M)$ satisfies condition (1). The above argument, combined with Lemmas 2.2 and 2.3 and Theorem 3.1, imply the following.

THEOREM 3.3. *Let G be a noncyclic group* of *order p^2 and let K be a field of characteristic p. Any endo-trivial KG-module is isomorphic to the direct sum of a free KG-module and $\Omega^n(1(G))$ for some n.*

Most of the remainder of this section is devoted to the proof of Theorem 3.1. We shall use the notation in the three conditions of the theorem without further reference.

LEMMA 3.4. *A KG-module M satisfies condition* (2) (*respectively,* (3)) *of Theorem 3.1 if and only if $\Omega^n(M)$ satisfies condition* (2) (*respectively,* (3)) *for all n.*

Proof. The statement about condition (2) is obvious. Suppose that M satisfies condition (3). For any $g \in G$, $g \neq 1$ we have a diagram

$$\begin{tikzcd} & & & P \arrow[d, shift left, "\sigma_g"] & \\ 0 \arrow[r] & \Omega^2(M) \arrow[r, "i_1"] & P_1 \arrow[r, "\psi_1"] & \Omega(M) \arrow[u, shift left, "i"] \arrow[r] & 0 \end{tikzcd}$$

where P_1 is a free KG-module. Since P is projective there exists a homomorphism $\theta\colon P \to P_1$ with $\psi_1\theta = \sigma_g$. Now let $\mu_g = \theta i$. Then $\psi_1\mu_g = \psi_1\theta i = \sigma_g i$ is multiplication by $g - 1$ on $\Omega(M)$. Now let $\phi\colon P_1 \to P_1$ by $\phi(f) = (g-1)f - \mu_g\psi_1(f)$, for all $f \in P_1$. Since $\psi_1\phi = 0$, $\phi(P_1) \subseteq i_1(\Omega^2(M))$. If $\nu_g = i_1^{-1}\phi$ then $\nu_g i_1$ is multiplication by $g - 1$ on $\Omega^2(M)$. Therefore $\Omega(M)$ satisfies condition (3). Remembering that every free KG-module is injective, we can prove by a similar argument, that $\Omega^{-1}(M)$ satisfies condition (3). An induction argument completes the proof of the lemma.

LEMMA 3.5. *Let M be a KG-module which satisfies condition* (2). *Let a be a $K\langle x\rangle$-generator for the $\Omega^k(1(\langle x\rangle))$ component of some decomposition of $M_{\langle x\rangle}$. Then M satisfies condition* (1) *if and only if every such element a satisfies the following.*

(i) *a is a generator for M; i.e. $a \notin \operatorname{Rad} M$.*

(ii) *If $\operatorname{Dim} M \equiv 1 \mod p$, (i.e. $k = 0$) then $(y-1)\,a \neq 0$, while if $\operatorname{Dim} M \equiv -1 \mod p$ then $(y-1)(x-1)^{p-2}a \neq 0$.*

Proof. Let $M_0 = \{m \in M \mid xm = m\}$. We have exact sequences

$$(3.6) \qquad 0 \to 1(G) \to M/(x-1)M \to (x-1)^{p-1}M \to 0,$$

where the homomorphism on the right is induced by multiplication by $(x-1)^{p-1}$, and

$$(3.7) \qquad 0 \to (x-1)^{p-1}M \to M_0 \to 1(G) \to 0.$$

The $1(G)$ part in (3.6) is generated by the class of a. The $1(G)$ factor in (3.7) is generated either by a if $k = 0$, or by $(x-1)^{p-2}a$ if $k = 1$. Since x acts trivially on the modules in these sequences, each is a direct sum of cyclic $K\langle y\rangle$-module. The rank of each is the number of $K\langle y\rangle$-components and is also the dimension of the socle of each.

Let $0 \to M \to Q \to \Omega^{-1}(M) \to 0$ be exact where Q is a free KG-module. Now

$$\operatorname{Rk} \Omega^{-1}(M) = \operatorname{Rk} Q = \operatorname{Dim} \operatorname{Soc} Q = \operatorname{Dim} \operatorname{Soc} M_0 \geq \operatorname{Rk}\,(x-1)^{p-1}M.$$

Also $\operatorname{Rk} M = \operatorname{Rk} M/(x-1)M \geq \operatorname{Rk}\,(x-1)^{p-1}M$. The only way that we could get $\operatorname{Rk} M > \operatorname{Rk} \Omega^{-1}(M)$ is to have sequence (3.6) split while sequence (3.7) does not split. This requires a to be a generator for M while neither a nor $(x-1)^{p-2}a$ (if $\operatorname{Dim} M \equiv -1 \mod p$) is in the socle of M_0.

(3.8) Let M be a KG-module which satisfies condition (2). Let a be an element of M which generates the $\Omega^k(1(\langle x\rangle))$ component of some decomposition of $M_{\langle x\rangle}$. Let b be the same for y instead of x. We shall say that M satisfies condition (4) if one of the following holds.

(i) $\operatorname{Dim} M \equiv 1 \mod p$, and a, b are generators for M. If m is any generator for M then either $(x-1)^{p-1}m \neq 0$ or $(y-1)^{p-1}m \neq 0$.

(ii) $\operatorname{Dim} M \equiv -1 \mod p$, and

$$(x-1)^{p-1}(y-1)^{p-2}b \neq 0 \neq (x-1)^{p-2}(y-1)^{p-1}a.$$

LEMMA 3.9. *Let M be a KG-module which has no free submodules and which satisfies condition* (4). *Then M satisfies* (1).

Proof. This follows directly from Lemma 3.5. For suppose that $\operatorname{Dim} M \equiv 1 \mod p$, and that a' is a generator for the $1(\langle x\rangle)$ component in some other decomposition of $M_{\langle x\rangle}$. Then $a' \equiv \alpha a \mod (x-1)^{p-1}M$ for some $\alpha \in K, \alpha \neq 0$.

So a' is also a generator for M, and $(y-1)^{p-1}a' = (y-1)^{p-1}a \neq 0$. If $\operatorname{Dim} M \equiv -1 \bmod p$, then $a' \equiv \alpha a \bmod (x-1)M$, and

$$(x-1)^{p-2}(y-1)^{p-1}a' = \alpha(x-1)^{p-2}(y-1)^{p-1}a \neq 0.$$

Thus a' is a generator for M since $\tilde{G}M = (x-1)^{p-1}(y-1)^{p-1}M = 0$.

PROPOSITION 3.10. *Let M be a KG-module with* $\operatorname{Dim} M \equiv -1 \bmod p$. *If M satisfies conditions* (1), (2), *and* (3), *then M also satisfies* (4) *and $\Omega(M)$ satisfies conditions* (1), (2) *and* (3).

Proof. Let a be a generator for the $\Omega(1(\langle x\rangle))$ component of $M_{\langle x\rangle}$. By Lemma 3.5, a is a generator for M. Choose an element $f \in P$ such that $\psi(f) = a$. Then f is a generator for P. Also $(x-1)^{p-1}f \in \Omega(M)$ while $(x-1)^{p-2}f \notin \Omega(M)$. Let $\tau_x : M \to P$ be defined as in condition (3). Now $\tau_x(a) = (x-1)f + u$ where $u \in \Omega(M) \subseteq P$. So $\tau_x((x-1)^{p-1}a) = (x-1)^{p-1}u = 0$. Thus $u \in (x-1)P$.

Now $(x-1)^{p-1}f$ must generate the $1(\langle x\rangle)$ component of some decomposition of $\Omega(M)_{\langle x\rangle}$. Therefore

$$\Omega(M) \cap (x-1)P = K \cdot (x-1)^{p-1}f + (x-1)\Omega(M).$$

Write $u = \alpha(x-1)^{p-1}f + (x-1)v$ for some $\alpha \in K$ and $v \in \Omega(M)$. Now

$$\tau_x((y-1)^{p-1}(x-1)^{p-2}a) = \tilde{G}f + \tilde{G}v$$

If $(y-1)^{p-1}(x-1)^{p-2}a = 0$, then $\tilde{G}f = -\tilde{G}v \neq 0$, and KGv is a free submodule of $\Omega(M)$. Since this is impossible, $(y-1)^{p-1}(x-1)^{p-2}a \neq 0$. The same argument for b proves that M satisfies condition (4).

We claim that $(x-1)^{p-1}f$ is a generator for $\Omega(M) \subseteq P$. Suppose not. Then there exists elements $l, m \in \Omega(M)$ such that

$$(x-1)^{p-1}f = (x-1)l + (y-1)m.$$

So $(x-1)^{p-1}(y-1)^{p-1}f = (y-1)^{p-1}(x-1)l$. Let

$$\omega = (x-1)^{p-2}(y-1)^{p-1}f - (y-1)^{p-1}l.$$

Since $(x-1)\omega = (y-1)\omega = 0$, the element ω is in $\operatorname{Soc}(P) = \tilde{G} \cdot P \subseteq \Omega(M)$. This is a contradiction since $(x-1)^{p-2}(y-1)^{p-1}f \notin \Omega(M)$. Therefore $(x-1)^{p-1}f$ is a generator for $\Omega(M)$.

If u is any generator for the $1(\langle x\rangle)$ component of $\Omega(M)_{\langle x\rangle}$, then we can write $u = \alpha(x-1)^{p-1}f + (x-1)^{p-1}v$ for some $\alpha \in K$, $\alpha \neq 0$ and some $v \in \Omega(M)$. Hence $(y-1)^{p-1}u = \alpha\tilde{G}f \neq 0$. By Lemma 3.5, $\Omega(M)$ satisfies condition (1). Lemma 3.4 completes the proof of the proposition.

PROPOSITION 3.11. *Let M be a KG-module which satisfies conditions* (1), (2) *and* (3). *For any $n > 0$, $\Omega^n(M)$ satisfies these conditions. If $n > 0$ and* $\operatorname{Dim} \Omega^n(M) \equiv -1 \bmod p$, *then $\Omega^n(M)$ also satisfies* (4).

Proof. By Lemma 3.4 and Proposition 3.10 it is only necessary to prove that if M satisfies conditions (1), (2) and (3) and if $\operatorname{Dim} M \equiv 1 \bmod p$, then $\Omega(M)$ satisfies (1). By Lemma 3.5, M has a generator a such that $(x-1)a = 0$ and $(y-1)a \neq 0$. Let f be a generator of P with $\psi(f) = a$. Then $(x-1)f$ is in $\Omega(M) \subseteq P$, and since f is a generator for P,

$$\tilde{G}f = (x-1)^{p-2}(y-1)^{p-1}[(x-1)f] \neq 0.$$

Thus $(x-1)f$ is a generator for $\Omega(M)$. Now $(x-1)f$ generates the $\Omega(1(\langle x\rangle))$ component of some decomposition of $\Omega(M)_{\langle x\rangle}$. Suppose u is another such generator by another decomposition. There exist elements $\alpha \in K$, $\alpha \neq 0$, and $v \in \Omega(M)$ such that $u = \alpha(x-1)f + (x-1)v$. Then $(x-1)^{p-2}(y-1)^{p-1}u = \alpha\tilde{G}f \neq 0$, and u must be a generator for $\Omega(M)$. By Lemma 3.5 we are done.

PROPOSITION 3.12. *Suppose that M satisfies conditions* (1), (2), (3) *and* (4) *and that* $\operatorname{Dim} M \equiv 1 \bmod p$. *Then $\Omega^{-1}(M)$ satisfies all four conditions.*

Proof. By Lemmas 3.4 and 3.9 it is only necessary to show that $\Omega(M)$ satisfies condition (4). Let $0 \to M \to Q \xrightarrow{\theta} \Omega^{-1}(M) \to 0$ be exact where Q is a free KG-module. We shall identify M with its image in Q. Now M has a generator a such that $(x-1)a = 0$. Hence there exists an element $f \in Q$ with $a = (x-1)^{p-1}f$. Then $(y-1)^{p-1}a = \tilde{G}f \neq 0$ and f must be a generator for Q. Now $\theta(f)$ generates the $\Omega(1(\langle x\rangle))$ component of some decomposition of $\Omega^{-1}(M)_{\langle x\rangle}$. By (3.8) we need only prove that $u = (x-1)^{p-2}(y-1)^{p-1}f$ is not in M.

Suppose that u is in M. Then $(x-1)^2u = 0 = (y-1)u$. From condition (4) we conclude that u is not a generator for M, and hence $u = (y-1)^{p-1}v$ for some $v \in M$. Let $f' = (x-1)^{p-2}f - v$. Since $(x-1)f' \equiv a \bmod (x-1)M$, $(x-1)f'$ is a generator for M. But then

$$(y-1)^{p-1}(x-1)f' = 0 \quad \text{and} \quad (x-1)^{p-1}(x-1)f' = 0,$$

and we have a contradiction to condition (4). Therefore $u \notin M$, and $\Omega^{-1}(M)$ satisfies (4).

PROPOSITION 3.13. *Suppose that M is a KG-module which satisfies conditions* (1), (2), (3) *and* (4). *Assume that* $\operatorname{Dim} M \equiv -1 \bmod p$. *Then either $\Omega^{-1}(M)$ satisfies all four conditions or $\Omega^{-1}(M)$ has a component which is isomorphic to* $1(G)$.

Proof. Let

$$0 \to M \to Q \xrightarrow{\psi} \Omega^{-1}(M) \to 0$$

be exact where Q is a free KG-module. Lemma 3.4 says that $\Omega^{-1}(M)$ satisfies conditions (2) and (3). For $g \in G$, $g \neq 1$, let $\sigma_g\colon Q \to M$ be a homomorphism as in condition (3).

Let $a \in M$ be a generator for the $\Omega(1(\langle x\rangle))$ component of some decomposition of $M_{\langle x\rangle}$. Then $a = (x-1)f$ for some $f \in Q$. Also

$$\tilde{G}f = (x-1)^{p-2}(y-1)^{p-1}a \neq 0.$$

So f is a generator for Q and $\psi(f)$ is a generator for $\Omega^{-1}(M)$. If $(y-1)\psi(f) = 0$ then $\psi(f) \in \operatorname{Soc}\Omega^{-1}(M)$ and $K \cdot \psi(f) \cong 1(G)$ is a direct summand of $\Omega^{-1}(M)$. So assume for the rest of this proof that for any such element f, $(y-1)f \notin M$. We shall prove that $\Omega^{-1}(M)$ satisfies all four conditions. By Lemma 3.9 we need only show that $\Omega^{-1}(M)$ satisfies condition (4).

Now $\sigma_y((x-1)f) = (y-1)(x-1)f$, and hence $\sigma_y(f) = (y-1)f + u \in M$ where $(x-1)u = 0$. Suppose that $(y-1)^{p-1}f \in M$. Then

$$\sigma_y((y-1)^{p-1}f) = 0 = (y-1)^{p-1}u.$$

Hence $u \in (x-1)^{p-1}(y-1)Q$ and there is an element $v \in (x-1)^{p-1}Q$ with $(y-1)v = u$. If $f' = f + v$, then f' is also a generator for Q and $(y-1)f' = \sigma_y(f) \in M$. But $(x-1)f' = (x-1)f = a \in M$. This contradicts our assumption and we can conclude that $(y-1)^{p-1}f \notin M$. Therefore $\Omega^{-1}(M)$ satisfies the first statement of condition (4) in (3.8, *i*).

Now suppose that m is a generator for $\Omega^{-1}(M)$ with the property that $(x-1)^{p-1}m = 0 = (y-1)^{p-1}m$. Let ω be a generator for Q with $\psi(\omega) = m$. Since $(y-1)^{p-1}\omega \in M$,

$$\sigma_x((y-1)^{p-1}\omega) = (x-1)(y-1)^{p-1}\omega \neq 0.$$

Write $\sigma_x(\omega) = (x-1)\omega + u$ where $u \in (y-1)Q$. Since $(x-1)^{p-1}\omega \in M$ we have

$$(x-1)^{p-1}u = \sigma_x((x-1)^{p-1}\omega) = 0.$$

So there exists an element $v \in (y-1)Q$ such that $(x-1)v = u$. Let $\omega' = \omega + v$. Then $(x-1)\omega' = (x-1)\omega + u = \sigma_x(\omega) \in M$ and

$$(y-1)^{p-1}\omega' = (y-1)^{p-1}\omega \in M.$$

Now $\psi(\omega')$ is a generator for $\Omega^{-1}(M)$. Also $(x-1)\psi(\omega') = 0 = (x-1)\psi(f)$. By condition (2), there exists a nonzero element $\alpha \in K$ such that $\psi(f - \alpha\omega')$ is in $(x-1)^{p-1}\Omega^{-1}(M)$. But then

$$0 \neq (y-1)^{p-1}\psi(f) = (y-1)^{p-1}\psi(f - \alpha\omega') \in \tilde{G}\Omega^{-1}(M).$$

Since $\Omega^{-1}(M)$ has no free direct summands, this is impossible. This completes the proof of the proposition.

Proof of Theorem 3.1. Let M be a KG-module satisfying conditions (1), (2), and (3). By Proposition 3.11, $\Omega^n(M)$ satisfies these three conditions for all $n \geq 0$, and either $\Omega(M)$ or $\Omega^2(M)$ satisfies condition (4). Note that condition (1) implies that neither M nor $\Omega(M)$ has a component which is isomorphic to $1(G)$.

Suppose no $\Omega^{-n}(M)$ has such a component for $n > 0$. By Propositions 3.11 and 3.12, $\Omega^{-n}(M)$ satisfies all four conditions for all $n > 0$. Hence

$$\mathrm{Rk}\,(M) > \mathrm{Rk}\,(\Omega(M)) > \mathrm{Rk}\,(\Omega^2(M)) > \cdots$$

This is clearly impossible. So there must be $n > 0$ such that $\Omega^{-n}(M) \cong 1(G) \oplus L$. By condition (2) and the Krull-Schmidt Theorem the restriction of L to any proper subgroup of G must be free. This completes the proof.

Recent results indicate that the assumption of condition (1) in the theorem is unnecessary. That is, we have the following.

THEOREM 3.14. *Let M be a KG-module which satisfies conditions* (2) *and* (3). *Then for some integer n, $M \cong \Omega^n(1(G)) \oplus L$ where the restriction of L to any proper subgroup of G is free.*

Proof. By [1], M is not periodic since otherwise p would divide the dimension of M. So there is no bound on the dimensions of the modules $\Omega^t(M)$ for $t > 0$ (see [4] or [7]). Consequently some $\Omega^t(M)$ satisfies condition (1). Now apply Theorem 3.1.

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