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A NONLINEAR VOLTERRA-STIELTJES INTEGRAL EQUATION AND A GRONWALL INEQUALITY IN ONE DIMENSION

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Introduction

In his article [27] T. H. Hildebrandt has given a complete treatment of linear differentio-Stieltjes integral equations involving H. S. Wall's theory of harmonic matrices [69] and the concept of the Young integral [74], [26]. This type of integral allows one to integrate any function of bounded variation with respect to another and to distinguish between the value of a function at some point and the right- as well as the left-hand limit at this point (e.g. if we regard the integral as a function of the upper limit). Thus, Hildebrandt derives a necessary and sufficient condition for the existence and uniqueness to both homogeneous and nonhomogeneous equations

$$Y(x) = Y_0 + \int_a^x dA(s)Y(s),$$

$$Y(x) = U(x) + \int_a^x dA(s)Y(s)$$

in which x varies in the closed interval $[a, b]$, Y and U are n-dimensional vector functions, and A is an $n \times n$ matrix function, defined also on $[a, b]$.

In the present paper we discuss mainly the case of only one dimension. Clearly, in this case Hildebrandt's results have an especially simple and explicit representation. Using this we are able to solve a nonlinear differentio-Stieltjes integral equation

$$y(x) = y_0 + \int_a^x f(s, y(s))\, dm(s) \quad (x \in [a, b]) \tag{0.1}$$

with a continuous Lipschitzian function f and a function m of bounded variation on $[a, b]$. Because of the discontinuities of m the usual proof of existence and uniqueness for the classical explicit first-order differential equations via Banach's fixed point principle is not applicable; in general the corresponding operator is not contractive. This problem is solved by introducing an appropriately weighted norm in generalization of the well-known very effective norm $\|f\| = \sup\{e^{-\alpha x}|f(x)|;\, x \in [a, b]\}$, first introduced by D. Morgenstern [52],

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which allows one to obtain the global solution in the case $m(x) = x$, compare e.g. with [70, pp. 48–50]. Another approach to solve the equation (0.1) was given in the paper of P. C. Das and R. R. Sharma [6] in which an assertion about local existence is proved. In [5] the same authors has investigated applications in deterministic control theory, compare also with R. W. Rishel [59].

In [43] and other articles J. S. MacNerney has extended the work of H. S. Wall [68], [69] about harmonic matrices to a more abstract setting by a consequent usage of product integral methods. This approach allows also the establishment of a nonlinear integral operation [48]. Many other articles are devoted to further development in various directions, see J. W. Neuberger [55], R. H. Cox [4], W. H. Ingram [33], B. W. Helton [18], J. V. Herod [23], C. W. Bitzer [1], D. L. Lovelady [43], J. C. Helton [21]. Compare also with the articles by D. B. Hinton [30], J. A. Reneke [58] and W. L. Gibson [13]. For connections between solutions to Stieltjes integral equations based on various types of integrals see [46], [23]. An extensive list of references can be found in the recent paper by J. C. Helton and S. Stuckwisch [22]. Using another integral concept J. Kurzweil has developed in the fiftieth also a general theory of differential equations with possibly left continuous solutions, see [38], [39], also [62] and the further work of Š. Schwabik. In this case there are connections between Kurzweil's and our approach, compare to the local existence theorem [39, p. 366].

To prove the continuous dependence of the solution to (0.1) on the initial value the classical Gronwall lemma [17, p. 24] is not applicable. We replace again the exponential function by a suitable discontinuous but "harmonic" [69] function and derive in this manner an appropriate generalized Gronwall inequality of the type described by J. V. Herod [24] for general linear Stieltjes integrals. Concerning other types of integrals see also D. B. Hinton [30, p. 318], W. W. Schmaedeke and G. R. Sell [61], Š. Schwabik [62, p. 401], B. W. Helton [19], [20], F. M. Wright, M. L. Klasi and D. R. Kennebeck [73], and J. R. Kroll and K. P. Smith [37]. For the purely discontinuous case see G. S. Jones [34], D. Willett and J. S. W. Wong [71], and J. Chandra and B. A. Fleishman [3].

From a detailed analysis of the above linear and nonlinear equations (which allow right- and left-hand discontinuities) we see that there are some defects: The existence and uniqueness of solutions depend on sometime troublesome conditions for the function m or the functions m and f, respectively. This gap will be omitted if we use another version of these equations. Thus we can solve uniquely the equation

$$y(x) = y_0 + \int_a^x y(s-0)\, dm(s) \quad (x \in [a, b])$$

without any conditions, and the existence and uniqueness of a solution to the nonlinear equation

$$y(x) = y_0 + \int_a^x f(s, y(s-0))\, dm(s) \quad (x \in [a, b])$$

requires only the usual Lipschitz condition on the function f. We remark that equations of this type occur in the theory of stochastic equations, based on the integral calcul of K. Itô, see e.g. C. Doléans-Dade [9], I. I. Gihman and A. V. Skorohod [14].

If we use the linear equations only with right continuous functions instead of arbitrary functions of bounded variation the situation is much simpler and some proofs are more elegant. For example we give a very short proof of our version of Gronwall's lemma in this case. Also the connections between the solutions of both versions of the homogeneous equation are more transparent.

Afterwards we consider an application of our results in stochastic control theory. A Bellman type equation is solved, which arises from the optimal control of one-dimensional quasi-diffusion processes [16].

Finally, we take a look at the case of more then one dimension. In a modified form, most of the previous results are valid also too.

In a short appendix the definition of the Young integral and some of its properties are listed. We give a Lebesgue type definition of the Young integral. Thus any measurable bounded function can be integrated with respect to an arbitrary weight function of bounded variation. This concept is useful in the treatment of nonlinear equations. But Hildebrandt [27] use a Riemann type Young integral. Nevertheless in the case that the integrand as well as the weight function are of bounded variation both integrals exist and are equal. Because the solutions of our integral equations are in fact of bounded variation all of Hildebrandt's results concerning linear equations remain applicable in our consideration. Most assertions made in the appendix can be found in [28]. For the higher-dimensional case see also the very detailed summary about this in the paper of O. Vejvoda and M. Tvrdý [67]. In further papers [65], [66] these authors have developed boundary value problems for integral equations with nondegenerate (time dependent) kernels.

Concerning mechanical interpretations of some Stieltjes integral equations see the monograph of F. R. Gantmacher and M. G. Kreĭn [12] and the nice appendix of the Russian translation of F. V. Atkinson's monograph, written by I. S. Kac and M. G. Kreĭn [35]. Compare also with H. Langer [42] and the work of W. T. Reid [57], W. F. Denny [8] and C. S. Hönig [31]. For classical nonlinear Volterra integral equations see R. K. Miller [51], Ja. M. Mamedov and S. A. Aširov [49].

The present paper is very influenced by T. H. Hildebrandt; the knowledge of his work as well as his kindly encouragement was very helpful during its preparation. Also the author wishes to express his hearty gratitude to the referee for the communicated improvements and suggestions concerning mathematics and style.

Preliminaries

With the exception of Section 7 we deal throughout this paper with real-valued bounded functions and especially with functions of bounded variation on the closed interval $[a, b]$ only. For such a function g of bounded variation and all points $x \in [a, b]$ we denote the differences $g(x) - g(x-0)$, $g(x+0) - g(x)$, and $g(x+0) - g(x-0)$ by $\Delta^- g(x)$, $\Delta^+ g(x)$, and $\Delta^{\pm} g(x)$, respectively. We make the convention $g(a-0) = g(a)$ and $g(b+0) = g(b)$. Clearly, the Δ^-, Δ^+ and $\Delta^{\pm}$ are linear operations. Further, let $|g|(x)$ be the total variation of g on the segment $[a, x]$.

Finally, let m be a fixed function of bounded variation on $[a, b]$. Without loss of generality we assume always $m(a) = 0$.

1. About linear equations

At first we consider the homogeneous equation

$$y(x) = y_0 + \int_a^x y(s)\, dm(s) \quad (x \in [a, b]) \tag{I}$$

with respect to an arbitrary real initial value y_0. The result of Hildebrandt has the following form.

Equation (I) *has a unique solution if and only if the relation* $1 - \Delta^- m(x) \neq 0$ *holds for all* $x \in [a, b]$. *Then the solution can be expressed by*

$$y(x) = y_0 e^{m(x)} \prod_{\tau < x} [1 + \Delta^+ m(\tau)] e^{-\Delta^+ m(\tau)} / \prod_{\tau \leq x} [1 - \Delta^- m(\tau)] e^{\Delta^- m(\tau)} \quad (x \in [a, b]).$$

Clearly, in case of $1 - \Delta^- m(x)$, $1 + \Delta^+ m(x) > 0$ $(x \in [a, b])$ the solution y is (strictly) *positive* on the whole interval $[a, b]$ whenever $y_0 > 0$. Because this assertion is one of the crucial points of the following considerations we repeat here some ideas of Hildebrandt's proof; for the full story see [27]. First let us consider equation (I) with a continuous weight function m. With the aid of the formula

$$\int_c^d m(s)^n\, dm(s) = (m(d)^{n+1} - m(c)^{n+1})/n + 1$$

$$(a \leq c < d \leq b;\ n = 0, 1, 2, \ldots) \tag{1.1}$$

we obtain, by a term by term integration, that $x \to y_0 e^{m(x)}$ $(x \in [a, b])$ is the solution to (I); compare with [68, p. 74]. Now we will consider the case of only one discontinuity of m at $a < x_1 < b$. On the interval $[a, x_1)$ we have $y(x) = y_0 e^{m(x)}$ again. At the point x_1, we have

$$\begin{aligned} y(x_1) &= y_0 + \int_a^{x_1} y(s)\, dm(s) \\ &= y_0 + \int_a^{x_1 - 0} y(s)\, dm(s) + \int_{x_1 - 0}^{x_1} y(s)\, dm(s) \\ &= y(x_1 - 0) + y(x_1)\Delta^- m(x_1). \end{aligned}$$

It follows that the value $y(x_1)$ is determined uniquely if and only if the term $1 - \Delta^- m(x_1)$ is nonvanishing. In this case

$$\begin{aligned} y(x_1) &= y_0 e^{m(x_1 - 0)}/[1 - \Delta^- m(x_1)] \\ &= y_0 e^{m(x_1)}/[1 - \Delta^- m(x_1)]e^{\Delta^- m(x_1)}. \end{aligned}$$

Further,

$$\begin{aligned} y(x_1 + 0) &= y_0 + \int_a^{x_1} y(s)\, dm(s) + y(x_1)\Delta^+ m(x_1) \\ &= [1 + \Delta^+ m(x_1)]y(x_1) \\ &= y_0 e^{m(x_1 + 0)}[1 + \Delta^+ m(x_1)]e^{-\Delta^+ m(x_1)}/[1 - \Delta^- m(x_1)]e^{\Delta^- m(x_1)} \end{aligned}$$

without any new condition. In the end, for $x_1 < x \leq b$ we have

$$y(x) = y(x_1 + 0) + \int_{x_1 + 0}^{x} y(s)\, dm(s).$$

Because of the continuity of m in $(x_1, b]$ we can write

$$\begin{aligned} y(x) &= y(x_1 + 0)e^{m(x) - m(x_1 + 0)} \\ &= y_0 e^{m(x)}[1 + \Delta^+ m(x_1)]e^{-\Delta^+ m(x_1)}/[1 - \Delta^- m(x_1)]e^{\Delta^- m(x_1)}. \end{aligned}$$

In the case of finitely many discontinuities of m the validity of (I) can be proven step by step. In general, we approximate the function m by suitable functions m_k ($k = 1, 2, \ldots$) with the same continuous part as m and with those discontinuities of m, which are greater than $1/k$. Now we solve the equations

$$y_k(x) = y_0 + \int_a^x y_k(s)\, dm_k(s) \quad (x \in [a, b])$$

and show that the limit of the sequence (y_k; $k = 1, 2, \ldots$) is a solution to equation (I).

The uniqueness of this solution can be shown with the help of some corresponding nonhomogeneous equations, which have weight functions with finitely many discontinuities only. But for this as well as for the proof of the following general assertion about the nonhomogeneous equation

$$y(x) = u(x) + \int_a^x y(s)\, dm(s) \quad (x \in [a, b]) \tag{II}$$

with arbitrary functions m and u of bounded variation we refer to Hildebrandt's original paper. For convenience we assume $1 - \Delta^- m(x) > 0$, $1 + \Delta^+ m(x) > 0$ ($x \in [a, b]$). Then the equation

$$h(x) = 1 + \int_a^x h(s)\, dm(s) \quad (x \in [a, b])$$

has a unique and positive solution h and we can formulate the following assertion.

The equation (II) *has a unique solution* y, *defined by the formula*

$$y(x) = h(x)\Bigg[u(a) + \int_a^x h(s)^{-1}\,du(s) + \sum_{a<\tau\le x} h(\tau-0)^{-1}\Delta^- m(\tau)\Delta^- u(\tau) - \sum_{a\le\tau<x} h(\tau+0)^{-1}\Delta^+ m(\tau)\Delta^+ u(\tau)\Bigg] \quad (x \in [a, b]).$$

If we interpret the interval $[a, b]$ as the time scale of some system, which is described for instance by the homogeneous equation (I), because of the relation $y(x) = y(x-0) + y(x)\Delta^- m(x)$ one can say that this system is anticipative. At the points x with $\Delta^- m(x) \neq 0$ we need for the further evolution of y information about the near future. In Section 4 we will give an alternative of this situation.

2. A Gronwall inequality

Let us assume additionally that the function m is nondecreasing on the interval $[a, b]$. Consequently, $1 + \Delta^+ m(x) > 0$, but we suppose in addition that $1 - \Delta^- m(x) > 0$ for all $x \in [a, b]$. Then the equation

$$h(x) = 1 + \int_a^x h(s)\,dm(s) \quad (x \in [a, b])$$

has a unique and positive solution. With the aid of this function we are able to formulate an appropriate analogue of Gronwall's lemma.

Let c *be a nonnegative constant and* y *a function of bounded variation on* $[a, b]$ *with*

$$0 \le y(x) \le c + \int_a^x y(s)\,dm(s) \quad (x \in [a, b]).$$

Then $y(x) \le c \cdot h(x)$ $(x \in [a, b])$.

Clearly, here the function h plays the role of the exponential function in the classical case. For the proof we can go the same somewhat tedious way as in the argument solving the equation (I): After consideration of a continuous m you have to deal with functions m, which have finitely many discontinuities. The general case can be treated by approximation. We omit this procedure, but we refer to J. V. Herod [24, p. 35] in a more general setting, and to Section 5, in which we give a very simple proof with respect to the special case of both right continuous functions y and m.

3. The nonlinear equation

Now we are able to formulate the main result of this paper. Let m be a function of bounded variation again. The function f, which is defined and continuous on the set $[a, b] \times \mathbf{R}$, satisfies a uniform Lipschitz condition

$$|f(x, y_1) - f(x, y_2)| \leq L|y_1 - y_2| \quad (x \in [a, b];\ y_1, y_2 \in \mathbf{R})$$

with some positive constant L.

THEOREM. *Assume there exists a (positive) constant α such that*

$$|\Delta^- m(x)|^{-1} > \alpha > L \tag{3.1}$$

for all points $x \in [a, b]$ with $\Delta^- m(x) \neq 0$. Then for every real y_0 the equation

$$y(x) = y_0 + \int_a^x f(s, y(s))\, dm(s) \quad (x \in [a, b]) \tag{III}$$

has a unique solution y, which depends continuously on the initial value $y(a) = y_0$.

To prove this assertion we fix first the solution h_α to the equation

$$h_\alpha(x) = 1 + \alpha \int_a^x h_\alpha(s)\, d|m|(s) \quad (x \in [a, b]),$$

which is positive because of condition (3.1) and the relation $\Delta^- |m|(x) = |\Delta^- m(x)|$ $(x \in [a, b])$. Further, it is clear that we have to seek for a solution of (III) in the set of all functions g which satisfy the conditions

$$m(x) = m(x - 0) \Rightarrow g(x) = g(x - 0),$$
$$m(x) = m(x + 0) \Rightarrow g(x) = g(x + 0)$$

for all points $x \in [a, b]$. This means that every point of left continuity of m is also one of the function g and similarly with respect to right continuity. To define a space of such functions let us introduce a new metric ρ_m on the set $[a, b]$ by

$$\rho_m(x, y) = |x - y| + |m(x) - m(y)| \quad (x, y \in [a, b]).$$

It should be remarked that the metric space $([a, b], \rho_m)$ is *not* complete and consequently not compact. Let us regard for example a point $x \in [a, b]$ with $\Delta^+ m(x) > 0, \Delta^- m(x) = 0$, and a sequence $x_n \to x, x_n > x$ which converges in the usual sense from the right to the point x. Then this sequence is not convergent in the space $([a, b], \rho_m)$ because there exist neighborhoods $(x - \delta, x]$ $(0 < \delta \leq \Delta^+ m(x))$ containing not any point of the sequence (x_n).

We denote by $\mathbf{C}_m[a, b]$ the space of all real-valued bounded functions which are continuous with respect to the topology obtained by the metric ρ_m. Because of the completeness of the real axis $\mathbf{R}$ the space $\mathbf{C}_m[a, b]$ with the supremum norm is complete itself; compare for example with [53, p. 246]. But for our considerations we must introduce another norm, which is defined for every $y \in \mathbf{C}_m[a, b]$ by

$$\|y\|_\alpha = \sup \{h_\alpha(x)^{-1} |y(x)|\,;\, x \in [a, b]\}.$$

Because of the positivity and finiteness of h_α this norm is equivalent to the supremum norm, see [70, p. 42] for instance. Consequently, $\mathbf{C}_m[a, b]$ with the norm $\|\cdot\|_\alpha$ is also a Banach space and we can apply Banach's fixed point principle. For this purpose we show that the operator T, defined for every $y \in \mathbf{C}_m[a, b]$ by

$$(Ty)(x) = y_0 + \int_a^x f(s, y(s))\, dm(s) \quad (x \in [a, b]),$$

is contractive on the space $(\mathbf{C}_m[a, b], \|\cdot\|_\alpha)$. Let y and z be arbitrary functions in $\mathbf{C}_m[a, b]$. Then, because of the Lipschitz condition and the definition of h_α for every $x \in [a, b]$, we have

$$\begin{aligned}
|(Ty)(x) - (Tz)(x)| &= \left| \int_a^x [f(s, y(s)) - f(s, z(s))]\, dm(s) \right| \\
&\le \int_a^x |f(s, y(s)) - f(s, z(s))|\, d|m|(s) \\
&\le L \int_a^x |y(s) - z(s)|\, d|m|(s) \\
&\le L\|y - z\|_\alpha \int_a^x h_\alpha(s)\, d|m|(s) \\
&= \alpha^{-1} L \|y - z\|_\alpha (h_\alpha(x) - 1) \\
&\le \alpha^{-1} L \|y - z\|_\alpha h_\alpha(x).
\end{aligned}$$

Consequently, we have the estimation

$$h_\alpha(x)^{-1} |(Ty)(x) - (Tz)(x)| \le \alpha^{-1} L \|y - z\|_\alpha;$$

this means $\|Ty - Tz\|_\alpha \le \alpha^{-1} L \|y - z\|_\alpha$. Because of (3.1) we have $\alpha^{-1} L < 1$ and the operator T is strict contractive. It follows that there exists a unique solution to equation (III).

Finally, let y_0 and z_0 be two initial values for equation (III), and y and z the corresponding solutions. By the Lipschitz condition then, for every $x \in [a, b]$, we have

$$|y(x) - z(x)| \le |y_0 - z_0| + L \int_a^x |y(s) - z(s)|\, d|m|(s).$$

The solution of equation (III) is of bounded variation and we can apply the Gronwall inequality from Section 2. We obtain the estimation

$$|y(x) - z(x)| \le |y_0 - z_0|\, h_L(x) \quad (x \in [a, b]),$$

in which the function h_L is the solution to the equation

$$h_L(x) = 1 + L \int_a^x h_L(s)\, d|m|(s) \quad (x \in [a, b]).$$

The function h_L is bounded and therefore the solution y to equation (III) depends continuously on its initial value y_0.

4. Another type of equation

At the end of Section 1 we observed that our equations are anticipative in some sense. Now we will consider equations which are possibly more realistic, if we think of concrete physical systems. This feeling will be emphasized by the fact that these equations are solvable uniquely under substantially weaker conditions. Indeed, let us consider the equations

$$w(x) = w_0 + \int_a^x w(s-0)\, dm(s) \qquad (x \in [a, b]) \tag{I'}$$

$$w(x) = u(x) + \int_a^x w(s-0)\, dm(s) \qquad (x \in [a, b]) \tag{II'}$$

$$w(x) = w_0 + \int_a^x f(s, w(s-0))\, dm(s) \quad (x \in [a, b]) \tag{III'}$$

in which the functions m and u are of bounded variation on $[a, b]$, w_0 is a real, and the function f is continuous and satisfies the Lipschitz condition from Section 3 with the same Lipschitz constant L.

Equation (I′) *has a unique solution* w, *which can be expressed by the formula*

$$w(x) = w_0 e^{m(x)}[1 + \Delta^- m(x)] e^{-\Delta^- m(x)} \prod_{\tau < x} [1 + \Delta^{\pm} m(\tau)] e^{-\Delta^{\pm} m(\tau)} \qquad (x \in [a, b]).$$

The convergence of the (possibly) infinite products follows from the absolute convergence of the series $\sum_{\tau < x} \Delta^{\pm} m(\tau)$; compare this with [56] or [36, p. 229]. In the sequel we consider only the essential difference between equations (I′) and (I) which occurs naturally at points x with $\Delta^- m(x) \neq 0$. At such points we have

$$w(x) = [1 + \Delta^- m(x)] w(x-0), \qquad w(x+0) = [1 + \Delta^{\pm} m(x)] w(x-0).$$

But to "solve" uniquely this equation we need no further condition, contrary to the case of equation (I) in which the equations

$$y(x) = [1 - \Delta^- m(x)]^{-1} y(x-0), \qquad y(x+0) = [1 + \Delta^+ m(x)] y(x)$$

must be fulfilled.

To simplify matters we consider the solution to (II′) in the case where

$$1 + \Delta^- m(x),\ 1 + \Delta^{\pm} m(x) > 0 \quad (x \in [a, b]).$$

In this case the homogeneous equation

$$k(x) = 1 + \int_a^x k(s-0)\,dm(s) \quad (x \in [a, b])$$

has a positive solution k and *we can write the unique solution to the nonhomogeneous equation* (II′) *in the form*

$$\begin{aligned} w(x) = k(x) \Bigg[u(a) &+ \int_a^x k(s-0)^{-1}\,du(s) \\ &- k(a+0)^{-1}\Delta^+ m(a)\Delta^+ u(a) \\ &- \sum_{a<\tau<x} k(\tau+0)^{-1}\Delta^\pm m(\tau)\Delta^\pm u(\tau) \Bigg] \\ &- \Delta^- m(x)\Delta^- u(x) \quad (x \in [a, b]). \end{aligned}$$

The proof of this assertion is similar to the argument in [27, Section 10, pp. 368–9], which verifies the solution of (II), using the Dirichlet formula.

Now we turn to the last equation.

The nonlinear equation (III′) *has a unique solution* w, *which depends continuously on the initial value* w_0.

The proof of this statement follows that in Section 3, but we choose in contrary here on the set $\mathbf{C}_m[a, b]$ the weighted norm

$$\|w\|_\alpha = \sup\{k_\alpha(x)^{-1}|w(x)|;\ x \in [a, b]\},$$

in which the weight function k_α is the (positive and bounded) solution to the equation

$$k_\alpha(x) = 1 + \alpha \int_a^x k_\alpha(s-0)\,d|m|(s) \quad (x \in [a, b]) \tag{4.1}$$

with $\alpha = 2L$. We show that the operator U, defined by

$$(Uw)(x) = w_0 + \int_a^x f(s, w(s-0))\,dm(s) \quad (w \in \mathbf{C}_m[a, b];\ x \in [a, b]),$$

is contractive. For $w, v \in \mathbf{C}_m[a, b]$ and $x \in [a, b]$ we have

$$\begin{aligned} |(Uw)(x) - (Uv)(x)| &\le \int_a^x |f(s, w(s-0)) - f(s, v(s-0))|\,d|m|(s) \\ &\le L\|w-v\|_\alpha \int_a^x k_\alpha(s-0)\,d|m|(s) \\ &\le 2^{-1}\|w-v\|_\alpha k_\alpha(x), \end{aligned}$$

and from here $\|Uw - Uv\|_\alpha \leq 2^{-1}\|w - v\|_\alpha$. To show the continuous dependence on the initial value w_0 we need the following version of Gronwall's lemma. On this occasion k is the (positive) solution to the equation (4.1) if we set $\alpha = 1$, compare the considerations in the next section for a special case.

Let $c \geq 0$ and w be a function of bounded variation on $[a, b]$ which satisfy the inequality

$$0 \leq w(x) \leq c + \int_a^x w(s-0)\, d|m|(s) \quad (x \in [a, b]).$$

Then we have the estimation $w(x) \leq c \cdot k(x)$ $(x \in [a, b])$.

5. A special case: right continuous solutions

In this section we simplify our considerations. To this end we assume the right continuity of the function m on the interval $[a, b]$. Also we assume $1 - \Delta^- m(x) > 0$, $1 + \Delta^- m(x) > 0$ for all $x \in [a, b]$. Especially, we are interested in an analogous form of the relation $e^x e^{-x} = 1$, where we interpret e^x and e^{-x} as solutions to the equations $y(x) = 1 + \int_0^x y(s)\, ds$ and $\bar{y}(x) = 1 - \int_0^x \bar{y}(s)\, ds$, respectively. For this matter, we consider the equations

$$y(x) = \eta + \int_a^x y(s)\, dm(s), \tag{I}$$

$$\bar{y}(x) = \eta - \int_a^x \bar{y}(s)\, dm(s), \tag{DI}$$

$$w(x) = \eta + \int_a^x w(s-0)\, dm(s), \tag{I$'$}$$

$$\bar{w}(x) = \eta - \int_a^x \bar{w}(s-0)\, dm(s), \tag{DI$'$}$$

with the common initial value $\eta \in \mathbf{R}$. Applying the results of Sections 1 and 4 we obtain the (positive) solutions to these equations for all $x \in [a, b]$ in the following form:

$$y(x) = \eta e^{m(x)} / \prod_{\tau \leq x} [1 - \Delta^- m(\tau)] e^{\Delta^- m(\tau)},$$

$$\bar{y}(x) = \eta e^{-m(x)} / \prod_{\tau \leq x} [1 + \Delta^- m(\tau)] e^{-\Delta^- m(\tau)},$$

$$w(x) = \eta e^{m(x)} \prod_{\tau \leq x} [1 + \Delta^- m(\tau)] e^{-\Delta^- m(\tau)},$$

$$\bar{w}(x) = \eta e^{-m(x)} \prod_{\tau \leq x} [1 - \Delta^- m(\tau)] e^{\Delta^- m(\tau)}.$$

From this it follow that $y(x) = \bar{w}(x)^{-1}$, $w(x) = \bar{y}(x)^{-1}$ $(x \in [a, b])$. For that reason, equations (I) and (DI$'$) as well as (I$'$) and (DI) are in a natural way *adjoint* to each other (compare with [27, Section 12, pp. 370–1]).

As a nice application we give as indicated previously a very simple proof of the Gronwall lemma from Section 2 with respect to right continuous functions m and y (m nondecreasing).

Let h and $\bar{k}$ be the solutions to the equations

$$h(x) = 1 + \int_a^x h(s)\, dm(s) \quad (x \in [a, b])$$

and

$$\bar{k}(x) = 1 - \int_a^x \bar{k}(s-0)\, dm(s) \quad (x \in [a, b]),$$

respectively. For a nonnegative constant c and all $x \in [a, b]$ we assume that $0 \leq y(x) \leq c + \int_a^x y(s)\, dm(s)$. Setting $z(x) = \int_a^x y(s)\, dm(s)$ we have $y(x) - z(x) \leq c$. We multiply by the integrating factor $\bar{k}(x-0)$ and integrate both sides, and obtain

$$\int_a^x y(s)\bar{k}(s-0)\, dm(s) - \int_a^x z(s)\bar{k}(s-0)\, dm(s) \leq c \int_a^x \bar{k}(s-0)\, dm(s),$$

where we have used the fact that m is nondecreasing. It follows

$$\int_a^x \bar{k}(s-0)\, dz(s) + \int_a^x z(s)\, d\bar{k}(s) \leq c(1 - \bar{k}(x)).$$

In this case the integration by parts is very simple (compare for example with [29, Satz 20.9, p. 132]). Using $z(a) = 0$ we have

$$\bar{k}(x)z(x) \leq c(1 - k(x)),$$

$$y(x) - c \leq z(x) \leq c(\bar{k}(x)^{-1} - 1) = c(h(x) - 1),$$

and, finally, the desired result

$$y(x) \leq c \cdot h(x) \quad (x \in [a, b]).$$

With the help of the solutions to (I′) and (DI) for $\eta = 1$ we can verify the version of Gronwall's lemma from Section 4 in the same fashion.

6. An application in stochastic control theory

Many problems within stochastic control theory lead to a so-called Bellman equation which allows one in principle to compute the minimal expected cost corresponding to the application of an optimal control policy. In case of the optimal control of one-dimensional nonconservative quasi-diffusion processes (see [16], and for classical diffusion processes [50, Section VI.3]) the Bellman equation has the form

$$(D_m D_p^+ v)(x) + \min_{z \in J} \{a(x, z)^{-1}[b(x, z)(D_p^+ v)(x-0) + c(x, z)]\} = 0$$
$$(x \in [a, b]), \qquad v(a) = \eta_0, \qquad (D_p^+ v)(a) = \eta_1, \tag{6.1}$$

in which $D_m D_p^+$ is Feller's generalized second order differential operator with a nondecreasing right continuous function m and a (strongly) isotone continuous function p (see for example [11], [50, pp. 21–2], [15]), D_p^+ stands for the right derivation with respect to the function p, J is a compact set in $\mathbf{R}$, and $a > 0$, b, c are continuous functions on $[a, b] \times J$.

In [50, Lemma 3, p. 161] it was shown that the function

$$\Psi(x, y) = -\min_{z \in J} \{a(x, z)^{-1}[b(x, z)y + c(x, z)]\} \quad (x \in [a, b],\ y \in \mathbf{R})$$

is continuous and satisfies the Lipschitz condition

$$|\Psi(x, y_1) - \Psi(x, y_2)| \leq L|y_1 - y_2| \quad (x \in [a, b];\ y_1, y_2 \in \mathbf{R})$$

where

$$L = \max\{|b(x, z)|/a(x, z);\ x \in [a, b],\ z \in J\}.$$

Integrating (6.1) we obtain the equation

$$(D_p^+ v)(x) = \eta_1 + \int_a^x \Psi(s, (D_p^+ v)(s-0))\, dm(s) \quad (x \in [a, b]).$$

According to Section 4 this equation has a unique solution z. Setting

$$v(x) = \eta_0 + \int_a^x z(s)\, dp(s) \quad (x \in [a, b])$$

we have solved (in principle) the Bellman equation (6.1).

7. A glance at the higher-dimensional case

In the original paper of T. H. Hildebrandt [27] all assertions about linear equations are formulated in the finite-dimensional case. Thus the homogeneous equation

$$Y(x) = Y_0 + \int_a^x dM(s)Y(s) \quad (x \in [a, b]),$$

in which $x \to M(x)$ is a matrix-valued function of bounded variation on $[a, b]$, has a unique solution if and only if the matrices $I - \Delta^- M(x)$ have reciprocals for all points of discontinuity of M.

Let us give the precise definitions. In the n-dimensional real vector space $\mathbf{R}^n$ we introduce for each vector $Y = (y_1, \ldots, y_n)'$ the norm $|Y| = \max_i |y_i|$; the norm of a vector function $x \to Y(x)$ ($x \in [a, b]$) is given by

$$\|Y\| = \sup\{|Y(x)|;\ x \in [a, b]\}.$$

The norm of the $n \times n$ matrix $M = (m_{ij})$ is defined by $|M| = \max_i \sum_j |m_{ij}|$. The matrix function $x \to M(x)$ ($x \in [a, b]$) is called of bounded variation if each of its components $m_{ij}(x)$ are of bounded variation; we set

$$V_c^d M = \sup_\sigma \sum_k |M(x_k) - M(x_{k-1})|,$$

where the least upper bound is taken with respect to all subdivisions

$$\sigma \equiv \{c = x_0 < x_1 < x_2 < \cdots < x_l = d\}$$

of the interval $[c, d] \subseteq [a, b]$. It follows immediately that $V_c^d m_{ij} \leq V_c^d M$, and the isotone function $x \to v_M(x) = V_a^x M$ is discontinuous if and only if some m_{ij} is discontinuous. Also we have for all $x \in [a, b]$ the relation

$$\Delta^- v_M(x) = \max_i \sum_j |\Delta^- m_{ij}(x)|. \tag{7.1}$$

Finally, we must introduce a counterpart to the space $\mathbf{C}_m[a, b]$ in the case of one dimension. Clearly, we are interested in the set of all bounded vector-valued functions $G(x) = (g_1(x), \ldots, g_n(x))'$ on $[a, b]$ which satisfy the conditions

$$\sum_j |\Delta^- m_{ij}(x)| = 0 \Rightarrow \Delta^- g_i(x) = 0,$$

$$\sum_j |\Delta^+ m_{ij}(x)| = 0 \Rightarrow \Delta^+ g_i(x) = 0$$

for all $x \in [a, b]$ and $i = 1, \ldots, n$. To define such a space we introduce on $[a, b]$ the metrics $\rho_{M(i)}$ $(i = 1, \ldots, n)$ by setting

$$\rho_{M(i)}(x, y) = |x - y| + \sum_j |m_{ij}(x) - m_{ij}(y)| \quad (x, y \in [a, b]).$$

Let us denote by $\mathbf{C}_M[a, b]$ the space of all bounded vector-valued functions $G = (g_1, \ldots, g_n)'$ such that $g_i \in \mathbf{C}_{M(i)}[a, b]$ $(i = 1, \ldots, n)$, where $\mathbf{C}_{M(i)}[a, b]$ $(i = 1, \ldots, n)$ is the space of all real valued bounded functions which are continuous on the metric space $([a, b], \rho_{M(i)})$.

Now we are able to formulate analogous assertions about the nonlinear systems

$$Y(x) = Y_0 + \int_a^x dM(s)F(s, Y(s)) \quad (x \in [a, b]) \tag{\textbf{III}}$$

$$W(x) = W_0 + \int_a^x dM(s)F(s, W(s-0)) \quad (x \in [a, b]) \tag{\textbf{III}$'$}$$

in the n-dimensional case. In this case F denotes a vector-valued continuous function on $[a, b] \times \mathbf{R}^n$ which satisfies the uniform Lipschitz condition

$$|F(x, U) - F(x, V)| \leq L|U - V| \quad (x \in [a, b];\ U, V \in \mathbf{R}^n).$$

Assume the existence of a positive constant α *such that*

$$\left(\max_i \sum_j |\Delta^- m_{ij}(x)|\right)^{-1} > \alpha > L. \tag{7.2}$$

Then, for every vector Y_0 *equation* **(III)** *has a unique solution* Y, *which depends continuously on the initial condition* $Y(a) = Y_0$.

We will prove only that $\mathfrak{T}$, defined by

$$(\mathfrak{T}Z)(x) = Y_0 + \int_a^x dM(s)F(s, Z(s)) \quad (x \in [a, b])$$

for all $Z \in \mathbf{C}_M[a, b]$, $x \in [a, b]$, is a contractive operator, if we choose an appropriate weighted norm. For $\alpha > 0$ let h_α be the solution of the equation

$$h_\alpha(x) = 1 + \alpha \int_a^x h_\alpha(s)\, dv_M(s) \quad (x \in [a, b]);$$

by (7.2) and (7.1) it is strictly positive. Then we define in $\mathbf{C}_M[a, b]$ the norm $\|\cdot\|_\alpha$ by

$$\|Z\|_\alpha = \sup\{h_\alpha(x)^{-1}|Z(x)|;\, x \in [a, b]\}.$$

The validity of the Lipschitz condition implies that for every $Y, Z \in \mathbf{C}_M[a, b]$ and $x \in [a, b]$ the estimation

$$\begin{aligned}
|(\mathfrak{T}Y)(x) - (\mathfrak{T}Z)(x)| &= \left|\int_a^x dM(s)[F(s, Y(s)) - F(s, Z(s))]\right| \\
&\leq \int_a^x |F(s, Z(s)) - F(s, Z(s))|\, dv_M(s) \\
&\leq L \int_a^x |Y(s) - Z(s)|\, dv_M(s) \\
&\leq L\|Y - Z\|_\alpha \int_a^x h_\alpha(s)\, dv_M(s) \\
&\leq \alpha^{-1}L\|Y - Z\|_\alpha h_\alpha(x).
\end{aligned}$$

Consequently,

$$h_\alpha(x)^{-1}|(\mathfrak{T}Y)(x) - (\mathfrak{T}Z)(x)| \leq \alpha^{-1}L\|Y - Z\|_\alpha \quad (x \in [a, b]),$$

and

$$\|\mathfrak{T}Y - \mathfrak{T}Z\|_\alpha \leq \alpha^{-1}L\|Y - Z\|_\alpha.$$

Because of (7.2), we have $\alpha^{-1}L < 1$. Also the other parts of the proof as well as the proof of the continuous dependence on the initial value are completely analogous to the one-dimensional problems of Sections 3 and 4. The same is true for the last assertion:

For any matrix valued function M of bounded variation, the system **(III′)** *has a unique solution W depending continuously on the initial vector W_0.*

Appendix

Each function g of bounded variation can be split into a continuous part g_c and a pure jump function g_b, the function of the breaks. This decomposition is unique up to an additive constant; we set

$$g_b(a) = 0,$$
$$g_b(x) = \Delta^+ g(a) + \sum_{a<\tau<x} \Delta^{\pm} g(\tau) + \Delta^- g(x) \quad (x \in [a, b]),$$
$$g_c(x) = g(x) - g_b(x) \quad (x \in [a, b]).$$

The verification of the continuity of g_c can be found for instance in [54].

For a function g of bounded variation on $[a, b]$ and a measurable, bounded function f there exists always the Lebesgue-Stieltjes integral $\int_c^d f(x) g_c(dx)$. Here and for the rest of the appendix let $a \le c < d \le b$. Now the (Lebesgue type) Young integral of the function f with respect to g is defined by the relation

$$\int_c^d f(x)\, dg(x) = \int_c^d f(x) g_c(dx) + f(c)\Delta^+ g(c) + \sum_{c<\tau<d} f(\tau)\Delta^{\pm} g(\tau) + f(d)\Delta^- g(d).$$

Additionally, we set $\int_c^c f(x)\, dg(x) = 0$. Clearly, we have the usual estimation

$$\left| \int_c^d f(x)\, dg(x) \right| \le \int_c^d |f(x)|\, d|g|(x).$$

Regarding the Young integral as a function of the upper limit

$$h(x) = \int_a^x f(s)\, dg(s) \quad (x \in [a, b])$$

we obtain afresh a function of bounded variation which has the properties

$$\Delta^- h(x) = f(x)\Delta^- g(x), \quad \Delta^+ h(x) = f(x)\Delta^+ g(x), \quad \Delta^{\pm} h(x) = f(x)\Delta^{\pm} g(x)$$

for all $x \in [a, b]$ (remember $g(a - 0) = g(a)$ and $g(b + 0) = g(b)$).

The connection of the Young integral to the Lebesgue-Stieltjes integral can be expressed by the formulas

$$(\mathrm{LS}) \int_{[c,d]} f(x) g(dx) = \int_{c-0}^{d+0} f(x)\, dg(x),$$

$$\int_c^d f(x)\, dg(x) = (\mathrm{LS}) \int_{(c,d)} f(x) g(dx) + f(c)\Delta^+ g(c) + f(d)\Delta^- g(d).$$

For more information about relationships of different types of Stieltjes integrals see [63], [60].

Henceforth let be all functions of bounded variation on $[a, b]$. Then our Young integral is equal to the Riemann type Young integral in [27]. If we set $h(x) = \int_a^x f\, dg$ we have the substitution theorem [28, p. 91] $\int_c^d k\, dh = \int_c^d kf\, dg$.

If we regard a pointwise convergent sequence $f_n(x) \to f(x)$ of uniformly bounded functions f_n on $[a, b]$ and another sequence of functions g_m which converge in variation to g then [27, p. 355]

$$\lim_{n,m} \int_c^d f_n \, dg_m = \int_c^d f \, dg.$$

If the function $(x, y) \to h(x, y)$ $(a \leq x, y \leq b)$ is bounded and moreover, is of bounded variation on in each variable $[a, b]$, then we have the Dirichlet formula [10], [27, p. 355]

$$\begin{aligned}
\int_c^d \int_c^x h(x, y) \, dg(y) \, df(x) \\
&= \int_c^d \int_y^d h(x, y) \, df(x) \, dg(y) - \Delta^+ f(c) h(c, c) \Delta^+ g(c) \\
&\quad - \sum_{c < \tau < d} [\Delta^+ f(\tau) h(\tau, \tau) \Delta^+ g(\tau) - \Delta^- f(\tau) h(\tau, \tau) \Delta^- g(\tau)] \\
&\quad + \Delta^- f(d) h(d, d) \Delta^- g(d).
\end{aligned}$$

Setting $h \equiv 1$, we then obtain the following integration by parts theorem:

$$\begin{aligned}
\int_c^d f \, dg + \int_c^d g \, df = f(d)g(d) - f(c)g(c) - \Delta^+ f(c) \Delta^+ g(c) \\
- \sum_{c < \tau < d} [\Delta^+ f(\tau) \Delta^+ g(\tau) - \Delta^- f(\tau) \Delta^- g(\tau)] + \Delta^- f(d) \Delta^- g(d).
\end{aligned}$$

If both functions f and g are right continuous then we obtain a very simple version of the integration by parts theorem [29, p. 132]:

$$\int_c^d f(x) \, dg(x) + \int_c^d g(x - 0) \, df(x) = f(d)g(d) - f(c)g(c).$$

For other types of integrals compare also with [60], [41], [40], [32], [47], [72].

References

1. C. W. Bitzer, *Stieltjes-Volterra equations*, Illinois J. Math., vol. 14 (1970), pp. 434–451.
2. J. A. Chatfield, *Equivalence of integrals*, Proc. Amer. Math. Soc., vol. 38 (1973), pp. 279–285.
3. J. Chandra and B. A. Fleishman, *On a generalization of the Gronwall-Bellman lemma in partially ordered Banach spaces*, J. Math. Anal. Appl., vol. 31 (1970), pp. 668–681.
4. R. H. Cox, *Integral equations on a Hilbert space*, Illinois J. Math., vol. 9 (1965), pp. 256–260.
5. P. C. Das and R. R. Sharma, *On optimal controls for measure delay-differential equations*, SIAM J. Control, vol. 9 (1971), pp. 43–61.
6. ———, *Existence and stability of measure differential equations*, Czechoslovak Math. J., vol. 22 (97), (1972), pp. 145–158.
7. W. P. Davis and J. A. Chatfield, *Concerning product integrals and exponentials*, Proc. Amer. Math. Soc., vol. 25 (1970), pp. 743–747.
8. W. F. Denny, *A linear Riemann-Stieltjes integral equation system*, J. Differential Equations, vol. 22 (1976), pp. 1–13.

9. C. Doléans-Dade, *Quelques applications de la formule de changement de variables pour les semimartingales*, Z. Wahrscheinlichkeitstheorie verw. Geb., vol. 16 (1970), pp. 181–194.
10. H. J. Ettlinger, *On the inversion of the order of integration of a two fold iterated integral*, Ann. of Math., (2), vol. 28 (1926), pp. 65–68.
11. W. Feller, *Generalized second order differential operators and their lateral conditions*, Illinois J. Math., vol. 1 (1957), pp. 459–504.
12. F. R. Gantmacher und M. G. Kreĭn, *Oszillationsmatrizen, Oszillationskerne und kleine Schwingungen mechanischer Systeme*, (Deutsche Übersetzung), Akademie-Verlag, Berlin, 1960.
13. W. L. Gibson, *Embedding Stieltjes-Volterra integral equations in Stieltjes integral equations*, Trans. Amer. Math. Soc., vol. 227 (1977), pp. 263–277.
14. I. I. Gihman and A. V. Skorohod, *Theory of stochastic processes*, vol. III (in Russian), Nauka, Moscow, 1975.
15. J. Groh, *Über eine Klasse eindimensionaler Markovprozesse*, Math. Nachrichten, vol. 65 (1975), pp. 125–136.
16. ———, *Optimal control of one dimensional non-conservative quasi-diffusion processes*, Trans. of Eighth Prague Conference on Information Theory etc., vol. A (1978), pp. 263–274.
17. P. Hartman, *Ordinary differential equations*, 2nd ed., Philip Hartman, Baltimore, 1973.
18. B. W. Helton, *Integral equations and product integrals*, Pacific J. Math., vol. 16 (1966), pp. 297–322.
19. ———, *A product integral representation for a Gronwall inequality*, Proc. Amer. Math. Soc., vol. 23 (1969), pp. 493–500.
20. ———, *A special integral and a Gronwall inequality*, Trans. Amer. Math. Soc., vol. 217 (1976), pp. 163–181.
21. J. C. Helton, *Solution of integral equations by product integration*, Proc. Amer. Math. Soc., vol. 49 (1975), pp. 401–406.
22. J. Helton and S. Stuckwisch, *Numerical approximation of product integrals*, J. Math. Anal. Appl., vol. 56 (1976), pp. 410–437.
23. J. V. Herod, *Solving integral equations by iteration*, Duke Math. J., vol. 34 (1967), pp. 519–534.
24. ———, *A Gronwall inequality for linear Stieltjes integrals*, Proc. Amer. Math. Soc., vol. 23 (1969), pp. 34–36.
25. ———, *A pairing of a class of evolution systems with a class of generators*, Trans. Amer. Math. Soc., vol. 157 (1971), pp. 247–260.
26. T. H. Hildebrandt, *Definitions of Stieltjes integrals of the Riemann type*, Amer. Math. Monthly, vol. 45 (1938), pp. 265–278.
27. ———, *On systems of linear differentio-Stieltjes-integral equations*, Illinois J. Math., vol. 3 (1959), pp. 352–373.
28. ———, *Introduction to the theory of integration*, Academic Press, New York, 1963.
29. K. Hinderer, *Grundbegriffe der Wahrscheinlichkeitstheorie*, 2. Aufl., Springer, New York, 1975.
30. D. B. Hinton, *A Stieltjes-Volterra integral equation theory*, Canad. J. Math., vol. 18 (1966), pp. 314–331.
31. C. S. Hönig, *Volterra Stieltjes-integral equations, Functional analytic methods; Linear constraints*, North-Holland Amsterdam; and American Elsevier, New York, 1975.
32. W. H. Ingram, *The j-differential and its integral*, Proc. Edinburgh Math. Soc. (II), vol. 12 (1960), pp. 85–93.
33. ———, *The linear j-differential equation*, Proc. Edinburgh Math. Soc. (II), vol. 14 (1965), pp. 211–219.
34. G. S. Jones, *Fundamental inequalities for discrete and discontinuous functional equations*, J. Soc. Ind. Appl. Math., vol. 12 (1964), pp. 43–57.
35. I. S. Kac and M. G. Kreĭn, *On the spectral function of the string* (in Russian), Appendix II of the Russian translation of the monograph: F. V. Atkinson, Discrete and continuous boundary problems, Mir, Moscow, 1968.
36. K. Knopp, *Theorie und Anwendung der Unendlichen Reihen*, 4. Aufl., Springer, Berlin, 1947.

37. J. R. KROLL and K. P. SMITH, *An eigenvalue problem for the Stieltjes mean sigma-integral related to Gronwall's inequality*, Proc. Amer. Math. Soc., vol. 33 (1972), pp. 384–388.
38. J. KURZWEIL, *Generalized ordinary differential equations and continuous dependence on a parameter*, Czechoslovak Math. J., vol. 7 (82) (1957), pp. 418–449.
39. ———, *Generalized ordinary differential equations*, Czechoslovak Math. J., vol. 8 (83) (1958), pp. 360–388.
40. ———, *On integration by parts*, Czechoslovak Math. J., vol. 8 (83) (1958), pp. 356–359.
41. R. E. LANE, *The integral of a function with respect to a function I, II*; Proc. Amer. Math. Soc., vol. 5 (1954), pp. 59–66; vol. 6 (1955), pp. 392–401.
42. H. LANGER, *Zur Spektraltheorie verallgemeinerter gewöhnlicher Differentialoperatoren zweiter Ordnung mit einer nichtmonotonen Gewichtsfunktion*, University of Jyväskylä, Dept. of Math., Report 14, 1972.
43. D. L. LOVELADY, *Multiplicative integration of infinite products*, Canad. J. Math., vol. 23 (1971), pp. 692–698.
44. J. S. MACNERNEY, *Stieltjes integrals in linear spaces*, Ann. of Math. (2), vol. 61 (1955), pp. 354–367.
45. ———, *Continuous products in linear spaces*, J. Elisha Mitchell Sci. Soc., vol. 71 (1955), pp. 185–200.
46. ———, *Integral equations and semigroups*, Illinois J. Math., vol. 7 (1963), pp. 148–173.
47. ———, *An integration-by-parts formula*, Bull. Amer. Math. Soc., vol. 69 (1963), pp. 803–805.
48. ———, *A nonlinear integral operation*, Illinois J. Math., vol. 8 (1964), pp. 621–638.
49. JA. D. MAMEDOV and S. A. AŠIROV, *Nonlinear Volterra equations* (in Russian), Ylym, Ašhabad, 1977.
50. P. MANDL, *Analytical treatment of* one-*dimensional Markov processes*, Springer, New York; and Academia, Prague, 1968.
51. R. K. MILLER, *Nonlinear Volterra integral equations*, W. A. Benjamin, Menlo Park, 1971.
52. D. MORGENSTERN, *Beiträge zur nichtlinearen Funktionalanalysis*, Dissertation, Technische Universität Berlin, 1952.
53. P. H. MÜLLER, *Höhere Analysis I*, Akademie-Verlag, Berlin, 1972.
54. I. P. NATANSON, *Theorie der Funktionen einer reellen Veränderlichen*, (Deutsche Übersetzung), 2. Aufl., Akademie-Verlag, Berlin, 1961.
55. J. W. NEUBERGER, *Continuous products and nonlinear integral equations*, Pacific J. Math., vol. 8 (1958), pp. 529–549.
56. A. PRINGSHEIM, *Ueber die Convergenz unendlicher Producte*, Math. Annalen, vol. 33 (1889), pp. 119–154.
57. W. T. REID, *Generalized linear differential systems*, J. Math. Mech., vol. 8 (1958), pp. 705–726.
58. J. A. RENEKE, *A product integral solution of a Stieltjes-Volterra integral equation*, Proc. Amer. Math. Soc., vol. 24 (1970), pp. 621–626.
59. R. W. RISHEL, *An extended Pontryagin principle for control systems whose control laws contain measures*, J. SIAM Control, Ser. A, vol. 3 (1965), pp. 191–205.
60. H. SCHÄRF, *Ueber links- und rechtsseitige Stieltjesintegrale und deren Anwendungen*, Portugaliae Mathematica, vol. 4 (1943), pp. 73–118.
61. W. W. SCHMAEDEKE and G. R. SELL, *The Gronwall inequality for modified Stieltjes integrals*, Proc. Amer. Math. Soc., vol. 19 (1968), pp. 1217–1222.
62. Š. SCHWABIK, *Stetige Abhängigkeit von einem Parameter und invariante Mannigfaltigkeiten für verallgemeinerte Differentialgleichungen*, Czechoslovak Math. J., vol. 19 (94) (1969), pp. 398–427.
63. ———, *On the relation between Young's and Kurzweil's concept of Stieltjes integral*, Časopis Pěst. Mat., vol. 98 (1973), pp. 237–251.
64. ———, *Note on Volterra-Stieltjes integral equations*, Časopis Pěst. Mat., vol. 102 (1977), pp. 275–279.
65. M. TVRDÝ and O. VEJVODA, *General boundary value problem for an integrodifferential system and its adjoint*, Časopis Pěst. Mat., vol. 98 (1973), pp. 26–42.

66. M. TVRDÝ, *Boundary value problems for generalized linear integrodifferential equations with left-continuous solutions*, Časopis Pěst. Mat., vol. 99 (1974), pp. 147–157.
67. O. VEJVODA and M. TVRDÝ, *Existence of solutions to a linear integro-boundary-differential equation with additional conditions*, Ann. di Mat. pura ed Appl. (4), vol. 89 (1971), pp. 169–216.
68. H. S. WALL, *Concerning continuous continued fractions and certain systems of Stieltjes integral equations*, Rend. Circ. Mat. Palermo (2), vol. 2 (1953), pp. 73–84.
69. ———, *Concerning harmonic matrices*, Arch. Math., vol. 5 (1954), pp. 160–167.
70. W. WALTER, *Gewöhnliche Differentialgleichungen*, 2. Aufl., Springer, New York, 1976.
71. D. WILLETT and J. S. W. WONG, *On the discrete analogues of some generalizations of Gronwall's inequality*, Monatsh. Math., vol. 69 (1965), pp. 362–367.
72. F. M. WRIGHT and J. D. BAKER, *On integration-by-parts for weighted integrals*, Proc. Amer. Math. Soc., vol. 22 (1969), pp. 42–52.
73. F. M. WRIGHT, M. L. KLASI and D. R. KENNEBECK, *The Gronwall inequality for weighted integrals*, Proc. Amer. Math. Soc., vol. 30 (1971), pp. 504–510.
74. W. H. YOUNG, *On integration with respect to a function of bounded variation*, Proc. London Math. Soc. (2), vol. 13 (1914), pp. 109–150.

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