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GROUPS WITH SOLVABLE CONJUGACY PROBLEMS

BY

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1. Main theorems

Let A and B be groups with $h \in A$ and $k \in B$. Suppose $gp(h)$, the *cyclic subgroup generated* by h, is isomorphic to $gp(k)$. We let $G = (A * B;\ h = k)$ denote the free product of A and B with $gp(h)$ and $gp(k)$ amalgamated by identifying h with k. Clearly B must have the following two properties if G is to have a solvable conjugacy problem:

(C_1) The conjugacy problem in B is solvable.
(C_2) The membership problem in B with respect to the amalgamated subgroup $gp(k)$ is solvable, i.e. for any $b \in B$ one can decide if $b \in gp(k)$.

The author proved the following in [5].

THEOREM 1. *Suppose A and B are free groups. Then $G = (A * B;\ h = k)$ has solvable conjugacy problem.*

This result was generalized by Comerford and Truffault in [2] as follows.

THEOREM 2. *Suppose A and B are sixth-groups and $h \in A$ and $k \in B$ have the same order. Then $G = (A * B;\ h = k)$ has solvable conjugacy problem.*

The main observation of this paper (stated below) and an analysis of the proofs of Theorems 1 and 2 show that the conditions on one of the factors, say B, are not necessary if h is a nonpower. Such generalizations are stated below. We say that x is a *nonpower* if there does not exist a y such that $x = y^n$ with $n > 1$, x is *nonselfconjugate* if $x^r \sim x^s$ implies $r = s$, and x is *seminonselfconjugate* if $x^r \sim x^s$ implies $|r| = |s|$. (Here $\sim$ is the conjugacy relation.) The definition of a sixth-group and Solitar's Theorem [8, p. 212] for the case $G = (A * B;\ h = k)$ appear in [2]. Any other terms or definitions appear in [8]. Lastly we note that if x has infinite order then (1) x is nonselfconjugate when x is in a free group, and (2) x is seminonselfconjugate when x is in a sixth-group, (cf. [7] and [1]).

THEOREM 3. *$G = (A * B;\ h = k)$ has solvable conjugacy problem if*

(a) *A is free and h is a nonpower and*
(b) *B satisfies $[C_1]$ and $[C_2]$ and k is nonselfconjugate.*

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THEOREM 4. *$G = (A * B;\ h = k)$ has solvable conjugacy problem if*

(a) *A is a sixth-group and h is a nonpower and nonselfconjugate and*
(b) *B satisfies (C_1) and (C_2) and k is nonselfconjugate.*

THEOREM 5. *$G = (A * B;\ h = k)$ has solvable conjugacy problem if*

(a) *A is a sixth-group and h is a nonpower and*
(b) *B satisfies (C_1) and (C_2), k is seminonselfconjugate, and for any $b \in B$ one can decide if there exists an n such that $b \sim k^n$.*

Proof of Theorems 3, 4 and 5. Let u and v be elements of G. We must show how to decide if $u \sim v$ in G. We can assume without loss in generality that u and v are cyclically reduced and have free product length n. As usual, the proof reduces to the cases $n > 1$ and $n = 1$.

Suppose $n > 1$. As noted in [5] and [2], $u \sim v$ in G *iff there exists an m* such that

$$h^m u_1 u_2 \cdots u_n h^{-m} = v_1 v_2 \cdots v_n \tag{1}$$

where $u_1 u_2 \cdots u_n$ and $v_1 v_2 \cdots v_n$ are normal forms of cyclic conjugates of u and v, respectively. The main observation of this paper follows.

Remark. We can assume without loss in generality that u_1 belongs to A. Otherwise u_n belongs to A and then we decide if $u^{-1} \sim v^{-1}$ in G.

In a free group or in a sixth-group (see Greendlinger [4]) h and u_1 commute if and only if h and u_1 are powers of the same element. But h is a nonpower and u_1 does not belong to $gp(h)$. Hence h and u_1 do not commute. Thus (1) holds if and only if

$$h^m u_1 h^r = v_1 \tag{2}$$

holds in A. The author showed in [5] that we can decide if (2) holds when A is free, and Comerford and Truffault showed in [2] that we can decide if (2) holds when A is a sixth-group. Thus we can decide if $u \sim v$ in G when $n > 1$.

Suppose $n = 1$. The proof of Theorems 3 and 4 is identical to the proof of Theorem 1. That is, suppose u and v belong to the same factor. Since h and k are nonselfconjugate, $u \sim v$ in G if and only if u and v are conjugate in the factor. On the other hand, suppose u and v are in different factors, say $u \in A$ and $v \in B$. Then $u \sim v$ in G if and only if $u \sim h^m$ in A and $v \sim k^m$ in B. However, in a free group or in a sixth-group one can decide if $u \sim h^m$ (cf. [6] and [2]), and for this m one can decide if $v \sim k^m$ in B since B has solvable conjugacy problem. Thus Theorems 3 and 4 are proved.

Theorem 5 is slightly more complicated since h and k need not be nonselfconjugate. However, h and k are both seminonselfconjugate, so there are only two possible powers of h and k that one has to consider. Otherwise, the proof is similar to the proof of Theorems 3 and 4.

2. Examples

We now give some examples of groups with solvable conjugacy problem.

(a) Garside [3] solved the conjugacy problem for the braid group B on $n + 1$ strings with generators $a_1, \ldots, a_n$ and defining relations

$$a_i a_j = a_j a_i \quad \text{when } |i - j| \geq 2,$$

$$a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1} \quad \text{for } i = 1, 2, \ldots, n - 1.$$

Let k be a braid in B. Then $k = W(a_i)$, a word in the a_i. By the *index* of the braid k, written $\operatorname{ind}(k)$, we mean the sum of the exponents of the a_i in W. Since the defining relations have index zero, $\operatorname{ind}(k)$ is independent of the particular word W. Clearly, $\operatorname{ind}(k^n) = n \cdot \operatorname{ind}(k)$ and if $k \sim k'$ then $\operatorname{ind}(k) = \operatorname{ind}(k')$.

Let A be any sixth-group and let $h \in A$ be any nonpower; and let k be any braid in B such that $\operatorname{ind}(k) \neq 0$. Clearly, the membership problem in B with respect to $gp(k)$ is solvable, k is nonselfconjugate, and for any braid b in B one can decide if there exists an n such that $b \sim k^n$. By Theorem 5, $G = (A * B; h = k)$ has solvable conjugacy problem.

(b) There is the natural generalization of (a). That is, let B be a group with solvable conjugacy problem whose defining relations have index zero, e.g. the groups discussed in Garside's paper [3]. Let k in B be an element with $\operatorname{ind}(k) \neq 0$. Then $G = (A * B; h = k)$ has solvable conjugacy problem where A is a sixth-group and $h \in A$ is a nonpower.

(c) First we need a lemma.

LEMMA. *Let A and B be groups with solvable membership problem with respect to any cyclic subgroup. Then $G = (A * B; h = k)$ has solvable membership problem with respect to any cyclic subgroup.*

Proof. Note first that G has solvable word problem. Given u and w in G we want to decide if u is a power of w. By choosing an appropriate inner automorphism, we can assume that w is cyclically reduced with free product length n. If $n = 1$, then u must lie in the same factor as w and the membership problem is solvable in the factor. If $n > 1$, then the length of w^k increases as $|k|$ increases. Consequently, a length argument can be used to determine if $u = w^k$ for some k. Thus the Lemma is proved.

Now let G be a finite tree product of sixth-groups where all amalgamated subgroups are cyclic and generated by nonpowers. A simple induction argument, Theorem 5, and the above Lemma show that G has solvable conjugacy problem.

(d) Let T be a tree product of groups with solvable conjugacy problem, e.g. the groups discussed in (c). Let w be an element of T which is not conjugate to an element in a vertex of T. Then the length of w^k increases as $|k|$ increases. In particular, w is seminonselfconjugate, the membership problem in T with respect to $gp(k)$ is solvable, and for any $u \in T$ one can decide if there exists an n such that $u = w^n$. Suppose A is a sixth-group and $h \in A$ is a nonpower. By Theorem 5, $G = (A * T; h = w)$ has solvable conjugacy problem.

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