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HOLOMORPHIC RETRACTS IN COMPLEX n-SPACE

BY

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1. Introduction

Let B_n denote the unit ball in C^n with sup norm (i.e., B_n is the unit polydisk in C^n centered at the origin) and let $\bar{B}^n$ be its closure. A holomorphic retract of B_n or $(\bar{B}_n)$ is a subset D of B_n (or $\bar{B}_n$) such that the identity function on D can be extended to a holomorphic function $F\colon B_n \to D$ (or a continuous function F: $\bar{B}_n \to D$ that is holomorphic on B_n). Thus $F(B_n) = D(F(\bar{B}_n) = D)$ and $F \circ F = F$. The function F is called a holomorphic retraction on B_n (or $\bar{B}_n$). In [3] Rudin gives the example $(z, w) \to (z, h(z))$ (where h is an arbitrary holomorphic function from the unit disk into itself) as an example of a retraction on B_2 and he points out that the retracts of B_n are unknown, $n \geq 3$. In this paper, we show that the retracts of B_n are all essentially of the type given in Rudin's example. In particular, let J be a subset of $\{1, 2, \ldots, n\}$ of cardinality $p \leq n$ and let $M = \{z \in B_n\colon z_j = 0 \text{ if } j \notin J\} \approx B_p$. For $1 \leq j \leq n$, let F_j be holomorphic on M and bounded by 1 with $F_j(z) = z_j$ whenever $j \in J$. If

$$D = \{(F_1(z), F_2(z), \ldots, F_n(z))\colon z \in M\} \tag{1}$$

then clearly D is a retract of B_n. What is not so clear is that conversely, if D is a retract of B_n then D has the form (1).

Notice that, as in the case of the Euclidean ball in C^n [3], even though the retracts of B_n are rather simple—aside from a permutation they are the graphs of holomorphic functions from B_p to B_{n-p}—the retractions may be quite complicated. For example, if $0 < t < 1$ and $F(z, w)$ is an arbitrary holomorphic function (complex valued) on B_2 such that $|F(z, w)| \leq t(1 - t)/2$ when $(z, w) \in B_2$ the function

$$[(1 - t)z + te^{i\alpha}w + (z - e^{i\alpha}w)^2 F(z, w)](1, e^{-i\alpha}) \tag{2}$$

is a holomorphic retraction of B_2 onto $\{z(1, e^{-i\alpha})\colon |z| < 1\}$.

The holomorphic retracts of $\bar{B}_n$ are also given by (1) except of course the functions F_j are continuous on $\bar{B}_n$ and the possibility exists that some of the F_j may be constants of modulus 1. A non-trivial example of a holomorphic retract on $\bar{B}_2$ is the map

$$(z, w) \to (1, zw). \tag{3}$$

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In [4] Shields proves that if f and g are continuous functions on the closed disk of the complex plane and are holomorphic on the open disk and map the closed disk into itself and if $f \circ g = g \circ f$, then f and g have a common fixed point. This result was extended to polydisks in C^2 by Eustice [2] and to the unit ball of a finite dimensional complex inner product space by Suffridge [5]. In each case, the method of proof was to consider the closure of the iterates $f^1 = f$, $f^2 = f \circ f, \ldots, f^n = f \circ f^{n-1}$ of f (denoted by $\Gamma(f)$) and to conclude that this compact topological semigroup ($\Gamma(f)$) contains a unique idempotent. The problem thus reduces to a study of idempotents in the semigroup of maps that are holomorphic on the open unit ball, continuous on the closed ball and map the closed ball into itself. The operation in this semigroup is, of course, function composition. It is clear that the idempotents in this semigroup are retractions of the closed unit ball and conversely.

In this paper, we find all holomorphic retracts of B_n and $\bar{B}_n$ (as remarked above). We also find all holomorphic retractions on B_n and $\bar{B}_n$, i.e., we find all holomorphic idempotents on B_n and $\bar{B}_n$ as described above, and we extend the result of Shields concerning common fixed points of commuting holomorphic maps to C^n.

2. Linear retractions on B_n

Let $\phi: B_n \to B_n$ be a biholomorphic map (i.e., ϕ is holomorphic and one-to-one, maps B_n onto B_n and has a holomorphic inverse). As observed by Eustice [2], $F: B_n \to B_n$ is a retraction if and only if $\phi^{-1} \circ F \circ \phi$ is a retraction. Given $a \in B_n$, it is well known and easy to see that the function ϕ_a defined by

$$\phi_a(z) = (w_1, w_2, \ldots, w_n), \qquad w_j = (z_j + a_j)/(1 + \bar{a}_j z_j) \tag{4}$$

is a biholomorphic map of B_n onto B_n and $\phi_a^{-1} = \phi_{-a}$. If F is a retraction and $0 \in F(B_n)$, then $F(0) = 0$ (because F is the identity on $F(B_n)$). Otherwise, assume $a \in F(B_n)$ and replace F by $G = \phi_a^{-1} \circ F \circ \phi_a$ so that $G(0) = 0$. Thus, in order to determine the retractions of B_n, it is sufficient to determine all those retractions F such that $F(0) = 0$. Note also that if F and G are as above, then $F(B_n)$ is of the form (1) if and only if $G(B_n)$ is of the form (1).

Assuming F is such a retraction of B_n i.e., $F(0) = 0$) we may expand F in a power series $F(z) = L(z) + (1/2)D^2F(0)(z, z) + \cdots$ where $L = DF(0): C^n \to C^n$ is linear. Clearly, $F \circ F = F \Rightarrow L \circ L = L$. Further, applying Schwarz's lemma we see that $\|F(z)\| \leq \|z\|$ (the norm is sup norm) so for $z \in B_n$, $0 < \|z\|$ and λ complex, $0 < \lambda < 1$, we have

$$\|F(\lambda(z/\|z\|)) \leq |\lambda| \quad \text{and} \quad \|(1/\lambda)F(\lambda(z/\|z\|))\| \leq 1.$$

Letting $\lambda \to 0$ we conclude $\|L(z)\| \leq \|z\|$ so L (restricted to B_n) is a retraction of B_n. As we will see, the nature of F is determined by L so we first determine all linear retractions of B_n.

Consider a linear map constructed as follows. Let $\{J_1, J_2, \ldots, J_{l+1}\}$ be a partition of $\{1, 2, \ldots, n\}$. Let $M_k = \{z \in C^n: z_j = 0 \text{ if } j \notin J_k\}$, $k = 1, 2, \ldots, l + 1$

and write $C^n = M_1 + M_2 + \cdots + M_{l+1}$. For each k, $1 \leq k \leq l$ we wish to choose a vector γ_k from M_k that satisfies

$$\gamma_k = (\gamma_{1k}, \gamma_{2k}, \ldots, \gamma_{nk}), \qquad |\gamma_{jk}| = \begin{cases} 0 & \text{if } j \notin J_k \\ 1 & \text{if } j \in J_k \end{cases}$$

(we are ignoring M_{l+1} for the moment). Also, for each k, $1 \leq k \leq l$ we wish to define a linear functional on C^n that is 0 on M_j if $j \neq k$. Let $\{a_{jk}: j \in J_k\}$ be a finite sequence of positive numbers such that $\sum_{j \in J_k} a_{jk} = 1$ and let

$$T_k(z) = \sum_{j \in J_k} \bar{\gamma}_{jk} a_{jk} z_j \tag{5}$$

where $\bar{\gamma}_{jk}$ denotes the complex conjugate of γ_{jk}. Note that

$$|T_k(z)| \leq \textstyle\sum_{j \in J_k} |\bar{\gamma}_{jk}| a_{jk} |z_j| \leq \sum_{j \in J_k} a_{jk} \|z\| = \|z\|.$$

Also, if we replace z_j on the right hand side of (5) by $\gamma_{jk} T_k(z)$ we obtain

$$\sum_{j \in J_k} \bar{\gamma}_{jk} a_{jk} (\gamma_{jk} T_k(z)) = \sum_{j \in J_k} a_{jk} T_k(z) = T_k(z).$$

Thus we have shown that the linear map $T(z) = \sum_{k=1}^{l} T_k(z)\gamma_k$ is a retraction of B_n. Further, if we now change the coordinate γ_{jk}, $j \in J_{l+1}$, $1 \leq k \leq l$ so that $\sum_{k=1}^{l} |\gamma_{jk}| \leq 1$, $j \in J_{l+1}$, then T is still a retraction of B_n. We will show that every linear retraction is of this form.

THEOREM 1. *If L (restricted to B_n) is a linear retraction of B_n then there exists*

(i) *a partition $\{J_1, J_2, \ldots, J_{l+1}\}$ of $\{1, 2, \ldots, n\}$ for some l (we allow $J_{l+1} = \emptyset$ but $J_k \neq \emptyset$ if $1 \leq k \leq l$),*

(ii) *vectors $\gamma_k = (\gamma_{1k}, \gamma_{2k}, \ldots, \gamma_{nk})$, $1 \leq k \leq l$ such that*

$$|\gamma_{jk}| = \begin{cases} 0 & \text{if } j \notin J_k \cup J_{l+1} \\ 1 & \text{if } j \in J_k \end{cases} \quad \text{and} \quad \sum_{k=1}^{l} |\gamma_{jk}| \leq 1 \quad \text{if } j \in J_{l+1}$$

and

(iii) *linear functionals T_k, $1 \leq k \leq l$, given by $T_k(z) = \sum_{j \in J_k} \bar{\gamma}_{jk} a_{jk} z_j$ where $a_{jk} > 0$ if $j \in J_k$ and $\sum_{j \in J_k} a_{jk} = 1$ such that*

$$L(z) = \sum_{k=1}^{l} T_k(z)\gamma_k. \tag{6}$$

In case $L \equiv 0$, we take $l = 0$ so that (ii) *and* (iii) *above are vacuous and the right hand side of* (6) *is taken to be* 0.

Proof. If $L \not\equiv 0$, choose $w \in L(B_n)$, $\|w\| > 0$ and let $J \subset \{1, 2, \ldots, n\}$ be such that $j \in J \Rightarrow |w_j| = \|w\|$. For $1 \leq j \leq n$, let L_j denote the jth coordinate function of L. Then $L_j(z) = \sum_{k=1}^{n} \beta_{jk} z_k$ and $|L_j(z)| \leq \|z\|$. Thus, $\|L_j\| \leq 1$ and $\sum_{k=1}^{n} |\beta_{jk}| \leq 1$. However, $w_j = L_j(w)$ so $j \in J$ implies

$$|w_j| = |L_j(w)| \leq \sum_{k=1}^{n} |\beta_{jk}| |w_k| \leq \sum_{k=1}^{n} |\beta_{jk}| \|w\| \leq \|w\| = |w_j|. \tag{7}$$

Thus equality must hold throughout (7) and we conclude $\sum_{k=1}^{n} |\beta_{jk}| = 1$, $\beta_{jk} = 0$ if $k \notin J$ and

$$(8) \qquad \beta_{jk} w_k = |\beta_{jk}| w_j, \quad j, k \in J.$$

Claim. For some $j \in J$, $\beta_{jj} > 0$.

Proof of claim. Fix $j \in J$ so that some $\beta_{pj} \neq 0$ and choose $p \in J$ to maximize $|\beta_{pj}|$. Since

$$\sum_{k \in J} \beta_{pk} z_k = L_p(z) = L_p(L(z)) = \sum_{k \in J} \beta_{pk} L_k(z) = \sum_{k \in J} \sum_{q \in J} \beta_{pk} \beta_{kq} z_q,$$

we conclude

$$(9) \qquad |\beta_{pj}| = \left| \sum_{k \in J} \beta_{pk} \beta_{kj} \right| \leq \sum_{k \in J} |\beta_{pk}| \, |\beta_{kj}| \leq \sum_{k \in J} |\beta_{pk}| \, |\beta_{pj}| = |\beta_{pj}|.$$

Thus equality must hold at each step of (9) so in particular, $|\beta_{pk}| \neq 0 \Rightarrow |\beta_{kj}| = |\beta_{pj}|$. Setting $k = j$ yields $|\beta_{jj}| = |\beta_{pj}|$ and using (8) yields $\beta_{jj} > 0$.

With $q \in J$ fixed so that $\beta_{qq} > 0$, let $J_1 \subset J$ be such that $\beta_{qj} \neq 0$ if and only if $j \in J_1$. For any $k \in J_1$, the proof of the claim shows that $\beta_{kk} \geq |\beta_{jk}|, 1 \leq j \leq n$. Further, applying the proof of the claim with $j = p = q$ yields $|\beta_{kq}| = \beta_{qq}$ when $k \in J_1$ (i.e., $|\beta_{qk}| \neq 0 \Rightarrow |\beta_{kq}| = \beta_{qq}$).

Let $M_1 = \{z \in C^n : z_j = 0 \text{ if } j \notin J_1\}$ and let $z \in M_1$ satisfy $z_j = w_j$, when $j \in J_1$. Arguing as in the derivation of (7) and (8) with w_j replaced by $L_j(z)$, $j \in J_1$ shows that for $k \in J_1$, $\beta_{kj} = 0$ if $j \notin J_1$. Thus, we have shown that $\beta_{kj} \neq 0$ if j, $k \in J_1$. Otherwise, for some $k \in J_1$, let J'_1 satisfy $\beta_{kj} \neq 0$ if and only if $j \in J'_1$. Since $\beta_{kq} \neq 0$, $q \in J'_1$ so $\beta_{qj} = 0$ if $j \neq J'_1$ by the above proof. By the choice of q we conclude $J'_1 = J_1$.

We may now apply the proof of the claim with $p = j \in J_1$ and $k \in J_1$ to conclude $|\beta_{kj}| = \beta_{jj}$ and by (8), $\beta_{jk} w_k = |\beta_{jk}| w_j = \beta_{kk} w_j$, $k, j \in J_1$.

In the notation of the statement of the theorem we want $a_{j1} = \beta_{jj}, j \in J_1$ and $T_1(z) = \sum_{j \in J_1} \bar{\gamma}_{j1} a_{j1} z_j$ where $\gamma_{j1} = w_j/w_1, j \in J_1$.

If $J_1 = \{1, 2, \ldots, n\}$ the proof is complete.

Otherwise, set $M_1^{\perp} = \{z \in C^n \colon z_j = 0 \text{ if } j \in J_1\} \approx C^q$ for some $q < n$. Clearly, $L(M_1^{\perp}) \subset M_1^{\perp}$. Let $L^{(1)}$ denote the restriction of L to $M_1^{\perp}$. Then $L^{(1)}$ is a retraction of B_q and we may continue as above to define $J_2, \ldots, J_l; \gamma_2, \ldots, \gamma_l$ and $T_2, \ldots, T_l$ as in the statement of the theorem where either $\bigcup_{k=1}^{l} J_k = \{1, 2, \ldots, n\}$ or $L^{(l)} \equiv 0$. Of course, as soon as J_j is identified, $1 \leq j \leq l$, this determines that $\gamma_{jk} = 0$ when $j \notin J_k$, $1 \leq k \leq l$.

If $\bigcup_{k=1}^{l} J_k = \{1, 2, \ldots, n\}$ the proof is complete (with $J_{l+1} = \emptyset$). Otherwise, it remains to set $J_{l+1} = \{1, 2, \ldots, n\} - \bigcup_{k=1}^{l} J_k$ and to determine L_j, $j \in J_{l+1}$.

For $1 \leq k \leq l + 1$, let $M_k = \{z \in C^n \colon z_j = 0 \text{ if } j \notin J_k\}$. Then $L(M_{l+1}) = 0$. Let $j \in J_{l+1}$ and $1 \leq k \leq l$. If we determine the value of L_j on M_k for each such j and k then L will be known. Let $\gamma'_k = (\gamma'_{1k}, \ldots, \gamma'_{nk})$ be defined by $\gamma'_{pk} = \gamma_{pk}$ if $p \notin J_{l+1}$, $\gamma'_{pk} = 0$ if $p \in J_{l+1}$ (note that γ_{pk} is not yet defined for $p \in J_{l+1}$). If

$z \in M_k$, since L is a retraction and $L(M_{l+1}) = 0$ we conclude

$$L_j(z) = L_j(L(z)) = L_j(T_k(z)\gamma'_k) = T_k(z)L_j(\gamma'_k).$$

Thus, set $\gamma_{jk} = L_j(\gamma'_k)$ and the proof is completed by observing that $\sum_{k=1}^{l} |\gamma_{jk}| \leq 1$ because $|L_j(z)| \leq \|z\|$ and hence $\|L_j\| \leq 1$.

It is interesting to observe the properties of the matrix A associated with a linear retraction L according to theorem 1. By replacing L by $\sigma \circ L \circ \sigma^{-1}$ for an appropriate permutation of coordinates σ, we may assume

$$J_1 = \{1, 2, \ldots, n_1\}, \qquad J_2 = \{n_1 + 1, \ldots, n_2\}, \ldots,$$
$$J_{l+1} = \{n_l + 1, \ldots, n\}.$$

Then A has the form

$$A = \left[\begin{array}{c|ccc|c} A_1 & & 0 & & \\ \hline & A_2 & 0 & & \\ & & \ddots & & 0 \\ 0 & 0 & & \ddots & \\ & & & A_l & \\ \hline & & B & & 0 \end{array}\right]$$

where each A_j is a square matrix with the following properties:

(i) each element on the main diagonal is positive,
(ii) each row of A_j is a multiple of the first row
(iii) the l^1 norm of each row of A_j is 1.

Further, each row of B is a linear combination of the preceding rows of A.

3. Holomorphic retractions on B_n

We now return to the problem of finding all holomorphic retractions on B_n. We will prove the following theorem.

THEOREM 2. *Suppose F: $B_n \to B_n$ is a retraction and $F(0) = 0$. If L is the linear part of F, then by Theorem 1, $L(z) = \sum_{k=1}^{l} T_k(z)\gamma_k$ where T_k and γ_k are described in Theorem 1. Using the notation of Theorem 1, there exist functions $G_1, G_2, \ldots, G_l$: $B_n \to B_1$ and functions H_j: $B_l \to B_1$, $j \in J_{l+1}$, such that the coordinates F_j of F satisfy $F_j(z) = G_k(z)\gamma_{jk}$ if $j \in J_k$, $1 \leq k \leq l$ and*

$$F_j(z) = H_j(G_1(z), G_2(z), \ldots, G_l(z)) \quad \text{if } j \in J_{l+1}.$$

The functions H_j, $j \in J_{l+1}$ are arbitrary (except that the range is in B_1). The functions G_k, $1 \leq k \leq l$, are arbitrary except for the following:

(i) *the linear part of* G_k *is* T_k *and*
(iii) *the non-linear part of* G_k *satisfies*

$$(G_k - T_k)(z) = \sum_{p,q \in J_k - \{j\}} (\bar{\gamma}_{jk} z_j - \bar{\gamma}_{qk} z_q)(\bar{\gamma}_{jk} z_j - \bar{\gamma}_{pk} z_p) g_{p,q}(z)$$

where j *is a fixed element of* J_k.

Thus, the nonlinear part of G_k *has a second order zero on*

$$\{z \in B_n : \text{for all } q \in J_k,\ z_q = \gamma_{qk} \bar{\gamma}_{jk} z_j\} = \left\{ z \in B_n : \left| \frac{T_k(z)}{z_j} \right| = 1 \quad \text{for all} \quad j \in J_k \right\}$$

Using Theorem 2, we can prove the following characterization of the retracts of B_n.

THEOREM 3. *Suppose* D *is a retract of* B_n. *Then there exist* $J \subset \{1, 2, \ldots, n\}$ *and functions*

$$F_j : M \to B_1,\ 1 \le j \le n, \quad \text{where} \quad M = \{z \in B_n : z_j = 0 \quad \text{if } j \notin J\},$$

such that $F_j(z) = z_j$ for $j \in J$ *and such that*

$$D = \{(F_1(z), F_2(z), \ldots, F_n(z)) : z \in M\}.$$

Proof of Theorem 3. As noted in Section 2, it is sufficient to prove the result under the assumption $F(0) = 0$ (where F is a retraction such that $F(B_n) = D$ and D is assumed to contain 0). We set $J = \{j_1, j_2, \ldots, j_l\}$ where $j_k \in J_k$ (the same notation as in Theorems 1 and 2) and the theorem clearly follows.

Proof of Theorem 2. As remarked before, the linear part of F is a retract and is therefore given by Theorem 1. We may assume

$$J_1 = \{1, 2, \ldots, n_1\}, \qquad J_2 = \{n_1 + 1, \ldots, n_2\}, \ldots,$$
$$J_{l+1} = \{n_l + 1, \ldots, n\}$$

(otherwise, replace F by $\sigma^{-1} \circ F \circ \sigma$ where σ is an appropriate permutation of coordinates). We know

$$L_1(z) = a_1 z_1 + \bar{\eta}_2 a_2 z_2 + \cdots + \bar{\eta}_{n_1} a_{n_1} z_{n_1}$$

where the a_k are the positive numbers a_{k1} and the η_k are the γ_{k1} of Theorem 1. Setting $z_k = \eta_k z_1$, $2 \le k \le n_1$ (denote the set of such $z \in B_n$ by M) we have $F_1(z) = z_1 + \sum_{j=1}^{\infty} P_{1j}(z)$ where P_{1j} is a homogeneous polynomial of degree j. Using the fact that $|F_1(z)| \le |z_1|$ when $\|z\| = |z_1|$ we readily see that $P_{1j}(z) \equiv 0$ on M. For example, with $z_k = \lambda_k z_1$, $n_1 < k \le n$, $z \in M$ we have

$$P_{1,2}(z) = z_1^2 h(\lambda_{n_1+1}, \ldots, \lambda_n)$$

and for $|z_1|$ small, we must have

$$1 \geq \left| \frac{z_1 + P_{1,2}(z)}{z_1} \right| = |1 + z_1 h(\lambda_{n_1+1}, \ldots, \lambda_n)|.$$

Hence $h \equiv 0$.

We wish to show $F_k(z) = \eta_k F_1(z)$ where $L_k(z) = \eta_k L_1(z)$, $2 \leq k \leq n_1$ (i.e., $F(z) \in M$). Write

$$F_k(z) = L_1(z)\eta_k + \sum_{j=2}^{\infty} P_{kj}(z)$$

where $2 \leq k \leq n_1$, and P_{kj} is homogeneous of degree j. As shown above for F_1, $P_{kj} \equiv 0$ on M. Let p be a minimum such that for some k, $2 \leq k \leq n_1$, we have $P_{kp} \not\equiv \eta_k P_{1p}(z)$. We now use the fact that $F_1(z) = F_1(F(z))$ and $F_k(z) = F_k(F(z))$.

By definition of p and the fact that P_{1j} and P_{kj} are zero on M, we conclude that $P_{1j}(F(z))$ and $P_{kj}(F(z))$ consist of terms of degree $\geq j + p - 1 > p$. Therefore,

$$P_{1p}(z) = L_1(P_{1p}(z), P_{2p}(z), \ldots, P_{n,p}(z))$$

and

$$P_{kp}(z) = L_k(P_{1p}(z), \ldots, P_{n,p}(z)) = \eta_k L_1(P_{1p}(z), \ldots, P_{n,p}(z)) = \eta_k P_{1p}(z).$$

This contradicts the choice of p and completes the proof of the existence of G_1 in the theorem satisfying (i). A similar argument for $J_2, \ldots, J_l$ completes the proof of the existence of the G_k in the theorem.

Now consider F_p, $p \in J_{l+1}$ (p fixed). Since $L_p(z) = \sum_{k=1}^{l} \tau_k T_k(z)$, $\sum_{k=1}^{l} |\tau_k| \leq 1$, we may write

$$F_p(z) = \sum_{k=1}^{l} \tau_k G_k(z) + h_p(z)$$

where h_p consists of terms of degree ≥ 2. Let q be a maximum positive integer such that F_p can be written as

$$F_p = \sum_{k=1}^{q-1} Q_k(G_1, G_2, \ldots, G_l) + \sum_{k=q}^{\infty} R_k$$

where R_k is homogeneous of degree $k \geq q$. Then

$$F_p = F_p(F) = \sum_{k=1}^{q-1} Q_k(G_1, G_2, \ldots, G_l) + \sum_{k=q}^{\infty} R_k(F).$$

We conclude that $R_q(z)$ consists of the terms of degree q in $R_q(F)$. These terms are clearly $R_q(L)$. In view of the nature of L, $R_q(L)$ is some function, say S_q, of $(T_1, T_2, \ldots, T_l)$. However, it is now clear that $S_q(T_1, T_2, \ldots, T_l)$ consists of the terms of degree q in $S_q(G_1, G_2, \ldots, G_l)$. Hence we may write

$$F_p = \sum_{k=1}^{q-1} Q_k(G_1, G_2, \ldots, G) + S_q(G_1, G_2, \ldots, G) + \left[\sum_{k=q}^{\infty} R_k - S_q(G_1, \ldots, G)\right]$$

where the last quantity consists of terms of degree $\geq q + 1$. This contradicts the choice of q and proves that $F_p(z) = H_p(G_1(z), G_2(z), \ldots, G_l(z))$ for some H_p.

It remains to prove that (ii) holds for $G_1, \ldots, G_l$. We can take

$$G_1(z) = a_1 z_1 + \bar{\eta}_2 a_2 z_2 + \cdots + \bar{\eta}_{n_1} a_{n_1} z_{n_1} + \sum_{k=2}^{\infty} P_{k1}(z)$$

and set $\bar{\eta}_k z_k = z_1$, $3 \leq k \leq n_1$, $\bar{\eta}_2 z_2 = e^{i\phi} z_1$. For such a z,

$$G_1(z) = z_1 + a_2 z_1(e^{i\phi} - 1) + z_1(e^{i\phi} - 1)f(z)$$

since $P_{k1}(z) = 0$ on $M = \{z \in B_n \colon \bar{\eta}_k z_k = z_1,\ 2 \leq k \leq n_1\}$. Statement (ii) will follow if $f(z) = 0$ when $\phi = 0$.

Observe that $|G_1(z)| \leq |z_1|$ for the z under consideration if we assume that $|z_j| \leq |z_1|$ for $j \geq n_1$. Therefore

$$|1 + a_2(e^{i\phi} - 1) + (e^{i\phi} - 1)f(z)| \leq 1.$$

We conclude that $(a_2 + \operatorname{Re} f(z))(\cos \phi - 1) - \sin \phi \operatorname{Im} f(z) \leq 0$. Now divide by $|\phi|$ and let $\phi \to 0$ separately through positive and negative values to see that $\operatorname{Im} f(z) \equiv 0$ when $\phi = 0$. Since $f(0) = 0$, we must have $f(z) = 0$ when $\phi = 0$ and the proof is completed by the following observations. Fix z_1, $1 > |z_1| > 0$ and expand

$$g(z_2, \ldots, z_n) = G_1(z_1, \ldots, z_n) - T_1(z_1, \ldots, z_n)$$

about the point $(\eta_2 z_1, \ldots, \eta_{n_1} z_1, w_{n_1+1}, \ldots, w_n)$ where $w_{n_1+1}, \ldots, w_n$ are chosen so that $|z_1| \geq |w_j|$, $n_1 < j \leq n$. Clearly

$$g(\eta_2 z_1, \ldots, \eta_{n_1} z_1, w_{n_1+1}, \ldots, w_n) = 0.$$

We have shown above that $\partial g(\eta_2 z_1, \ldots, w_n)/\partial z_2 = 0$ and by a similar argument $\partial g(\eta_2 z_1, \ldots, w_n)/\partial z_j = 0$, $3 \leq j \leq n_1$.

Since $w_{n_1+1}, \ldots, w_n$ are arbitrary, it follows that derivatives of all orders of g and $\partial g/\partial z_j (2 \leq j \leq n_1)$ with respect to the variables z_k, $n_1 < k \leq n$ are zero at the point under consideration. Thus (ii) is proved for $G_1 - T_1$. A similar argument for $G_k - T_k$, $2 \leq k \leq l$, completes the proof.

Example. Suppose F is a retraction of B_2 and $F(0) = 0$. Then (see [2]) except for a possible permutation of coordinates F is one of the following:

(i) $(z_1, z_2) \to (z_1, z_2)$,
(ii) $(z_1, z_2) \to (z_1, f(z_1))$,
(iii) $(z_1, z_2) \to [tz_1 + (1 - t)e^{-i\alpha} z_2 + (e^{-i\alpha} z_2 - z_1)^2 f(z_1, z_2)](1, e^{i\alpha})$ where α is real and $0 < t < 1$,
(iv) $(z_1, z_2) \to (0, 0)$.

The reader might find it helpful to determine the retractions of B_3 assuming 0 maps to 0. Aside from permutations, there are seven different types of such retractions: The identity has three dimensional range, there are two different types with two dimensional range (i.e., a two manifold, not necessarily affine), there are three different types with one dimensional range and the zero map is the seventh type.

4. Holomorphic retractions on $\bar{B}_n$

Now assume $F: \bar{B}_n \to \bar{B}_n$ is continuous, F restricted to B_n is holomorphic and F is a retraction. If $F(B_n) \subset B_n$, then Theorem 2 applies and we have the additional condition that all functions involved extend to continuous functions on $\bar{B}_n$. Therefore, we assume $F(B_n) \not\subset B_n$. This means $|F_j(z)| = 1$ for some j, $1 \leq j \leq n$ and $z \in B_n$ so by the maximum principle, $F_j(z) = C_j$ (constant) for some j. By replacing F by $\sigma^{-1} \circ F \circ \sigma$ for an appropriate linear map σ that permutes coordinates, we may assume

$$F = (C_1, C_2, \ldots, C_k, F_{k+1}, \ldots, F_n)$$

where $|C_j| = 1$, $1 \leq j \leq k$, and $F_j: B_n \to B_1$, $k+1 \leq j \leq n$. Let

$$M = \{z \in C^n: z_j = 0,\ 1 \leq j \leq k\} \approx B_{n-k}.$$

Then $\hat{F}: M \to M$ defined by

$$\hat{F} = G = (0, 0, \ldots, 0, G_1, G_2, \ldots, G_{n-k}) \tag{10}$$

where $G_j(z_{k+1}, \ldots, z_n) = F_{k+j}(C_1, C_2, \ldots, C_k, z_{k+1}, \ldots, z_n)$ is a retraction (that does not necessarily take 0 to 0). Thus, the nature of G was determined in Section 3. It now follows that

$$F_{k+j}(z_1, z_2, \ldots, z_n) = G_j(z_{k+1}, \ldots, z_n) + \sum_{p=1}^{k} (z_p - C_p) h_{pj}(z)$$

for some complex valued holomorphic h_{pj} defined on B_n. For $1 \leq p \leq k$, set

$$H_p(z) = (0, 0, \ldots, 0, h_{p_1}(z), h_{p_2}(z), \ldots, h_{p_{n-k}}(z)), \tag{11}$$

$$F(z) = (C_1, C_2, \ldots, C_k, 0, 0, \ldots, 0) + G(z_{k+1}, \ldots, z_n) + \sum_{p=1}^{k} (z_p - C_p) H_p(z). \tag{12}$$

We have proved most of the following theorem.

THEOREM 4. *If $F: \bar{B}_n \to \bar{B}_n$ is a holomorphic retraction on $\bar{B}_n$ (continuous on $\bar{B}_n$ and holomorphic on B_n), then there is a permutation of coordinates such that if F is replaced by $\sigma^{-1} \circ F \circ \sigma$, then F is given by* (12) *where G and H_p are given by* (10) *and* (11) *respectively and G is a retraction of*

$$M = \{z: z_j = 0,\ 1 \leq j \leq k\} \approx B_{n-k}.$$

If each of the functions involved is continuous on $\bar{B}_n$ then a necessary and sufficient condition on the functions H_p for F given by (12) *to be a retraction is that*

$$G(z_{k+1}, \ldots, z_n) + \sum_{p=1}^{k} (z_p - C_p)H_p(z) \subset G(M) \quad \textit{for all } z \in B_n.$$

Proof. All that remains to complete the proof is to check $F = F \circ F$ to verify the last statement in the theorem. This easy verification is left to the reader.

Note that the map

$$(z_1, z_2) \to (e^{i\alpha}, e^{-i\alpha}z_1 z_2) = (e^{i\alpha}, z_2 + (z_1 - e^{i\alpha})(e^{-i\alpha}z_2))$$

is an idempotent on $\bar{B}_2$ that has zero linear part. This can only happen for retractions on B_2 when the map is constant.

The following theorem clearly follows from Theorems 3 and 4.

THEOREM 5. *Suppose D is a retract of $\bar{B}_n$. Then there exist*

$$J \subset \{1, 2, \ldots, n\}, \quad M = \{z \in \bar{B}_n\colon z_j = 0 \quad \text{if } j \notin J\}$$

and functions $F_j\colon M \to \bar{B}_1$, $1 \le j \le n$, such that $F_j(z) = z_j$ if $j \in J$ and such that $D = \{(F_1(z), F_2(z), \ldots, F_n(z))\colon z \in M\}$.

5. Common fixed points of commuting maps

We will prove the following extension of Shields [4] and Eustice [2] results.

THEOREM 6. *Let f and g be continuous maps of $\bar{B}_n$ into itself that are holomorphic on B_n and assume $f \circ g = g \circ f$. Then f and g have a common fixed point in $\bar{B}_n$.*

Proof. Assume the theorem is true for all positive integers $k < n$. Following Shields method [4], let $\Gamma(f)$ be the closure of the iterates of f (i.e., set $f^1 = f$, $f^n = f \circ f^{n-1}$, $n = 2, 3, \ldots$, so $\Gamma(f)$ is the closure of $\{f^n\colon n = 1, 2, \ldots\}$ in the topology of uniform convergence on compact subsets of B_n). Then $\Gamma(f)$ contains a unique idempotent H that is therefore a retraction of $\bar{B}_n$ (see [6]).

If $D = H(\bar{B}_n)$ then since $f(z) = f(H(z)) = H(f(z)) \in D$ whenever $z \in D$, f maps D into D and similarly g maps D into D. Suppose $D \ne \bar{B}_n$. Then

$$M = \{z \in \bar{B}_n\colon z_j = 0 \quad \text{if } j \notin J\}$$

given by Theorem 5 has dimension $k < n$. Define $\hat{f}\colon M \to M$ by

$$\hat{f} = (\hat{f}_1, \hat{f}_2, \ldots, \hat{f}_n)$$

where $\hat{f}_j = 0$ if $j \notin J$ and $\hat{f}_j(z) = f_j(F_1(z), F_2(z), \ldots, F_n(z))$ when $j \in J$ (where F_1, $F_2, \ldots, F_n$ are given by Theorem 5) and define $\hat{g}$ similarly, Then $\hat{f}$ and $\hat{g}$ commute and have a common fixed point $c = (c_1, c_2, \ldots, c_n) \in M$ by the induction hypothesis. Clearly, $(F_1(c), F_2(c), \ldots, F_n(c))$ is a common fixed point for f and g.

If $D = \bar{B}_n$, then H is the identity and f is a biholomorphic map of $\bar{B}_n$ onto itself (see [6]). Then $f(z) = (\phi_1(z_{\tau_1}), \phi_2(z_{\tau_2}), \ldots, \phi_n(z_{\tau_n}))$ where $\tau_1, \tau_2, \ldots, \tau_n$ is a permutation of $\{1, 2, \ldots, n\}$ and each ϕ_j is a linear fractional transformation that maps the unit disk onto itself. Note that if P is the fixed point set of f then $g(P) \subset P$ for $z \in P \Rightarrow g(z) = g(f(z)) = f(g(z))$. Thus, if $\tau_j \neq j$, for some j, P lies on the manifold $z_j = \phi_j(z_{\tau_j})$ and the induction hypothesis implies that f and g have a common fixed point. Therefore, we need only consider the case

$$f(z) = (\phi_1(z_1), \phi_2(z_2), \ldots, \phi_n(z_n)).$$

If any of the ϕ_j has a unique fixed point c_j in the open disk, then again, P lies on a manifold of dimension $< n$ and the induction hypothesis yields a common fixed point. If some ϕ_j has two fixed points on the boundary, then the iterates of ϕ_j converge to one of these points (see [1], [7], and [8]) contradicting the assumption that $\Gamma(f)$ contains the identity. The only remaining case is f the identity. Clearly, any fixed point of g is a common fixed point for f and g in this case. This completes the proof.

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