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A NOTE ON THE RANGE OF THE OPERATOR $X \to AX - XB$

BY

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1. Introduction

Let $\mathscr{L}(\mathscr{H})$ denote the algebra of all bounded linear operators on an infinite dimensional complex Hilbert space $\mathscr{H}$. For A and B in $\mathscr{L}(\mathscr{H})$, let τ_{AB} (or simply τ) denote the operator on $\mathscr{L}(\mathscr{H})$ defined by $\tau(X) = AX - XB$. The purpose of this note is to give a characterization of the case when the range of τ, $\mathscr{R}(\tau)$, is norm dense in $\mathscr{L}(\mathscr{H})$.

For a bounded operator T on a Banach space, let $\sigma(T)$, $\sigma_p(T)$, $\sigma_r(T)$, and $\sigma_l(T)$ denote, respectively, the spectrum, point spectrum, right spectrum, and left spectrum of T. Following [2], let

$$\sigma_\delta(T) = \{\lambda\colon T - \lambda \text{ is not surjective}\}$$

and let

$$\sigma_\pi(T) = \{\lambda\colon T - \lambda \text{ is not bounded below}\}.$$

(For T in $\mathscr{L}(\mathscr{H})$, $\sigma_r(T) = \sigma_\delta(T)$ and $\sigma_l(T) = \sigma_\pi(T)$.) In [2], C. Davis and P. Rosenthal proved that

$$\sigma_\delta(\tau_{AB}) = \sigma_\delta(A) - \sigma_\pi(B) \equiv \{\alpha - \beta\colon \alpha \in \sigma_\delta(A),\ \beta \in \sigma_\pi(B)\}$$

$$\text{and} \quad \sigma_\pi(\tau) = \sigma_\pi(A) - \sigma_\delta(B);$$

moreover, it was shown in [6] that $\sigma_\delta(\tau) = \sigma_r(\tau)$ and $\sigma_\pi(\tau) = \sigma_l(\tau)$.

The preceding results show that τ is surjective if and only if $\sigma_r(A) \cap \sigma_l(B) = \emptyset$. In [9], D. A. Herrero gave an example of the case when $\mathscr{R}(\tau)$ is proper but norm dense, and he raised the question of characterizing this case. Part of the motivation for this question arises in the study of the closure of similarity orbits. Let $\mathscr{S}(T)^-$ denote the norm closure of the similarity orbit of an operator T; operators R and T are *asymptotically similar* if $T \in \mathscr{S}(R)^-$ and $R \in \mathscr{S}(T)^-$ (equivalently, $\mathscr{S}(R)^- = \mathscr{S}(T)^-$) [1]. In [9] it is noted that if $\mathscr{R}(\tau_{AB})$ is dense, then $A \oplus B$ is asymptotically similar to each operator on $\mathscr{H} \oplus \mathscr{H}$ whose operator matrix is of the form

$$\begin{pmatrix} A & Y \\ 0 & B \end{pmatrix}.$$

(When τ is surjective these operators are similar [11, p. 9].)

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If $A = B$, τ is called the inner derivation induced by A. In his study of derivations [14], J. G. Stampfli proved that the range of an inner derivation is not norm dense [14, Theorem 1]. D. A. Herrero noted in [9] that by modifying Stampfli's proof, it can be shown that if $\mathscr{R}(\tau_{AB})$ is dense, then A and B satisfy the following spectral condition, hereafter referred to as property (H) (see below for notation):

(H) (i) $\sigma_{re}(A) \cap \sigma_{le}(B) = \emptyset$;
(ii) $\sigma_p(A^*)^* \cap \sigma_p(B) = \emptyset$.

D. A. Herrero inquired whether, conversely, condition (H) insures that $\mathscr{R}(\tau)$ is dense. In this note we provide the following characterization of the case when τ has dense range; as a consequence it will be shown that condition (H) does not always imply that $\mathscr{R}(\tau)$ is dense.

THEOREM 1.1. *The following are equivalent:*

(1) τ_{AB} *has dense range;*
(2) (i) $\sigma_{re}(A) \cap \sigma_{le}(B) = \emptyset$ *and*
(ii) *There exists no nonzero trace class operator* X *such that* $BX = XA$;
(3) *If* $Y \in \mathscr{L}(\mathscr{H})$ *and* $\varepsilon > 0$, *then there exists* $X \in \mathscr{L}(\mathscr{H})$ *such that* $AX - XB - Y$ *is compact and has norm less than* ε.

Section 2 contains the proof of Theorem 1.1 and some of its consequences. Section 3 contains several examples, one of which shows that when $\mathscr{R}(\tau)$ is dense, $\sigma_r(A) \cap \sigma_l(B)$ may have nonempty interior. In Section 4 we consider the question as to whether $\mathscr{R}(\tau)^- = \mathscr{L}(\mathscr{H})$ implies the injectivity of τ_{BA}; such is the case, for example, if A and B are normal.

We conclude this section with some notation. Let $\mathscr{K}(\mathscr{H})$ denote the ideal of all compact operators in $\mathscr{L}(\mathscr{H})$, and let $\mathscr{C}$ denote the Calkin algebra $\mathscr{L}(\mathscr{H})/\mathscr{K}(\mathscr{H})$. For T in $\mathscr{L}(\mathscr{H})$, let $\tilde{T}$ denote the image of T under the canonical projection of $\mathscr{L}(\mathscr{H})$ onto the Calkin algebra. Let $\sigma_{le}(T) = \sigma_l(\tilde{T})$ (the left essential spectrum of T), and let $\sigma_{re}(T) = \sigma_r(\tilde{T})$ (the right essential spectrum of T). Let $\tilde{\tau}_{AB}$ or $\tilde{\tau}$ denote the operator on $\mathscr{C}$ defined by $\tilde{\tau}(\tilde{X}) = \tilde{A}\tilde{X} - \tilde{X}\tilde{B}$; $\tilde{\tau}$ is surjective if and only if $\sigma_{re}(A) \cap \sigma_{le}(B) = \emptyset$ [5, Theorem 3.8].

Let $(C_1, \|\cdot\|_1)$ denote the Banach space of all trace class operators in $\mathscr{L}(\mathscr{H})$; for K in C_1, $\|K\|_1 = \text{trace}\,((K^*K)^{1/2})$. Let $\mathscr{B} = \{e_i\}_{i \in I}$ denote an othonormal basis for $\mathscr{H}$. Let $K' \colon \mathscr{B} \to \mathscr{H}$ denote a function such that $\sum_{i \in I} \|K'e_i\| < \infty$. Then K' has a unique extension to a (bounded) trace class operator K. Indeed, if $x = \sum a_i e_i$ is in $\mathscr{H}$, then

$$\|\textstyle\sum a_i K'e_i\| \le \sum |a_i|\,\|K'e_i\| \le (\sum \|K'e_i\|^2)^{1/2}(\sum |a_i|^2)^{1/2},$$

so a unique bounded extension K exists. Moreover,

$$\text{trace}\,((K^*K)^{1/2}) = \textstyle\sum ((K^*K)^{1/2}e_i, e_i) \le \sum \|(K^*K)^{1/2}e_i\| = \sum \|Ke_i\| < \infty$$

[13, pg. 37],

so K is trace class and $\|K\|_1 \le \sum \|Ke_i\|$. (Note that the latter inequality may be strict.) In the sequel we will use this inequality, and the above method for defining trace class operators, without further reference.

Let $(C_\infty, \|\cdot\|)$ denote the Banach space of all compact operators on $\mathscr{H}$. Thus $(C_\infty)^* = C_1$ and $(C_1)^* = \mathscr{L}(\mathscr{H})$ [13, page 48]. Under these identifications we have $(\tau_{AB}|C_\infty)^* = -\tau_{BA}|C_1$ and $(-\tau_{BA}|C_1)^* = \tau_{AB}$ [5]. It follows from standard duality results that if $\mathscr{R}(\tau_{AB})$ is dense, then $\tau_{BA}|C_1$ is injective, and that $\tau_{BA}|C_1$ is injective if and only if $\tau_{AB}|C_\infty$ has dense range [12, pp. 94–96]. For $X \subset \mathbf{C}$, X^* denotes the set of complex conjugates of the elements of X.

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2. The case when the range of τ is dense

We begin with a proof of the main result.

Proof of Theorem 1.1. For the proof of the implication (1) ⇒ (2), we first show that if $\sigma_{re}(A) \cap \sigma_{le}(B) \neq \emptyset$, then τ does not have dense range. Let $\lambda \in \sigma_{re}(A) \cap \sigma_{le}(B)$; thus there exist orthonormal sequences $\{e_n\}_{n=1}^\infty$, $\{f_n\}_{n=1}^\infty \subset \mathscr{H}$ such that

$$\|(A-\lambda)^* e_n\| < 1/n \text{ and } \|(B-\lambda)f_n\| < 1/n \quad \text{for } n \ge 1 \text{ [7]}.$$

Let Y denote the partial isometry defined by $Yf_n = e_n$ $(n \ge 1)$ and $Yg = 0$ if $(g, f_n) = 0$ for all n; we will show that Y is not in the closure of the range of τ.

Let X be in $\mathscr{L}(\mathscr{H})$; then

$$\|(A-\lambda)Xf_n - e_n\|^2 = \|(A-\lambda)Xf_n\|^2 + 1 - 2\,\mathrm{Re}\,(Xf_n, (A-\lambda)^*e_n) \ge 1 - 2\|X\|/n.$$

Now for $n > 2\|X\|$, $1 - 2\|X\|/n > 0$, and so

$$\begin{aligned} \|AX - XB - Y\| &\ge \sup_{n > 2\|X\|} \|(AX - XB - Y)f_n\| \\ &\ge \sup_{n > 2\|X\|} \|(A-\lambda)Xf_n - e_n\| - \|X(B-\lambda)f_n\| \\ &\ge \sup_{n > 2\|X\|} (1 - 2\|X\|/n)^{1/2} - \|X\|/n \\ &= 1; \end{aligned}$$

thus $Y \notin \mathscr{R}(\tau)^-$. Next, if $\tau_{AB}|C_1$ is not injective, then since $(-\tau_{BA}|C_1)^* = \tau_{AB}$, it follows that $\mathscr{R}(\tau_{AB})$ is not dense.

We next prove the implication (2) ⇒ (3). Let Y be in $\mathscr{L}(\mathscr{H})$. From condition (2)(i), $\sigma_{re}(A)$ and $\sigma_{le}(B)$ are disjoint, and thus $\tilde{\tau}$ is surjective [5, Theorem 3.8]; in particular, there exists $X \in \mathscr{L}(\mathscr{H})$ and $K \in \mathscr{K}(\mathscr{H})$ such that $AX - XB = Y + K$. Since, from condition (2)(ii), $\tau_{BA}|C_1$ is injective, then $\tau_{AB}|C_\infty$ has dense range (see the discussion at the conclusion of the introduction). Thus there exists a sequence $\{K_n\} \subset \mathcal{K}(\mathcal{H})$ such that $AK_n - K_n B \to K$. Now

$$A(X - K_n) - (X - K_n)B \to Y$$

and

$$A(X - K_n) - (X - K_n)B - Y = K - (AK_n - K_n B) \in \mathcal{K}(\mathcal{H}).$$

Thus (2) $\Rightarrow$ (3), and since (3) $\Rightarrow$ (1) is obvious, the proof is complete.

The first application extends J. G. Stampfli's result that the range of an inner derivation is not norm dense (op. cit.). An operator X in $\mathcal{L}(\mathcal{H})$ is said to be *quasi-invertible* if X is injective and has dense range. Let A and B be in $\mathcal{L}(\mathcal{H})$; B is said to be a *quasiaffine transform* of A if there exists a quasi-invertible operator X such that $AX = XB$. Operators A and B are *quasisimilar* if they are quasiaffine transforms of one another.

COROLLARY 2.1. *If A and B are quasisimilar, then τ_{AB} and τ_{BA} do not have dense range.*

Proof. If A and B are quasisimilar, then $\sigma_{re}(A) \cap \sigma_{le}(B) \neq \emptyset$ and $\sigma_{re}(B) \cap \sigma_{le}(A) \neq \emptyset$ [5, Theorem 2.1], so the result follows from Theorem 1.1.

Remark. For the case $A = B$, J. G. Stampfli also proved that $\mathcal{L}(\mathcal{H})/\mathcal{R}(\tau)^-$ is nonseparable [14, Theorem 1, Corollary]. D. A. Herrero has observed that an analogous result holds for τ_{AB} whenever $\sigma_{re}(A) \cap \sigma_{le}(B) \neq \emptyset$. Note also that if B is merely a quasiaffine transform of A, then τ_{AB} may be surjective (see [5, Example 3.11]).

It follows from [5, Theorem 3.8] that $\tilde{\tau}_{AB}$ is bounded below if and only if it is injective. The next result is an analogue for the case when $\tilde{\tau}_{AB}$ is surjective.

COROLLARY 2.2. *$\tilde{\tau}_{AB}$ is surjective if and only if it has dense range.*

Proof. In one direction the proof is obvious; it thus suffices to assume that $\tilde{\tau}_{AB}$ is not surjective and to prove that $\mathcal{R}(\tilde{\tau})$ is not dense. If $\tilde{\tau}_{AB}$ is not surjective, then $\sigma_{re}(A) \cap \sigma_{le}(B) \neq \emptyset$ [5, Theorem 3.8]. Under this assumption, let Y and $\{f_n\}$ be as defined in the proof of Theorem 1.1. It was shown in that proof that Y is not in $\mathcal{R}(\tau)^-$; we now observe that $\tilde{Y} \notin \mathcal{R}(\tilde{\tau})^-$. Indeed, if $K \in \mathcal{K}(\mathcal{H})$, then $Kf_n \to 0$, so for each X in $\mathcal{L}(\mathcal{H})$ we have

$$\|AX - XB - Y - K\| \geq \sup_{n > 2\|X\|} (1 - 2\|X\|/n)^{1/2} - \|X\|/n - \|Kf_n\| = 1.$$

Thus $\|AX - XB - Y - K\| \geq 1$, and since K is an arbitrary compact operator, then $\|\tilde{A}\tilde{X} - \tilde{X}\tilde{B} - \tilde{Y}\| \geq 1$. Since X is also arbitrary, we conclude that $\tilde{Y} \notin \mathcal{R}(\tilde{\tau})^-$.

Let A, B, and Y be in $\mathcal{L}(\mathcal{H})$ and let $S = S(A, B, Y)$ denote the operator on

$\mathscr{H} \oplus \mathscr{H}$ whose operator matrix is of the form

$$\begin{pmatrix} A & Y \\ 0 & B \end{pmatrix}.$$

Whether S has a proper hyperinvariant subspace is an open problem, even in the case when $\sigma(A)$ is disconnected [4, Section 2] [10, problem 10]. On the other hand, it follows easily from Rosenblum's Theorem [17] that if $\sigma(A)$ and $\sigma(B)$ are disjoint, then $\mathscr{H} \oplus \{0\}$ is a hyperinvariant subspace for S (cf. [11, page 8]). The following result is a mild extension of this fact.

COROLLARY 2.3. *If $\sigma_e(A) \cup \sigma_e(B)$ is disconnected and τ_{AB} has dense range, then S has a proper hyperinvariant subspace.*

Proof. Since $\mathscr{R}(\tau_{AB})$ is dense, then S is asymptotically similar to $T = A \oplus B$ (op. cit.). It follows readily that $\sigma_e(S) = \sigma_e(T)$ (see [7] for the relevant properties of essential spectra). In particular, since $\sigma_e(T) = \sigma_e(A) \cup \sigma_e(B)$, then $\sigma_e(S)$ is disconnected. If $\sigma_e(S) = \sigma(S)$, then $\sigma(S)$ is disconnected, so the existence of a proper hyperinvariant subspace follows from the Riesz decomposition [11, page 32]. If $\sigma_e(S) \neq \sigma(S)$, then S or S^* has an eigenvalue, and thus S has a proper hyperinvariant subspace in this case also.

Remark. In Example 3.3 (below) we give an example of operators A and B such that $\mathscr{R}(\tau_{AB})$ is proper and dense, $\sigma_e(A) \cup \sigma_e(B)$ is disconnected, and $\sigma_e(A) \cap \sigma_e(B) \neq \emptyset$. In this example, the existence of a proper hyperinvariant subspace for every $S(A, B, Y)$ is, however, obvious, since A has eigenvalues and $\sigma_p(A) \subset \sigma_p(S)$. This example suggests the following question.

Question 2.4. If $\mathscr{R}(\tau_{AB})$ is proper and dense, does A or B^* have an eigenvalue?

With regard to this question, note the following. If $\mathscr{R}(\tau_{AB})$ is proper, then $\sigma_r(A) \cap \sigma_l(B) \neq \emptyset$ [2], while if $\mathscr{R}(\tau_{AB})$ is dense, then $\sigma_{re}(A) \cap \sigma_{le}(B) = \emptyset$ (Theorem 1.1). If both conditions hold, then either $\sigma_{re}(A) \neq \sigma_r(A)$ or $\sigma_{le}(B) \neq \sigma_l(B)$. Thus, if $\mathscr{R}(\tau_{AB})$ is proper and dense, either A^* and B has an eigenvalue.

3. Examples

The first example will be used to show that the spectral property (H) does not imply that $\mathscr{R}(\tau)$ is dense.

Example 3.1. We show that if U is a nonunitary isometry and $r(A) \leq 1$ (where $r(A)$ denotes the spectral radius of A), then $\mathscr{R}(\tau_{AU^*})$ is not dense. Since U is a nonunitary isometry, the von Neumann decomposition implies that there exists a reducing subspace $\mathscr{M} \subset \mathscr{H}$ such that $U \mid \mathscr{M}$ is a unilateral shift of multiplicity one. Let $\{e_n\}_{n=1}^{\infty}$ denote an orthonormal basis for $\mathscr{M}$ such that $Ue_n = e_{n+1}$ $(n \geq 1)$.

We first consider the case $r(A) < 1$, and we will show that in this case ker (τ_{U^*A}) contains a nonzero trace class operator. Let P denote the orthogonal projection of $\mathscr{H}$ onto $\mathscr{M}$ and let h denote a nonzero vector in $\mathscr{H}$. We define an operator Y by the following relations:

$$Y|(1-P)\mathscr{H} = 0; \quad Ye_1 = h; \quad Ye_{n+1} = A^{*n}h \text{ for } n \geq 1.$$

We extend $\{e_n\}$ to an orthonormal basis $\{e_n\} \cup \{f_i\}_{i \in I}$ for $\mathscr{H}$. Since $r(A) < 1$, then $\sum_{n=1}^{\infty} \|A^n\| < \infty$ (cf. [11, page 54]); in particular,

$$\sum_{i \in I} \|Yf_i\| + \sum_n \|Ye_n\| = \sum_n \|Ye_n\| = \sum_n \|A^{*n-1}h\| < \infty.$$

Thus, as discussed in Section 1, Y is trace class. Now $YUe_n = Ye_{n+1} = A^{*n}h = A^*(A^{*n-1}h) = A^*Ye_n$; moreover, since P commutes with U, $YU(1-P) = Y(1-P)U = 0 = A^*Y(1-P)$. Thus $YU = A^*Y$, so Y^* is a nonzero element of ker $(\tau_{U^*A} | C_1)$. It follows from Theorem 1.1 that $\mathscr{R}(\tau_{AU^*})$ is not dense.

We next consider the case $r(A) = 1$. If $\sigma_{re}(A) \cap \sigma_{le}(U^*) \neq \emptyset$, the conclusion that $\mathscr{R}(\tau_{AU^*})$ is not dense follows from Theorem 1.1. We may thus assume that $\sigma_{re}(A)$ and $\sigma_{le}(U^*)$ are disjoint. Now $\sigma_{le}(U^*)$ contains the unit circle, T, and $\sigma(A^*) \cap T \subset$ bdry $(\sigma(A^*))$ (since $r(A) = 1$). It follows that if $z \in \sigma(A^*) \cap T$, then z is an isolated point of $\sigma(A^*)$ and an eigenvalue of A^* with finite multiplicity. (For otherwise, since $z \in$ bdry $(\sigma(A^*))$, it follows from [7, Theorem 3.3] that $z \in \sigma_{le}(A^*)$; thus $\bar{z} \in \sigma_{re}(A) \cap \sigma_{le}(U^*)$, which is a contradiction.) Thus $\sigma(A^*) \cap T$ is a finite, isolated part of $\sigma(A^*)$ whose corresponding Riesz subspace is finite dimensional. Since $\mathscr{H}$ is infinite dimensional, it follows from the Riesz decomposition that A^* has an (infinite dimensional) invariant subspace $\mathscr{M}$ such that $r(A^*|\mathscr{M}) < 1$. In particular, there is a nonzero vector $h \in \mathscr{M}$ such that $\sum_{n=1}^{\infty} \|A^{*n-1}h\| < \infty$. Now we may procede as in the above case to show that ker (τ_{U^*A}) contains a nonzero trace class operator, which implies that $\mathscr{R}(\tau_{AU^*})$ is not dense. The proof of the example is now complete.

If we apply the preceding example in the case when $r(A) < 1$, $\sigma_p(A^*) = \emptyset$, and index $(U) \neq -\infty$, then we conclude that τ_{AU^*} does not have dense range, although condition (H) is satisfied (with $\sigma_p(A^*)^* = \sigma_{re}(A) \cap \sigma_{le}(U^*) = \emptyset$). (Of course, $\mathscr{R}(\tau_{AU^*})$ is not dense even if index $(U) = -\infty$, but in this case condition (H) is not satisfied since $\sigma_{le}(U^*)$ coincides with the closed unit disk.) The preceding example shows that the condition $\sigma_p(A^*)^* \cap \sigma_p(B) = \emptyset$ does not imply that ker $(\tau_{BA} | C_1) = \{0\}$; on the other hand, it is not difficult to verify that the converse implication is valid (cf. the proof of Theorem 1.1).

The next two examples show that when $\mathscr{R}(\tau)$ is dense, then $\sigma_r(A) \cap \sigma_1(B)$ may be "large" in certain senses. For an arbitrary operator T, let $\delta_r(T) = \sigma_r(T) \backslash \sigma_{re}(T) = \{\lambda: T - \lambda$ has closed range and $0 < \dim \ker ((T-\lambda)^*) < \infty\}$, and let

$$\begin{aligned}\delta_1(T) &= \sigma_l(T) \backslash \sigma_{le}(T) \\ &= \{\lambda: T - \lambda \text{ has closed range and } 0 < \dim \ker (T) < \infty\}.\end{aligned}$$

If $\mathscr{R}(\tau_{AB})$ is dense, then from Theorem 1.1 (or condition (H)) we have

$$\sigma_r(A) \cap \sigma_l(B) = (\sigma_{re}(A) \cap \delta_l(B)) \cup (\sigma_{le}(B) \cap \delta_r(A)).$$

We next show that both $\sigma_{re}(A) \cap \delta_l(B)$ and $\sigma_{le}(B) \cap \delta_r(A)$ may be nonempty.

Example 3.2. Let $\mathscr{H}_1$ denote a finite dimensional Hilbert space and let $\mathscr{H}_2$ denote a separable, infinite dimensional Hilbert space. Let $Q \in \mathscr{L}(\mathscr{H}_2)$ denote a quasinilpotent operator with dense range, and let U denote an isometry such that $1 \notin \sigma_p(U)$ and index $(U) \neq -\infty$. Let A and B denote the operators on $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$ given by $A = 1_{\mathscr{H}_1} \oplus Q$ and $B = 0_{\mathscr{H}_1} \oplus U$. Then $\sigma_{re}(A) \cap \delta_l(B) = \{0\}$ and $\sigma_{le}(B) \cap \delta_r(A) = \{1\}$.

It suffices to show that τ_{AB} has dense range. Let $X \in \mathscr{L}(\mathscr{H})$ and let $(X_{ij})_{1 \leq i,j \leq 2}$ denote the operator matrix of X. A calculation shows that the operator matrix of $AX - XB$ has the form

$$\begin{pmatrix} X_{11} & X_{12}(1 - U) \\ QX_{21} & QX_{22} - X_{22}U \end{pmatrix}.$$

Suppose dim $\mathscr{H}_1 = n < \infty$ and let $\{e_1, \ldots, e_n\}$ denote an orthonormal basis for $\mathscr{H}_1$. Let $\varepsilon > 0$ and let $Y = (Y_{ij})_{1 \leq i,j \leq 2}$ denote the operator matrix of an operator Y. Since Q has dense range, there exists $f_i \in \mathscr{H}_2$ such that $\|Qf_i - Y_{21}e_i\| < \varepsilon/n$ $(1 \leq i \leq n)$. We define an operator $X_{21} \in \mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$ by the relations $X_{21}e_i = f_i$ $(1 \leq i \leq n)$. Then

$$\|QX_{21} - Y_{21}\| \leq \sum_{i=1}^{n} \|(QX_{21} - Y_{21})e_i\| \leq \sum_{i=1}^{n} \|Qf_i - Y_{21}e_i\| < \varepsilon.$$

Similarly, since $(1 - U^*)$ has dense range, there exists an operator $X_{12} \in \mathscr{L}(\mathscr{H}_2, \mathscr{H}_1)$ such that

$$\|X_{12}(1 - U) - Y_{12}\| = \|(1 - U^*)X_{12}^* - Y_{12}^*\|$$
$$\leq \sum_{i=1}^{n} \|((1 - U^*)X_{12}^* - Y_{12}^*)e_i\| < \varepsilon.$$

Since $\sigma_r(Q) \cap \sigma_l(U) = \emptyset$, τ_{QU} is surjective, and there exists $X_{22} \in \mathscr{L}(\mathscr{H}_2)$ such that $QX_{22} - X_{22}U = Y_{22}$. Finally, let $X_{11} = Y_{11}$. If X is defined by the operator matrix $(X_{ij})_{1 \leq i,j \leq 2}$, then the above calculations show that $AX - XB - Y$ is a finite rank operator with norm less than 2ε. This completes the proof that τ_{AB} has dense range; note that the proof also shows that $\|AX - XB - Y\|_1 < 2\varepsilon$.

Our final example shows that when $\mathscr{R}(\tau_{AB})$ is dense, $\sigma_r(A) \cap \sigma_l(B)$ may have nonempty interior and $\sigma_e(A) \cap \sigma_e(B)$ may be infinite.

Example 3.3. Let U denote the unilateral shift of multiplicity one. For $i > 0$ let $\mathscr{H}_i$ denote a separable Hilbert space with orthonormal basis $\{e_n^i\}_{n=1}^{\infty}$. Let U_i be the unilateral shift on $\mathscr{H}_i$ defined by $U_i e_n^i = e_{n+1}^i$ for $n \geq 1$. Let $\{r_i\}_{i=1}^{\infty}$ denote a countable dense subset of $[3/4, 1]$. Let A denote the operator on $\mathscr{H} = \sum_{i=1}^{\infty} \oplus \mathscr{H}_i$ of the form $A = \sum_{i=1}^{\infty} \oplus r_i U_i^*$. Let $\{e_n\}_{n=-\infty}^{+\infty}$ denote an orthonormal basis for $\mathscr{H}$. Let B denote the bilateral weighted shift defined by $Be_n = (1/2)e_{n+1}$ for $n \geq 0$, and $Be_n = 2e_{n+1}$ for $n < 0$. To show that τ_{AB} has dense range we first show that τ_{BA} is injective. Suppose $X \in \mathscr{L}(\mathscr{H})$ and $BX = XA$. Then for $n \geq 1$, $B^n X = XA^n$ and so $B^n X e_n^i = XA^n e_n^i = 0$ for each $i \geq 1$. Since B is injective, $Xe_n^i = 0$ for i, $n \geq 1$; thus $X = 0$ and so τ_{BA} is injective.

Let $\sigma = \{z\colon 3/4 \leq |z| \leq 1\}$. We show next that $\sigma_r(A) = \sigma_{re}(A) = \sigma$. Since $A^* = \sum_{i=1}^\infty \oplus r_i U_i$, $\delta_r(A) = \emptyset$ and so $\sigma_r(A) = \sigma_{re}(A)$. Since $\sigma_{re}(U_i^*) = \{z\colon |z| = 1\}$ and $\{r_i\}_{i=1}^\infty$ is dense in $[3/4, 1]$, it follows that $\sigma \subset \sigma_{re}(A)$. Since $r(A) = 1$, it suffices to show that if $|\lambda| < 3/4$, then $\lambda \notin \sigma_r(A)$. Note that if $|\alpha| < 1$, then $U \sum_{i=0}^\infty \oplus (\alpha U)^i$ converges in norm to an operator R_α such that $(U^* - \alpha)R_\alpha = 1$ and $\|R_\alpha\| < 1/(1 - |\alpha|)$. Let $|\lambda| < 3/4$. For each i, let $\alpha_i = \lambda/r_i$, so that $|\alpha_i| < 1$. Now

$$\|(1/r_i)R_{\alpha_i}\| \leq 1/(r_i - |\lambda|) \leq 1/(3/4 - |\lambda|).$$

It follows that $S_\lambda = \sum_{i=1}^\infty \oplus (1/r_i)R_{\alpha_i}$ is a bounded right inverse for $A - \lambda$; thus $\sigma_{re}(A) = \sigma_r(A) = \sigma$.

Next, it is easy to see that there is a rank one operator F such that $B + F$ is unitarily equivalent to $B' = 2U^* \oplus (1/2)U$. Thus

$$\sigma_{le}(B) = \sigma_{le}(B') = \{z\colon |z| = 2\} \cup \{z\colon |z| = 1/2\};$$

in particular, $\sigma_{re}(A)$ and $\sigma_{le}(B)$ are disjoint. Since also τ_{BA} is injective it follows from Theorem 1.1 that $\mathscr{R}(\tau_{AB})$ is dense.

Note that if $1/2 < |\lambda| < 2$, then $B' - \lambda$ is Fredholm with index $(B' - \lambda) = 1$; thus $\{\lambda\colon 1/2 < |\lambda| < 2\} \subset \delta_l(B)$. It follows that $\sigma_r(A) \cap \sigma_l(B)$ contains the open annulus $\{z\colon 3/4 < |z| < 1\}$. Finally, note that

$$\{z\colon |z| = 1/2\} \subset \sigma_{le}(A) \cap \text{bdry}\,(\sigma_e(B)).$$

4. On the injectivity of τ

If τ_{AB} is surjective, then τ_{BA} is bounded below [2]; moreover, if $(\mathscr{J}, \|\cdot\|_{\mathscr{J}})$ is any norm ideal, then $\tau_{AB}|\mathscr{J}$ is surjective (and right invertible in $\mathscr{L}(\mathscr{J})$), and $\tau_{BA}|\mathscr{J}$ is bounded below (and left invertible in $\mathscr{L}(\mathscr{J})$) [6]. It is thus natural to inquire whether analogous results hold when τ_{AB} has dense range; does it follow that τ_{BA} is injective? does $\tau_{AB}|\mathscr{J}$ have dense range in $\mathscr{J}$?

The first result of this section allows several reformulations of these questions. We begin by recalling some results on norm ideals. In the sequel, for $1 \leq p \leq \infty$, $(C_p, \|\cdot\|_p)$ denotes the Schatten p-ideal in $\mathscr{L}(\mathscr{H})$ [8], [13]. Thus $(C_1, \|\cdot\|_1)$ coincides with the space of trace class operators and $(C_\infty, \|\cdot\|_\infty)$ coincides with the space of all compact operators on $\mathscr{H}$; moreover, $C_1 = \bigcap_{p=1}^\infty C_p$. Let $\mathscr{F}$ denote the ideal of all finite rank operators on $\mathscr{H}$. Recall that for each p, $\mathscr{F}$ is p-norm dense in C_p [13, pp. 72–73]; moreover, if $(\mathscr{J}, \|\cdot\|_{\mathscr{J}})$ is any norm ideal and $F \in \mathscr{F}$, then $\|F\| \leq \|F\|_{\mathscr{J}} \leq \|F\|_1$ [8, page 69].

PROPOSITION 4.1. *Let A and B be in $\mathscr{L}(\mathscr{H})$. The following are equivalent:*

(i) $\tau_{AB} \mid C_1$ *has dense range;*
(ii) $\tau_{AB} \mid C_p$ *has dense range for each* p, $1 \le p \le \infty$;
(iii) τ_{BA} *is injective.*

Moreover, if τ_{AB} *has dense range, the above properties are equivalent to each of the following properties:*

(iv) $\tau_{BA} \mid C_p$ *is injective for each* p, $1 \le p \le \infty$;
(v) $\tau_{BA} \mid C_\infty$ *is injective.*

Proof. The equivalence of (i) and (iii) follows immediately from the identity $\tau_{BA} = (-\tau_{AB} \mid C_1)^*$. The implication (ii) $\Rightarrow$ (i) is obvious. To complete the first part, we suppose that $\mathscr{R}(\tau_{AB} \mid C_1)$ is dense in C_1, and we prove that (ii) holds. Let $Y \in C_p$ and let $\varepsilon > 0$. Let F be a finite rank operator such that $\|F - Y\|_p < \varepsilon/2$. Since $F \in C_1$, there exists $X \in C_1$ such that $\|AX - XB - F\|_1 < \varepsilon/2$. Then $X \in C_p$, and

$$\begin{aligned}\|AX - XB - Y\|_p &\le \|AX - XB - F\|_p + \|F - Y\|_p \\ &\le \|AX - XB - F\|_1 + \|F - Y\|_p \\ &< \varepsilon.\end{aligned}$$

Thus, $\mathscr{R}(\tau_{AB} \mid C_p)$ is dense.

We now assume that τ_{AB} has dense range. To prove (ii) $\Rightarrow$ (iv) we first let $1 < p < \infty$. If $1/p + 1/q = 1$, then $\tau_{BA} \mid C_p = (-\tau_{AB} \mid C_q)^*$ [5]. Since $\tau_{AB} \mid C_q$ has dense range, then $\tau_{BA} \mid C_p$ is injective. Also, since $(\tau_{BA} \mid C_\infty)^* = -\tau_{AB} \mid C_1$, which has dense range, then $\tau_{BA} \mid C_\infty$ is injective. Finally, since τ_{AB} has dense range, Theorem 1.1 implies that $\tau_{BA} \mid C_1$ is injective.

The implication (iv) $\Rightarrow$ (v) is obvious. To complete the proof we prove that (v) $\Rightarrow$ (iii). Since τ_{AB} has dense range, Theorem 1.1 implies that $\sigma_{le}(B) \cap \sigma_{re}(A) = \emptyset$, so $\tilde{\tau}_{BA}$ is bounded below [5, Theorem 3.8]. Thus ker $(\tau_{BA}) \subset \mathscr{K}(\mathscr{H})$, and since $\tau_{BA} \mid C_\infty$ is injective, the result follows.

Remark. The equivalence of (i)–(v) fails if it is not assumed that $\mathscr{R}(\tau_{AB})$ is dense. Thus, if U is a unilateral shift of multiplicity one, then $\tau_{UU} \mid C_\infty$ is injective, but τ_{UU} is obviously not injective. Note also that the injectivity of τ_{BA} does not imply that τ_{AB} has dense range; indeed [4] contains an example of normal operators A and B such that $\sigma_{re}(A) = \sigma(A) = \sigma(B) = \sigma_{le}(B)$ but τ_{AB} and τ_{BA} are injective.

The next result shows that if A and B are normal, then the condition that τ_{AB} has dense range is equivalent to property (H), which also implies the injectivity of τ_{BA}.

PROPOSITION 4.2. *If A and B are normal operators in $\mathcal{L}(\mathcal{H})$, the following are equivalent:*

(1) *τ_{AB} has dense range;*
(2) *A and B satisfy property* (H);
(3) *If $Y \in \mathcal{L}(\mathcal{H})$ and $\varepsilon > 0$, then there exists $X \in \mathcal{L}(\mathcal{H})$ such that $AX - XB - Y$ is a finite rank operator with trace norm less than ε.*

In this case both τ_{AB} and τ_{BA} are injective. Moreover, if $(\mathcal{J}, \|\cdot\|_{\mathcal{J}})$ is a norm ideal, $Y \in \mathcal{J}$, and $\varepsilon > 0$, then there exists $X \in \mathcal{J}$ such that $AX - XB - Y$ is a finite rank operator and $\|AX - XB - Y\|_{\mathcal{J}} < \varepsilon$.

Proof. Recall that if N is normal, then $\sigma_{re}(N) = \sigma_{le}(N) = \sigma_e(N)$ and $\sigma_p(N^*)^* = \sigma_p(N)$: in particular, $\sigma(N) = \sigma_e(N) \cup \sigma_p(N)$. Thus, if τ_{AB} has dense range, Theorem 1.1 implies that

$$\text{(i) } \sigma_e(A) \cap \sigma_e(B) = \emptyset \quad \text{and} \quad \text{(ii) } \sigma_p(A) \cap \sigma_p(B) = \emptyset,$$

which conditions are equivalent to (H) in this case.

For (2) $\Rightarrow$ (3), we assume that (i) and (ii) hold. If, additionally, $\sigma_p(A) \cap \sigma_e(B) = \sigma_p(B) \cap \sigma_e(A) = \emptyset$, then $\sigma(A) \cap \sigma(B) = \emptyset$, so τ is invertible and the result follows from [6]; indeed, in this case each $\tau \mid \mathcal{J}$ is invertible [6, Section 3]. We assume that $\sigma_p(A) \cap \sigma_e(B) \neq \emptyset$ and $\sigma_p(B) \cap \sigma_e(A) \neq \emptyset$, and we omit the proof of the intermediate cases, which can be treated similarly.

Since each limit point of the spectrum of a normal operator is in the essential spectrum, condition (i) implies that $\sigma_p(A) \cap \sigma_e(B)$ and $\sigma_p(B) \cap \sigma_e(A)$ are finite, proper, isolated parts of $\sigma(A)$ and $\sigma(B)$ respectively. Let $\alpha_1, \ldots, \alpha_n$ denote the distinct elements of $\sigma_p(A) \cap \sigma_e(B)$ and let $\beta_1, \ldots, \beta_k$ denote the distinct elements of $\sigma_p(B) \cap \sigma_e(A)$. Condition (i) implies that

$$0 < \dim \ker (A - \alpha_i) < \infty \ (1 \le i \le n) \quad \text{and} \quad 0 < \dim \ker (B - \beta_i) < \infty \ (1 \le i \le k);$$

moreover, from (ii) it follows that $\alpha_i \neq \beta_j$ for $1 \le i \le n$ and $1 \le j \le k$. Let

$$\mathcal{H}_1 = \bigvee_{i=1}^{n} \ker (A - \alpha_i), \quad \mathcal{H}_2 = \mathcal{H}_1^{\perp},$$

$$\mathcal{K}_1 = \bigvee_{j=1}^{k} \ker (B - \beta_j) \quad \text{and} \quad \mathcal{K}_2 = \mathcal{K}_1^{\perp};$$

thus $\mathcal{H}_1$ and $\mathcal{K}_1$ are finite dimensional reducing subspaces for A and B respectively. Let $A_i = A \mid \mathcal{H}_i$ and $B_i = B \mid \mathcal{K}_i$ for $i = 1, 2$. Clearly $\sigma(A_1) \cap \sigma(B_1) = \emptyset$; also, conditions (i) and (ii) and the preceding definitions imply that $\sigma(A_2) \cap \sigma(B_2) = \emptyset$. Let P and Q denote, respectively, the projections onto $\mathcal{H}_1$ and $\mathcal{K}_1$.

Let $Y \in \mathcal{L}(\mathcal{H})$ and let $\varepsilon > 0$. Consider

$$Y_{11} = PYQ \mid \mathcal{K}_1 \in \mathcal{L}(\mathcal{K}_1, \mathcal{H}_1).$$

Since A_1 and B_1 have disjoint spectra, a modification of Rosenblum's Theorem [17] (cf. [11, Corollary 0.13]) implies that there exists $X_{11} \in \mathscr{L}(\mathscr{K}_1, \mathscr{H}_1)$ such that $A_1 X_{11} - X_{11} B_1 = Y_{11}$. Similarly, if

$$Y_{22} = (1 - P)Y(1 - Q)\,|\,\mathscr{K}_2,$$

then there exists $X_{22} \in \mathscr{L}(\mathscr{K}_2, \mathscr{H}_2)$ such that $A_2 X_{22} - X_{22} B_2 = Y_{22}$.

Let $Y_{21} = (1 - P)YQ\,|\,\mathscr{K}_1 \in \mathscr{L}(\mathscr{K}_1, \mathscr{H}_2)$. Let $\{f_1, \ldots, f_q\}$ denote an orthonormal basis for $\mathscr{K}_1$ such that $B_1 f_i = \beta_{j_i} f_i$, $i = 1, \ldots, q$. Since $\sigma_p(A_2) \subset \sigma_p(A)$ and $\sigma_p(A) \cap \sigma_p(B) = \emptyset$, it follows that $A_2 - \beta_{j_i}$ is injective and thus has dense range. Let $x_i \in \mathscr{H}_2$ be such that $\|(A_2 - \beta_{j_i})x_i - Y_{21} f_i\| < \varepsilon/q$ $(1 \le i \le q)$. Define $X_{21} \in \mathscr{L}(\mathscr{K}_1, \mathscr{H}_2)$ by $X_{21} f_i = x_i$ $(1 \le i \le q)$. Thus

$$\begin{aligned}\|A_2 X_{21} - X_{21} B_1 - Y_{21}\| &\le \sum_{i=1}^{q} \|A_2 X_{21} f_i - X_{21} B_1 f_i - Y_{21} f_i\| \\ &= \sum_{i=1}^{q} \|A_2 x_i - \beta_{j_i} x_i - Y_{21} f_i\| \\ &< \varepsilon.\end{aligned}$$

Similarly, if $Y_{12} = PY(1 - Q)\,|\,\mathscr{K}_2 \in \mathscr{L}(\mathscr{K}_2, \mathscr{H}_1)$, there exists $X_{12} \in \mathscr{L}(\mathscr{K}_2, \mathscr{H}_1)$, and there exists an orthonormal basis $\{e_1, \ldots, e_p\}$ for $\mathscr{H}_1$, such that

$$\begin{aligned}\|A_1 X_{12} - X_{12} B_2 - Y_{12}\| &= \|X_{12}^* A_1^* - B_2^* X_{12}^* - Y_{12}^*\| \\ &\le \sum_{i=1}^{p} \|(X_{12}^* A_1^* - B_2^* X_{12}^* - Y_{12}^*)e_i\| \\ &< \varepsilon.\end{aligned}$$

If $X = X_{11} Q + X_{21} Q + X_{12}(1 - Q) + X_{22}(1 - Q)$, then $AX - XB - Y$ has finite rank, and the above estimates imply that $\|AX - XB - Y\|_1 < 2\varepsilon$: thus (3) holds, and the implication (3) $\Rightarrow$ (1) is clear.

To obtain the conclusion about norm ideals, we modify the preceding argument as follows. Let $Y \in \mathscr{J}$ and let $\varepsilon > 0$. Since $\sigma(A_2) \cap \sigma(B_2) = \emptyset$, there exist operators $A_1' \in \mathscr{L}(\mathscr{H}_1)$ and $B_1' \in \mathscr{L}(\mathscr{K}_1)$ such that

$$A' = A_1' \oplus A_2 \;(\in \mathscr{L}(\mathscr{H}_1 \oplus \mathscr{H}_2))$$

and

$$B' = B_1' \oplus B_2 \;(\in \mathscr{L}(\mathscr{K}_1 \oplus \mathscr{K}_2))$$

have disjoint spectra. Thus [6] implies that there exists $X' \in \mathscr{J}$ such that $A'X' - X'B' = Y$. Since

$$(1 - P)A' = A'(1 - P) = A(1 - P)$$

and

$$(1 - Q)B' = B'(1 - Q) = (1 - Q)B,$$

it follows that

$$A(1-P)X'(1-Q) - (1-P)X'(1-Q)B = (1-P)Y(1-Q).$$

Let X_{11}, X_{21}, and X_{12} be the finite rank operators previously defined, and let

$$X = X_{11}Q + X_{21}Q + X_{12}(1-Q) + (1-P)X'(1-Q).$$

Since $X' \in \mathscr{J}$, then $X \in \mathscr{J}$. A direct calculation shows that

$$\begin{aligned} &AX - XB - Y \\ &= (A_2X_{21} - X_{21}B_1 - Y_{21})Q + (A_1X_{12} - X_{12}B_2 - Y_{12})(1-Q). \end{aligned}$$

The estimates given above in the definitions of X_{21} and X_{12} imply that both terms in the last expression have trace norm less than ε. Thus $AX - XB - Y$ is a finite rank operator such that

$$\|AX - XB - Y\|_{\mathscr{J}} \leq \|AX - XB - Y\|_1 < 2\varepsilon,$$

and $\tau_{AB} | \mathscr{J}$ has dense range. Since $\tau_{AB} | C_1$ has dense range, Proposition 4.1 implies that τ_{BA} is injective; an application of Fuglede's Theorem [11, pp. 19–20] implies that τ_{AB} is also injective, and the proof is complete.

The above results (particularly Example 3.2 and Proposition 4.2) suggest the following questions, which we have been unable to resolve.

Question 4.3. If $\mathscr{R}(\tau_{AB})$ is dense, $\varepsilon > 0$, and $Y \in \mathscr{L}(\mathscr{H})$, does there exist $X \in \mathscr{L}(\mathscr{H})$ such that $AX - XB - Y$ is trace class (or even finite rank) and $\|AX - XB - Y\|_1 < \varepsilon$?

Question 4.4. If $\mathscr{R}(\tau_{AB})$ is dense, is τ_{BA} injective?

We include the following results for the sake of completeness. We omit the proofs, which may be based entirely on those of J. P. Williams for the case $A = B$ [16].

PROPOSITION 4.5 [16, Theorem 2]. *$\mathscr{R}(\tau_{AB})$ is dense in the weak operator topology if and only if there exists no nonzero finite rank operator F such that $BF = FA$.*

PROPOSITION 4.6 [16, Section 3, Corollary 1]. *$\mathscr{R}(\tau_{AB})$ is ultraweakly dense in $\mathscr{L}(\mathscr{H})$ if and only if there exists no nonzero trace class operator K such that $BK = KA$.*

The preceding results show that if $\mathscr{R}(\tau)$ is norm dense, then $\mathscr{R}(\tau)$ is ultraweakly dense, which in turn implies that $\mathscr{R}(\tau)$ is dense in the weak operator topology. It is not difficult to construct examples which show that neither of the converse implications is valid.

Our characterization of the case when $\mathscr{R}(\tau)$ is dense depends on the spectral condition $\sigma_{re}(A) \cap \sigma_{le}(B) = \theta$ and on the determination that $\tau_{BA} | C_1$ is injective. The injectivity of τ_{BA} appears to depend on the detailed structures of the operators A and B, not merely on their spectral properties. For the case when A and B are normal, a characterization of the injectivity of τ_{AB} is given in [4, Proposition 2.4]; for the case of analytic Toeplitz operators, see [3]. Related results for decomposable operators and hyponormal operators are contained in [5, Section 4] and [15, Appendix].

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