ILLINOIS JOURNAL OF MATHEMATICS
Volume 25, Number 2, Summer 1981

ON A CONNECTIVITY PROPERTY INDUCED BY THE θ-CLOSURE OPERATOR

BY

JESSE P. CLAY AND JAMES E. JOSEPH

In memory of Professor W. W. S. Claytor

1. Introduction

The notion of θ-closure of a subset A, $\mathrm{cl}_\theta(A)$, of a topological space was introduced by Veličko for the purpose of studying the important class of H-closed spaces in terms of arbitrary filterbases [23] and in order to generalize the Taimanov Extension Theorem [24]. Recently, the operator cl_θ has been studied by a number of researchers (See [1], [2], [4], [7], [8]–[10], [11]–[15].) Herrington and Long [10] have characterized minimal Hausdorff and C-compact spaces in terms of this operator. Dickman and Porter [1] have utilized the operator to show that while H-closed spaces are not necessarily of the second category, these spaces do satisfy a property of "second category type"; they also used cl_θ to give a characterization of those Hausdorff spaces in which the Fomin H-closed extension operator commutes with the projective cover (absolute) operator [1]; these same authors used the operator in [2] to study the extension function problem for the θ-continuous functions of Fomin [5] between Hausdorff spaces. It has also been shown recently [11] that $\mathrm{cl}_\theta(A)$ is quasi H-closed relative to an H-closed space in the sense of Porter and Thomas [21] for each subset A of the space. In this paper, we use cl_θ to introduce a collection of subsets of a space which we call θ-connected relative to the space, and we study this collection of subsets. This collection of subsets of a space always contains the collection of connected subsets of the space but these two classes do not always coincide. All spaces are assumed to be Hausdorff.

In Section 2, we provide some elementary properties of subsets which are θ-connected relative to a space and use these subsets to generalize the well-known theorem that the intersection of continua directed by inclusion is a continuum. Noiri [20] has established that if a space X is connected and the function $g: X \to Y$ is a surjection which is weakly-continuous in the sense of Levine, then Y is connected. It is interesting to find that if g is weakly-continuous, then $g(A)$ is θ-connected relative to Y for each connected $A \subset X$. Another of the results in this section is a parallel of the well-known Wallace Theorem. This parallel is provided for the θ-rigid subsets of Dickman and

Received May 23, 1979.

Porter and is applied to produce some additional properties of θ-connected subsets relative to a space.

In Section 3, we introduce the notions of θ-component relative to a space and θ-quasicomponent of a subset of a space relative to the space. The main result in this section is an improvement of the well-known and frequently useful fact that components and quasicomponents coincide in compact Hausdorff spaces.

2. θ-connectivity relative to a space

Let X be a space and let $A \subset X$; we denote the closure, interior and boundary of A by cl (A), int (A) and bd (A), respectively; we say that $x \in X$ is in the *θ-closure* of A ($\mathrm{cl}_\theta(A)$) if each closed neighborhood V of x satisfies $V \cap A \neq \phi$; A is *θ-closed* if $\mathrm{cl}_\theta(A) = A$; a pair (P, Q) of nonempty subsets of X is a *θ-separation relative to X* if

$$(P \cap \mathrm{cl}_\theta(Q)) \cup (Q \cap \mathrm{cl}_\theta(P)) = \phi;$$

A is *θ-connected relative to X* if A is not the union of P and Q where (P, Q) is a θ-separation relative to X. In the first three theorems in this section we offer some additional characterizations of θ-connected relative to X subsets and some elementary properties of such subsets. We give the proofs of Theorem 2.2(3), 2.2(7) and Theorem 2.3 only since the proofs closely parallel those for properties of connected subsets.

THEOREM 2.1. *The following statements are equivalent for a space X and $A \subset X$:*

(1) *A is θ-connected relative to X.*

(2) *For each two points $x, y \in A$, there is a θ-connected relative to X subset B of A with $x, y \in B$.*

(3) *If (P, Q) is a θ-separation relative to X and $A \subset P \cup Q$ then either $A \subset P$ or $A \subset Q$.*

THEOREM 2.2. *The following properties hold for spaces X and Y:*

(1) *Connected subsets of X are θ-connected relative to X.*

(2) *Any pair (V, W) of nonempty disjoint open subsets of X is a θ-separation relative to X.*

(3) *If A is θ-connected relative to X and $A \subset B \subset \mathrm{cl}_\theta(A)$ then B is θ-connected relative to X.*

(4) *If Ω is a family of θ-connected relative to X subsets no pair of which is a θ-separation relative to X then $\bigcup_\Omega F$ is θ-connected relative to X.*

(5) *If Ω is a family of θ-connected relative to X subsets and there is a nonempty $F_0 \in \Omega$ such that (F_0, F) fails to be a θ-separation relative to X for each $F \in \Omega$ then $\bigcup_\Omega F$ is θ-connected relative to X.*

(6) *If $\{F(n)\}$ is a sequence of nonempty θ-connected relative to X subsets and $(F(n), F(n+1))$ fails to be a θ-separation relative to X for all n, then $\bigcup F(n)$ is θ-connected relative to X.*

(7) *If A is θ-connected relative to X and B is θ-connected relative to Y then $A \times B$ is θ-connected relative to $X \times Y$.*

(8) *If A is θ-connected relative to X and $X^* \subset X$ satisfies*

$$A \cap \operatorname{int}(X^*) \neq \phi \quad \text{and} \quad A \cap \operatorname{int}(X - X^*) \neq \phi,$$

then $A \cap \operatorname{bd}(X^*) \neq \phi$.

Proof of (3). Let (P, Q) be a θ-separation relative to X and suppose that $B \subset P \cup Q$. If $A \cap P \neq \phi$ then $A \subset P$ from Theorem 2.1(3); consequently, we have $B \subset \mathrm{cl}_\theta(A) \subset \mathrm{cl}_\theta(P)$. Hence $B \cap Q = \phi$ and $B \subset P$. A similar argument shows that $B \subset Q$ when $A \cap Q \neq \phi$. If $A = \phi$ then $B = \phi$. Thus (3) follows from Theorem 2.1(3).

Proof of (7). Let $(x, y), (u, v) \in A \times B$. Then $\{x\} \times B$ and $A \times \{v\}$ are both θ-connected relative to $X \times Y$ since θ-connectivity relative to a space is a topological invariant. We see that

$$(x, v) \in (A \times \{v\}) \cap (\{x\} \times B)$$

and

$$(x, y), (u, v) \in (A \times \{v\}) \cup (\{x\} \times B) \subset A \times B.$$

Therefore, by combining Theorems 2.2(4) and 2.1(2) we see that $A \times B$ is θ-connected relative to $X \times Y$. Q.E.D.

THEOREM 2.3. *If $\{X(n)\}_\Lambda$ is an infinite family of spaces and $A(n) \subset X(n)$ is θ-connected relative to $X(n)$ for each $n \in \Lambda$ then the product, $\prod_\Lambda A(n)$, is θ-connected relative to $\prod_\Lambda X(n)$.*

Proof. Let (P, Q) be a θ-separation relative to $\prod_\Lambda X(n)$ and suppose that

$$\prod_\Lambda A(n) \subset P \cup Q.$$

Let $x \in (\prod_\Lambda A(n)) \cap P$ and, for each $n \in \Lambda$, let π_n: $\prod_\Lambda X_n \to X_n$ be the projection. There is a basic open set $W = \bigcap_{\Gamma(x)} \pi_n^{-1}(V(n))$ with $x \in W$ and $Q \cap \mathrm{cl}(W) = \theta$. Let $y \in \prod_\Lambda A(n)$ and let

$$H(y) = \left\{ z \in \prod_\Lambda A(n) : z(n) = y(n) \text{ for all } n \in \Lambda - \Gamma(x) \right\}.$$

$H(y)$ is known to be homeomorphic to $\prod_{\Gamma(x)} A(n)$ and Theorem 2.2(7) may be used with induction to show that $\prod_{\Gamma(x)} A(n)$ is θ-connected relative to $\prod_{\Gamma(x)} X(n)$. It follows that $H(y)$ is θ-connected relative to $\prod_\Lambda X(n)$. This implies that $H(y) \subset P$ since we see easily that v defined by $v(n) = x(n)$ when $n \in \Gamma(x)$ and $v(n) = y(n)$ for $n \in \Lambda - \Gamma(x)$ satisfies $v \in H(y) \cap W \subset P$. Q.E.D.

We now provide two examples. The first example establishes that subsets which are θ-connected relative to a space need not be connected. In this connection, it is well known that closure and θ-closure coincide in regular spaces, so in regular spaces connected subsets and subsets θ-connected relative to the space coincide. Our second example utilizes the classical example of a noncompact minimal Hausdorff space to show that these two collections may coincide in spaces which are not regular.

Example 2.4. Let I be the unit interval $[0, 1]$ let $Y = I \times \{0\}$ and let $X = I \times I$ with the topology generated by the following base for the open sets: (1) the relative open sets from the plane in $X - Y$ and (2) for $x \in Y$, sets of the form $(V \cap (X - Y)) \cup \{x\}$ where V is open in the plane with $x \in V$. Y is discrete in the relative topology from X and hence Y is not connected. Suppose that (P, Q) is a θ-separation relative to X and that $Y = P \cup Q$. Choose $(r, 0) \in P$ and, without loss of generality, assume that there is an $s \in I$ with $r < s$ and $(s, 0) \in Q$. Let $c = \sup \{r \in I\colon r < s \text{ and } (r, 0) \in P\}$. We see easily that $(c, 0) \in \mathrm{cl}_\theta (P)$. Hence $(c, 0) \in P$. Since it is readily seen that $(c, 0) \in \mathrm{cl}_\theta (Q)$, we obtain a contradiction and Y is θ-connected relative to X.

Example 2.5. Let E be the set of even positive integers, F the set of odd positive integers and let $X = \{0\} \cup E \cup F \cup \{j + n^{-1}\colon j,\ n \in (E \cup F) - \{1\}\}$ be endowed with the topology generated by the following open set base: (1) the relative open sets from the reals in $X - \{0, 1\}$, (2) all subsets of the form $\{0\} \cup \{j + n^{-1}\colon j \geq j_0,\ n \in E\}$, where $j_0 > 1$, and (3) all subsets of the form $\{1\} \cup \{j + n^{-1}\colon j \geq j_0,\ n \in F - \{1\}\}$, where $j_0 > 1$. It is known that this space is not regular; it is not difficult to see that the collection of θ-connected relative to X subsets and hence the collection of connected subsets is precisely the collection of singletons.

In [20], Noiri has shown that if X is connected and $g\colon X \to Y$ is a weakly-continuous surjection then Y is connected. We recall that a function $g\colon X \to Y$ is *weakly-continuous* at $x \in X$ if for each open set W about $g(x)$, there is an open set V about x with $g(V) \subset \mathrm{cl}\ (W)$; g is *weakly-continuous* if g is weakly-continuous at each $x \in X$ [17]. We may establish the following result.

THEOREM 2.6. *If $g\colon X \to Y$ is weakly-continuous and $K \subset X$ is connected then $g(K)$ is θ-connected relative to Y.*

Proof. Suppose (P, Q) is a θ-separation relative to Y with $g(K) = P \cup Q$. Let

$$A = K \cap g^{-1}(P) \quad \text{and} \quad B = K \cap g^{-1}(Q).$$

Since K is connected we may suppose, without loss of generality, that $x \in A \cap \mathrm{cl}\ (B)$. Then

$$g(x) \in P \subset Y - \mathrm{cl}_\theta\ (Q).$$

There is a W open about $g(x)$ with $\mathrm{cl}\,(W) \subset Y - Q$ and, since g is weakly-continuous, there is a V open about x with $g(V) \subset \mathrm{cl}\,(W)$. Since $x \in \mathrm{cl}\,(B)$ we have $V \cap g^{-1}(Q) \neq \phi$. Hence $g(V) \cap Q \neq \phi$, a contradiction. Q.E.D.

A function $g\colon X \to Y$ is *θ-continuous* at $x \in X$ if for each W open about $g(x)$, there is a V open about x satisfying $g(\mathrm{cl}\,(V)) \subset \mathrm{cl}\,(W)$; g is *θ-continuous* if g is θ-continuous at each $x \in X$ [5]. It is clear that θ-continuous functions are weakly-continuous. We note in our next result that θ-connectivity relative to a space is preserved by θ-continuous functions. The proof is similar to that of Theorem 2.6 and is omitted.

THEOREM 2.7. *If $g\colon X \to Y$ is θ-continuous and $K \subset X$ is θ-connected relative to X then $g(K)$ is θ-connected relative to Y.*

The *θ-adherence* of a filterbase Ω ($\mathrm{ad}_\theta\,\Omega$) on a space X is $\bigcap_\Omega \mathrm{cl}_\theta\,(F)$. A subset A of a space is *quasi H-closed* (QHC) *relative to X* if each filterbase Ω on A satisfies $A \cap \mathrm{ad}_\theta\,\Omega \neq \phi$; A is *H-closed* if A is QHC relative to A [21]. It is known that $\mathrm{cl}_\theta\,(A)$ is the intersection of all closed neighborhoods of A for any subset A of a space [11] and, consequently, that $\mathrm{ad}_\theta\,\Omega$, for each filterbase Ω, is the adherence of the filterbase of open neighborhoods of elements of Ω. Theorem 2.8 generalizes the result that $\mathrm{cl}_\theta\,(A)$ is QHC relative to the space for any subset A of an H-closed space.

THEOREM 2.8. *If X is an H-closed space and Ω is a filterbase on X, then $\mathrm{ad}_\theta\,\Omega$ is* QHC *relative to X.*

Proof. Let ∇ be a filterbase on $\mathrm{ad}_\theta\,\Omega$. Then for open neighborhoods V, W, elements of Ω and ∇, respectively, we have $V \cap W \neq \phi$. So the collection of such $V \cap W$ is an open filterbase, call it Ω^*, on X. Since X is H-closed we have $\phi \neq \mathrm{ad}\,\Omega^* \subset \mathrm{ad}_\theta\,\nabla \cap \mathrm{ad}_\theta\,\Omega$. Q.E.D.

A space is *C-compact* if each closed subset of the space is QHC relative to the space. This class of spaces has been studied by a number of mathematicians (See [3], [6], [10], [18], [22], [26], [27], [29].) It is immediate that C-compact spaces are H-closed. It has been shown in [4] that subsets of H-closed spaces with disjoint θ-closures are separated by disjoint open neighborhoods. These facts along with Proposition 2.9 (stated without proof) will be used in Theorem 2.10 to generalize the result that the intersection of continua which are ordered by inclusion is a continuum.

PROPOSITION 2.9. *Let X be a space and let $A \subset X$ be θ-closed. If (P, Q) is a θ-separation relative to X and $A = P \cup Q$ then P and Q are θ-closed in X.*

THEOREM 2.10. *Let X be a C-compact space and let Ω be a filterbase of subsets which are θ-connected relative to X. Then* $\mathrm{ad}_\theta\, \Omega$ *is θ-connected relative to X if* $\mathrm{ad}_\theta\, \Omega$ *is θ-closed in X.*

Proof. Let (P, Q) be a θ-separation relative to X and suppose that $\mathrm{ad}_\theta\, \Omega = P \cup Q$. Then P and Q are disjoint and θ-closed in X from Proposition 2.9. Since X is C-compact there are open neighborhoods V, W of P and Q, respectively, with $V \cap W = \phi$. Since $\mathrm{ad}_\theta\, \Omega \subset V \cup W$ and X is C-compact there is an $F_0 \in \Omega$ with $F_0 \subset V \cup W$. Since (V, W) is a θ-separation relative to X, and since F_0 is θ-connected relative to X, we must have $F_0 \subset V$ or $F_0 \subset W$. Hence

$$\mathrm{cl}_\theta\,(F_0) \subset \mathrm{cl}\,(V) \quad \text{or} \quad \mathrm{cl}_\theta\,(F_0) \subset \mathrm{cl}\,(W).$$

This gives $\mathrm{ad}_\theta\, \Omega \subset \mathrm{cl}\,(V)$ or $\mathrm{ad}_\theta\, \Omega \subset \mathrm{cl}\,(W)$. Therefore, $Q \cap \mathrm{ad}_\theta\, \Omega = \phi$ or $P \cap \mathrm{ad}_\theta\, \Omega = \phi$, a contradiction. Q.E.D.

The proof of the following corollary is straightforward.

COROLLARY 2.11 [28]. *Let $\{K(n): n \in \Lambda\}$ be a nonempty collection of continua directed by inclusion. Then $\bigcap_\Lambda K(n)$ is a continuum.*

Theorem 2.12 is similar to Theorem 2.10.

THEOREM 2.12. *Let X be a C-compact space and let Ω be a filterbase on X consisting of connected subsets. Then* $\mathrm{ad}_\theta\, \Omega$ *is connected if* open and closed subsets of $\mathrm{ad}_\theta\, \Omega$ *are θ-closed in X.*

Proof. Let (P, Q) be a separation of $\mathrm{ad}_\theta\, \Omega$ with $P \times Q \neq \phi$. Then P and Q are open and closed in $\mathrm{ad}_\theta\, \Omega$. Thus there are disjoint open neighborhoods V, W of P, Q, respectively. By arguments similar to those in the proof of Theorem 2.10 we obtain a contradiction. Q.E.D.

The Wallace Theorem (see [16, Theorem 12, p. 142]) states that if A and B are compact subsets of spaces X and Y, respectively, and $A \times B \subset W$ where W is open in $X \times Y$ then some open neighborhood M of A and open neighborhood N of B satisfy $M \times N \subset W$. The final results of this section are parallels of this theorem for the θ-rigid subsets of Dickman and Porter [2] and applications. A subset A of a space X is *θ-rigid* if for each cover Ω of A by open subsets there is a finite $\Omega^* \subset \Omega$ satisfying $A \subset \mathrm{int}\,(\bigcup_{\Omega^*} \mathrm{cl}\,(V))$. It is known that a subset A of a space X is θ-rigid if and only if $A \cap \mathrm{ad}_\theta\, \Omega \neq \phi$ whenever Ω is a filterbase on X satisfying $F \cap \mathrm{cl}\,(V) \neq \phi$ for all $F \in \Omega$ and open neighborhoods V of A [14]. We make use of Proposition 2.13 to produce the promised result.

PROPOSITION 2.13. *Let X and Y be spaces and let $B \subset Y$ be θ-rigid. Let $x \in X$ and $K \subset X \times Y$ satisfy $(\{x\} \times B) \cap \mathrm{cl}_\theta\,(K) = \phi$. Then there are open sets V about x in X and W containing B in Y such that* $\mathrm{cl}\,(V \times W) \cap K = \phi$.

Proof. For each $y \in B$ there are open sets $V(y)$ about x in X and $W(y)$ about y in Y with $\text{cl}\,(V(y) \times W(y)) \cap K = \phi$. Since B is θ-rigid there is a finite $B^* \subset B$ with

$$B \subset \text{int}\left(\text{cl}\left(\bigcup_{B^*} W(y)\right)\right) = W.$$

Let $V = \bigcap_{B^*} V(y)$. Then V is open about x and W is open containing B; moreover,

$$\text{cl}\,(W) = \text{cl}\left(\bigcup_{B^*} W(y)\right)$$

so $\text{cl}\,(V \times W) \cap K = \phi$. Q.E.D.

THEOREM 2.14. *Let X and Y be spaces and let A, B be θ-rigid in X, Y respectively. Let $K \subset X \times Y$ satisfy $(A \times B) \cap \text{cl}_\theta(K) = \phi$. Then there are open sets V in X containing A and W in Y containing B satisfying* $\text{cl}\,(V \times W) \cap K = \phi$.

Proof. It follows from Proposition 2.13 that for each $x \in A$ there are open sets $V(x)$ in X about x and $W(x)$ in Y containing B with $\text{cl}\,(V(x) \times W(x)) \cap K = \phi$. There is a finite $A^* \subset A$ with $A \subset \text{int}\,(\text{cl}\,((\bigcup_{A^*} V(x))) = V$. Let $W = \bigcap_{A^*} W(x)$. Then V and W have the required properties. Q.E.D.

Corollary 2.15 and Theorem 2.16 are offered without proof.

COROLLARY 2.15. *Let $A \subset X$ and $B \subset Y$ be compact and θ-rigid, respectively. Let $K \subset X \times Y$ satisfy $(A \times B) \cap \text{cl}_\theta(K) = \phi$. Then there are open sets V in X containing A and W in Y containing B satisfying* $\text{cl}\,(V \times W) \cap K = \phi$.

THEOREM 2.16. *Let X and Y be spaces and let A, B be θ-rigid in X, Y, respectively. Let Q be open in $X \times Y$ and let $A \times B \subset Q$. Then there are open sets, V in X containing A and W in Y containing B with* $\text{cl}\,(V \times W) \subset \text{cl}\,(Q)$.

We now offer some applications of the foregoing results. Let X and Y be spaces and $R \subset X \times Y$. If $A \subset X$ we denote $\{y\colon (x, y) \in R \text{ for some } x \in A\}$ by $R[A]$ ($R[x]$ if $A = \{x\}$) and if $A \subset Y$ we denote $\{x \in X\colon R[x] \cap A \neq \phi\}$ by $R^{-1}[A]$. If $y \in Y$ we let

$$T^{-1}(R;\, y) = \{x \in X\colon x \in \text{cl}_\theta\,(R^{-1}(\text{cl}\,(W))) \text{ for each } W \text{ open about } y\}$$

and we let $T^{-1}(R;\, A) = \bigcup_{y \in A} T^{-1}(R;\, y)$ for $A \subset Y$. The following results parallel for relative θ-connectivity results of Long [19] for connectivity.

THEOREM 2.16. *Let X be an H-closed space, Y be a space and $R \subset X \times Y$ be a relation so that $R[x]$ is θ-rigid for each $x \in X$ and $R[A]$ is θ-connected relative to Y whenever A is θ-connected relative to X. Then $R \cap (K \times Y)$ is θ-connected* *relative to $X \times Y$ for each θ-connected relative to X subset K if each such subset K and $x \in K$ satisfy*

$$T^{-1}(R; R[x]) \cap \mathrm{cl}_\theta (K) = \{x\}.$$

Proof. Suppose that $K \subset X$ is θ-connected relative to X and assume the given conditions. Suppose $R \cap (K \times Y) = P \cup Q$ where (P, Q) is a θ-separation relative to $X \times Y$. Let

$$A = \pi_X(P) \cap K \quad \text{and} \quad B = \pi_X(Q) \cap K.$$

If $x \in A$, then $\{x\} \times R[x]$ is θ-connected relative to $X \times Y$ by Theorem 2.2(7),

$$\{x\} \times R[x] \subset P \cup Q \quad \text{and} \quad (\{x\} \times R[x]) \cap P \neq \phi;$$

hence $\{x\} \times R[x] \subset P$ and, since $\{x\} \times R[x]$ is θ-rigid, it follows from Proposition 2.13 that there are open sets V about x and W containing $R[x]$ with

$$(\mathrm{cl}\,(V) \times \mathrm{cl}\,(W)) \cap Q = \phi.$$

It follows easily by a similar argument that $R[\mathrm{cl}\,(V) \cap B] \cap \mathrm{cl}\,(W) = \phi$. Moreover,

$$R[K] = R[A] \cup R[B], \quad R[A] \neq \phi \quad \text{and} \quad R[B] \neq \phi.$$

Suppose $y \in \mathrm{cl}_\theta (R[B]) \cap R[A]$ and let $x \in A$ with $y \in R[x]$. Then

$$\Omega = \{R^{-1}[\mathrm{cl}\,(H)] \cap B \colon H \text{ open about } y\}$$

is a filterbase on the H-closed space X. Hence $\mathrm{ad}_\theta\, \Omega \neq \theta$ and we see readily that

$$\mathrm{ad}_\theta\, \Omega \subset \mathrm{cl}_\theta (K) \cap T^{-1}(R; R[x]).$$

By the argument above we have open sets W about y and V about x such that

$$R[\mathrm{cl}\,(V) \cap B] \cap \mathrm{cl}\,(W) = \phi;$$

so $x \notin \mathrm{ad}_\theta\, \Omega$ and this is a contradiction. Therefore,

$$R[A] \cap \mathrm{cl}_\theta (R[B]) = \phi$$

and, similarly,

$$R[B] \cap \mathrm{cl}_\theta (R[A]) = \phi.$$

This means that $R[K]$ is not θ-connected relative to Y, a contradiction. Q.E.D.

Our last two theorems in this section are offered without proof.

THEOREM 2.17. *Let X be C-compact, Y be a space and $R \subset X \times Y$ be a relation which satisfies $R[x]$ is θ-rigid for each $x \in X$ and $R[A]$ is θ-connected relative to Y whenever A is θ-connected relative to X. Then $R \cap (K \times Y)$ is θ-connected relative to $X \times Y$ for each θ-connected relative to X subset K if each* *such K and $x \in K$ satisfy*

$$T^{-1}(R; R[x]) \cap \mathrm{cl}\,(K) = \{x\}.$$

THEOREM 2.18. *Let X be H-closed (C-compact), Y be a space and $R \subset X \times Y$ be a relation so that $R[x]$ is θ-rigid for each $x \in X$ and $R[A]$ is θ-connected relative to Y whenever A is connected in X. Then $R \cap (K \times Y)$ is θ-connected relative to $X \times Y$ for each connected $K \subset X$ if each such K and $x \in K$ satisfy*

$$T^{-1}(R; R[x]) \cap \mathrm{cl}_\theta\,(K) = \{x\} \quad (T^{-1}(R; R[x]) \cap \mathrm{cl}\,(K) = \{x\}).$$

3. θ-components and θ-quasicomponents relative to a space

Let X be a space. A *θ-component relative to X* is a θ-connected relative to X subset which is not properly contained in any θ-connected relative to X subset. It follows easily from Theorem 2.2(3) that θ-components relative to X are θ-closed in X; and from Theorem 2.2(4) we see that each θ-connected relative to X subset is contained in a θ-component relative to X. It is not difficult to see that if H and K are distinct θ-components relative to X then either $H \cap \mathrm{cl}_\theta\,(K) = \phi$ or $K \cap \mathrm{cl}_\theta\,(H) = \phi$. If A is a nonempty subset of X we say that $x, y \in A$ are equivalent if whenever $A = P \cup Q$, where (P, Q) is a θ-separation relative to X, we have $x, y \in P$ or $x, y \in Q$. If $A[x]$ denotes the equivalence class of x with respect to this equivalence relation on A we call $A[x]$ a *θ-quasicomponent of A relative to X*. It is clear that a θ-quasicomponent of A relative to A is a quasicomponent of A; if $A \subset X$ and $x \in A$ we denote by $\theta S(A, x)$ the set of all $P \subset X$ such that for some $Q \subset X$, (P, Q) is a θ-separation relative to X satisfying $A = P \cup Q$ and $x \in P$. We may then give the following characterization of $A[x]$ for $x \in A$. We omit the proof.

PROPOSITION 3.1. *If X is a space and $A \subset X$ is nonempty then, for each $x \in A$, $A[x]$ is the intersection of the elements of $\theta S(A, x)$.*

PROPOSITION 3.2. *If X is a space and $\phi \neq B \subset A$ then $B[x] \subset A[x]$ for each $x \in B$.*

Proof. Let $x \in B$, let $y \in B[x]$ and let (P, Q) be a θ-separation relative to X with $A = P \cup Q$. If $P \cap B = \phi$ ($Q \cap B = \phi$) then $x, y \in Q$ ($x, y \in P$). Otherwise $(P \cap B, Q \cap B)$ is a θ-separation relative to X and $B = (P \cap B) \cup (Q \cap B)$. So $x, y \in P \cap B$ or $x, y \in Q \cap B$. Q.E.D.

The following proposition improves the well-known fact that quasicomponents are closed.

PROPOSITION 3.3. *If X is a space and A is a nonempty subset of X then $A[x]$ is θ-closed in A for each $x \in A$.*

Proof. Let $x \in A$ and let $y \in A - A[x]$. There are sets P, Q such that $A = P \cup Q$, (P, Q) is a θ-separation relative to X, $y \in P$ and $A[x] \subset Q$. Hence $y \notin \mathrm{cl}_\theta\,(A[x])$ in X and it follows that $y \notin \mathrm{cl}_\theta\,(A[x])$ in the subspace A. Q.E.D.

COROLLARY 3.4. *Quasicomponents are θ-closed.*

The last corollary along with Theorem 3 of [14] enables us to identify a collection of subsets of H-closed spaces which are θ-rigid.

COROLLARY 3.5. *Quasicomponents of H-closed spaces are θ-rigid.*

$P \subset A \subset X$ is a *θ-component of A relative to X* if P is θ-connected relative to X and no θ-connected relative to X subset of A properly contains P.

PROPOSITION 3.6. *If X is a space and A is a nonempty subset of X then $A[x]$ is a θ-component of A relative to X for each $x \in A$ for which $A[x]$ is θ-connected relative to X.*

Proof. Suppose $x \in A$ and $A[x]$ is θ-connected relative to X. Let $C \subset A$ be θ-connected relative to X and assume that $A[x] \subset C$. If $y \in C$ and $A = P \cup Q$, where (P, Q) is a θ-separation relative to X, then $C \subset P$ or $C \subset Q$, so $x, y \in P$ or $x, y \in Q$. Hence $y \in A[x]$. Q.E.D.

COROLLARY 3.7. *Connected quasicomponents are components.*

It is well known and frequently useful that components and quasicomponents coincide in compact Hausdorff spaces. Theorem 3.8 represents an improvement of this result.

THEOREM 3.8. *Let X be a C-compact space and let $A \subset X$ be θ-closed. Then each θ-quasicomponent of A relative to X is a θ-component of A relative to X.*

Proof. Let $x \in A$. We will show that $A[x]$ is θ-connected relative to X. The desired conclusion will then follow from Proposition 3.6. Let $y \in A[x]$ and let

$$\Omega = \{B \subset A\colon B \text{ is } \theta\text{-closed in } X,\ x \in B \text{ and } y \in B[x]\}.$$

$\Omega \neq \phi$ since $A \in \Omega$. Order Ω by inclusion, let Ω^* be a chain in Ω and let $C = \bigcap_{\Omega^*} F$. Then C is θ-closed in X and $x \in C$. Let (P, Q) be a θ-separation relative to X with $C = P \cup Q$, $x \in P$ and $y \in Q$. P and Q are θ-closed in X; since X is C-compact, there are disjoint open subsets V, W of X with $P \subset V$ and $Q \subset W$. If $F - (V \cup W) \neq \phi$ for every $F \in \Omega^*$ then $C - (V \cup W) \neq \phi$ since X is C-compact and for each $F \in \Omega^*$, F is θ-closed in X. Thus there is an $F_0 \in \Omega^*$ with $F_0 \subset V \cup W$. However, this means that $(F_0 \cap V, F_0 \cap W)$ is a θ-separation relative to X with $F_0 = (F_0 \cap V) \cup (F_0 \cap W)$, $x \in F_0 \cap V$ and $y \in F_0 \cap W$, a contradiction. Therefore $C \in \Omega$ and, by Zorn's Lemma, Ω has a minimal element, C_0. We will show that C_0 is θ-connected relative to X and that $C_0 \subset A[x]$. It will then follow from Theorem 2.1(2) that $A[x]$ is θ-connected relative to X. Suppose (P, Q) is a θ-separation relative to X with $C_0 = P \cup Q$. Without loss assume that $x, y \in P$. Since $P \neq C_0$ and P is θ-closed in X there are sets P^*, P^{**}, satisfying $P = P^* \cup P^{**}$ where (P^*, P^{**}) is a θ-separation relative to X, $x \in P^*$ and $y \in P^{**}$. We see that $(P^*, P^{**} \cup Q)$ is a θ-separation relative to X with $C_0 = P^* \cup (P^{**} \cup Q)$, $x \in P^*$ and $y \in P^{**} \cup Q$. This is a contradiction. Thus C_0 is θ-connected relative to X. It is immediate that $C_0 = C_0[x] \subset A[x]$. Q.E.D.

COROLLARY 3.9. *Quasicomponents and components coincide in a compact space.*

Veličko [25] has called a subset of a space *θ-connected* if it is not the union of two nonempty subsets with disjoint closed neighborhoods in X. If $B \subset X$ is not θ-connected we have nonempty B_1, B_2 such that $B = B_1 \cup B_2$ and open V, W such that $B_1 \subset V$, $B_2 \subset W$ and $\text{cl}\,(V) \cap \text{cl}\,(W) = \emptyset$. Hence $B_1 \cap \text{cl}_\theta\,(B_2) = B_2 \cap \text{cl}_\theta\,(B_1) = \emptyset$, so (B_1, B_2) is a θ-separation relative to X and B is not θ-connected relative to X. Hence the collection of subsets θ-connected relative to X is contained in the collection of θ-connected subsets. In Example 3.9, we note that this inclusion is proper for some spaces and the paper is complete.

Example 3.9. Let X be the space of Example 2.5. The set $\{0, 1\}$ is θ-connected.

REFERENCES

1. R. F. DICKMAN, JR. and J. R. PORTER, *θ-closed subsets of Hausdorff spaces*, Pacific J. Math., vol. 59 (1975), pp. 407–415.
2. ———, *θ-perfect and θ-absolutely closed functions*, Illinois J. Math., vol. 21 (1977), pp. 42–60.
3. R. F. DICKMAN, JR. and A. ZAME, *Functionally compact spaces*, Pacific J. Math., vol. 31 (1969), pp. 303–311.
4. M. S. ESPELIE and J. E. JOSEPH, *Some properties of θ-closure*, Canadian J. Math., to appear.
5. S. FOMIN, *Extensions of topological spaces*, Ann. of Math., vol. 44 (1943), pp. 471–480.
6. G. GOSS and G. VIGLINO, *Some topological properties weaker than compactness*, Pacific J. Math., vol. 35 (1970), pp. 635–638.
7. L. L. HERRINGTON, *Some remarks on $H(i)$ spaces and strongly-closed graphs*, Proc. Amer. Math. Soc., vol. 58 (1976), pp. 277–283.
8. L. L. HERRINGTON and P. E. LONG, *Characterizations of H-closed spaces*, Proc. Amer. Math. Soc., vol. 48 (1975), pp. 469–475.
9. ———, *A characterization of minimal Hausdorff spaces*, Proc. Amer. Math. Soc., vol. 57 (1976), pp. 373–374.
10. ———, *Characterizations of C-compact spaces*, Proc. Amer. Math. Soc., vol. 52 (1975), pp. 417–426.
11. J. E. JOSEPH, *Multifunctions and cluster sets*, Proc. Amer. Math. Soc., vol. 74 (1979), pp. 329–337.
12. ———, *Multifunctions and graphs*, Pacific J. Math., vol. 79 (1978), pp. 509–529.
13. ———, *θ-closure and θ-subclosed graphs*, Math. Chronicle, vol. 8 (1979), pp. 99–117.
14. ———, *Some remarks on θ-rigidity*. Kyungpook Math. J., to appear.
15. ———, *Multifunctions and inverse cluster sets*, Canad. Math. Bull., vol. 23 (1980), pp. 161–171.
16. J. L. KELLEY, *General topology*, Van Nostrand, Princeton, N.J., 1955.

17. NORMAN LEVINE, *A decomposition of continuity in topological spaces*, Amer. Math. Monthly, vol. 68 (1961), pp. 44–46.
18. T. LIM and K. TAN, *Functional compactness and C-compactness*, J. London Math. Soc., (2), vol. 9 (1974), pp. 371–377.
19. P. E. LONG, *Connected mappings*, Duke Math. J., vol. 35 (1968), pp. 677–682.
20. T. NOIRI, *On weakly continuous mappings*, Proc. Amer. Math. Soc., vol. 46 (1974), pp. 120–124.
21. J. PORTER and J. THOMAS, *On H-closed and minimal Hausdorff spaces*, Trans. Amer. Math. Soc., vol. 138 (1969), pp. 159–170.
22. S. SAKAI, *A note on C-compact spaces*, Proc. Japan Acad., vol. 46 (1970), pp. 917–920.
23. N. V. VELIČKO, *H-closed topological spaces*, Mat. Sb., vol. 70 (1966), pp. 98–112 = Amer. Math. Transl., vol, 78 (1968), pp. 103–118.
24. ———, *On extension of mappings of topological spaces*, Sibirsk. Mat. Z., vol. 6 (1965), pp. 64–69 = Amer. Math. Transl., vol. 98 (1970), pp. 41–47.
25. ———, *On the theory of H-closed topological spaces*, Sibirsk. Mat. Z., vol. 8 (1967), pp. 754–763 = Siberian Math. J., vol. 8 (1967), pp. 569–579.
26. G. VIGLINO, *C-compact spaces*, Duke Math. J., vol. 36 (1969), pp. 761–764.
27. ———, *Seminormal and C-compact spaces*, Duke Math. J., vol. 38 (1971), pp. 57–61.
28. S. W. WILLARD, *General topology*, Addison-Wesley, Reading Mass., 1970.
29. ———, *Functionally compact spaces, C-compact spaces and mappings of minimal Hausdorff spaces*, Pacific J. Math., vol. 38 (1971), pp. 267–272.

HOWARD UNIVERSITY
WASHINGTON, D.C.