ZEROS OF CERTAIN COMPOSITE POLYNOMIALS IN ALGEBRAICALLY CLOSED FIELDS

BY

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The study of the zeros of composite polynomials has mostly been confined to polynomials in the complex plane. The object of this paper is to study the zeros of the composite polynomials which arise as linear combinations of a polynomial and its (formal) derivatives in an algebraically closed field K of characteristic zero. Our main theorem concerning the zeros of such composite polynomials gives certain interesting results which, when applied to the complex plane, furnish improved versions of the corresponding classical results due to Walsh, Marden, and Kakeya. At the end we show that our results cannot be further generalized in certain directions.

1. Introduction

If K is an algebraically closed field of characteristic zero, then we know (cf. [1, pp. 38–40], [6, pp. 248–255]) that there exists a maximal ordered subfield K_0 of K such that $K = K_0(i)$, where $i^2 = -1$. Once we fix the subfield K_0, we have a conjugation defined in K and we shall, therefore, denote in the usual manner the real and the imaginary parts and the absolute value of an element in K. A subset A of K is K_0-convex if $\sum_{j=1}^{n} \mu_j a_j \in A$ for every $a_j \in A$ and $\mu_j \in K_{0+}$ with $\sum_{j=1}^{n} \mu_j = 1$, where $K_{0+} = \{\alpha \in K_0 : \alpha \leq 0\}$. Let K_ω denote the compactification $K \cup \{\omega\}$ of K by adjoining a single element ω (called infinity) with the following operations:

(a) The subset K of K_ω preserves its initial field operations (e.g. addition and multiplication); and

(b) $a + \omega = \omega + a = \omega$, $\forall a \in K$, $a_\bullet \omega = \omega_\bullet a = \omega$, $\forall a \in K - \{0\}$, and $\omega^{-1} = 0$, $0^{-1} = \omega$.

For a fixed $\zeta \in K$, we define a permutation ϕ_ζ of K_ω by $\phi_\zeta(z) = 1/(z - \zeta)$ for every $z \in K_\omega$. We say that a subset A of K_ω is a *generalized circular region* ("*d.e.*" according to Zervos [9, p. 353]) of K_ω if either A is one of the sets ϕ, K, K_ω, or A satisfies the following two conditions:

(a) $\phi_\zeta(A)$ is K_0-convex for every $\zeta \in K - A$;

(b) $\omega \in A$ if A is not K_0-convex.

We shall denote by $D(K_\omega)$ the class of all generalized circular regions of K_ω. The empty set ϕ, K, K_ω, and the single point sets (and their complements in K_ω)

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are trivial members of $D(K_\omega)$. The characterization due to Zervos [9, pp. 372–287] of the class $D(K_\omega)$ when $K = \mathbf{C}$ (the field of complex numbers), leads to the following:

PROPOSITION 1.1 (Zervos [9, p. 352]).   *The non-trivial members of $D(\mathbf{C}_\omega)$ are the open interior (or exterior) of circles or the open half-planes, adjoined with a connected subset (possibly empty) of their boundary.*

The members of $D(\mathbf{C}_\omega)$ with all or no boundary points included are called (*classical*) circular regions of $D(\mathbf{C}_\omega)$. A homographic transformation [9, p. 353] T is a permutation on K defined by

$$T(z) = (az + b)/(cz + d)$$

for some elements $a, b, c, d \in K$ such that $ad - bc \neq 0$. In relation to homographic transformation an important result is the following:

PROPOSITION 1.2 (Zervos [9, p. 353]).   *Every   homographic   transformation permutes the class $D(K_\omega)$.*

The *ball* $S(a, r)$ and the *closed ball* $D(a, r)$, with center $a \in K$ and radius $r \in K_{0+}$ are defined, respectively, by

$$S(a, r) = \{z \in K : |z - a| < r\} \quad \text{and} \quad D(a, r) = \{z \in K : |z - a| \le r\}.$$

It can be seen (cf. [8, Proposition (3.4)']) that the ball and the closed ball are necessarily members of $D(K_\omega)$.

$\mathscr{F}_n(K)$ and Z_f will respectively, denote the class of all polynomials of degree n from K to K and the set of all zeros of a given polynomial f. If $f \in \mathscr{F}_n(K)$ such that $f(z) = \sum_{k=0}^{n} a_k z^k$, we define the *derivative* f' of f formally by $f'(z) = \sum_{k=1}^{n} k a_k z^{k-1}$ and the *successive* derivatives $f^{(j)}(z)$ of f by $\sum_{k=j}^{n} (j!)C(k, j)a_k z^{k-j}$, $j = 0, 1, \ldots$, with the convention that $f^{(0)} \equiv f$. Clearly the derivative of a constant polynomial is identically zero and $f^{(j)} \in \mathscr{F}_{n-j}(K)$ for every $j \le n$. If $P(z) = \lambda z^n$, $\lambda \in K$, we easily verify that

$$(1.1) \qquad P^{(k)}(z) = (k!)C(n, k)\lambda z^{n-k}, \quad k = 0, 1, \ldots, n.$$

From the definition of derivatives, it is also obvious that

$$(1.2) \qquad (f + g)' = f' + g'$$

for any polynomials f and g.

Let $C(n, k) = n!/k!(n - k)!$. If $f, g \in \mathscr{F}_n(K)$ and are given by

$$f(z) = \sum_{k=0}^{n} C(n, k)A_k z^k, \qquad g(z) = \sum_{k=0}^{n} C(n, k)B_k z^k,$$

we say [9, p. 362] that f and g are (mutually) *apolar* (written briefly as $f \perp g$)

if the following condition is satisfied:

$$\sum_{k=0}^{n} (-1)^k C(n, k) A_k B_{n-k} = 0.$$

This concept is too well known in the case when $K = \mathbf{C}$. The following result due to Zervos [9, p. 363] gives the relative location of the zeros of two apolar polynomials. For $K = \mathbf{C}$ and $A \in D(\mathbf{C}_\omega)$, this theorem is a fundamental result due to Grace [2] (see also Marden [4, Theorem (15.3)]).

THEOREM 1.3. *Let* $f, g \in \mathscr{F}_n(K)$ *and* $A \in D(K_\omega)$. *If* $Z_f \subseteq A$ *and* $f \perp g$, *then* $Z_g \cap A \neq \phi$.

Quite a number of mathematicians have studied the problem of determining the location of the zeros of composite polynomials (i.e. the polynomials derived in various ways by composition of two or more given polynomials). But this study has mostly been confined to polynomials in $\mathscr{F}_n(\mathbf{C})$ (a wide coverage of these topics can be seen in Marden [4]). However, in relation to polynomials in more general spaces, this problem has so far been tackled only by Zervos [9, p. 363] and Zaheer [8, Theorem (22.2)] in their attempts to generalize a (classical) result due to Szegö [5, Section 2, Theorem 2]. Our object, in this paper is to study certain composite polynomials in $\mathscr{F}_n(K)$.

2. Main theorem and its consequences

We shall primarily study the zeros of linear combinations of a polynomial and its (formal) derivatives. From our main theorem we shall obtain certain interesting results concerning the zeros of such composite polynomials. In order to prove our main theorem, we establish the following two lemmas. Lemma (2.1) offers a simple method for constructing polynomials apolar to a given polynomial and generalizes a classical result of Szegö [5] (see also Marden [4, Theorem (15, 2), p. 61]) and Lemma (2.2) is related to a result as given in Marden for $K = \mathbf{C}$ [4, p. 65].

LEMMA 2.1. *If a polynomial* $f(z) = \sum_{m=0}^{n} a_m z^m$ *in* $\mathscr{F}_n(K)$ *satisfies the relation* $\sum_{m=0}^{n} a_m \mu_m = 0$, *with* $\mu_0 \neq 0$, *then the polynomial* $G(z) = \sum_{m=0}^{n} (-1)^{n-m} C(n, m) \mu_m z^{n-m}$ *is apolar to* $f(z)$.

Proof. Trivial in view of the definition of apolarity.

LEMMA 2.2. *Let* $f \in \mathscr{F}_n(K)$ *and* $P(t) = t^n$ *for every* $t \in K$. *If* γ *is a zero of the polynomial*

$$(2.1) \qquad h(z) = \sum_{k=0}^{p} l_k f^{(k)}(z), \quad l_k \in K \ (l_0, l_p \neq 0, 0 \leq p \leq n),$$

then the polynomial $G(z) = \sum_{k=0}^{p} l_k P^{(k)}(\gamma - z)$ *is apolar to* $f(z)$.

Proof. Throughout the proof we shall follow the convention that $k! = 0$ if $k < 0$ and that $C(n, k) = 0$ for $k > n$ or $k < 0$. If $f(z) = \sum_{m=0}^{n} a_m z^m$, then (cf. (1.1) and (1.2))

$$f^{(k)}(z) = \sum_{m=k}^{n} (k!)C(m, k)a_m z^{m-k} = \sum_{m=0}^{n} (k!)C(m, k)a_m z^{m-k}.$$

From (2.1) we therefore obtain

$$h(z) = \sum_{k=0}^{p} l_k \left[\sum_{m=0}^{n} (k!)C(m, k)a_m z^{m-k} \right]$$

$$= \sum_{m=0}^{n} a_m \left[\sum_{k=0}^{p} (k!)C(m, k)l_k z^{m-k} \right].$$

This implies that $h(\gamma) = \sum_{m=0}^{n} a_m \mu_m = 0$, where

$$\mu_m = \sum_{k=0}^{p} (k!)C(m, k)l_k \gamma^{m-k}, \quad \mu_0 = l_0 \neq 0.$$

Hence by Lemma (2.1), the polynomial

$$G(z) = \sum_{m=0}^{n} (-1)^{n-m} C(n, m)\mu_m z^{n-m}$$

is apolar to $f(z)$. Now, we compute $G(z)$ in the desired form as follows:

$$G(z) = \sum_{m=0}^{n} (-1)^{n-m} C(n, m) z^{n-m} \left[\sum_{k=0}^{p} (k!)C(m, k)l_k \gamma^{m-k} \right]$$

$$= \sum_{k=0}^{p} l_k \left[\sum_{m=k}^{n} (-1)^{n-m} C(n, m)(k!)C(m, k)\gamma^{m-k} z^{n-m} \right]$$

$$= \sum_{k=0}^{p} l_k \left[\sum_{m=k}^{n} (k!)C(n, k)C(n-k, m-k)\gamma^{m-k}(-z)^{n-m} \right]$$

$$= \sum_{k=0}^{p} (k!)C(n, k)l_k \left[\sum_{m=0}^{n} C(n-k, m-k)\gamma^{m-k}(-z)^{(n-k)-(m-k)} \right]$$

$$= \sum_{k=0}^{p} (k!)C(n, k)l_k (\gamma - z)^{n-k}$$

$$= \sum_{k=0}^{p} l_k P^{(k)}(\gamma - z) \quad \text{(due to (1.1)).}$$

This completes the proof.

Now, we proceed to prove our main result.

THEOREM 2.3.   *Given $f \in \mathscr{F}_n(K)$, let us define*

$$h(z) = \sum_{k=0}^{p} l_k f^{(k)}(z), \quad l_k \in K \ (l_0, l_p \neq 0, 0 \leq p \leq n).$$

If $A \in D(K_\omega)$ such that $Z_f \subseteq A$, then every zero γ of $h(z)$ is of the form $\gamma = \alpha + \beta$ for some suitable choice of elements $\alpha \in A$ and $\beta \in Z_F$, where

$$F(z) = \sum_{k=0}^{p} (k!)C(n, k)l_k z^{n-k}.$$

Proof.   If $\gamma \in Z_h$, Lemma (2.2) implies that $f(z)$ is apolar to the polynomial

$$G(z) = \sum_{k=0}^{p} l_k P^{(k)}(\gamma - z),$$

where $P(t) = t^n$ for every $t \in K$. Since

$$P^{(k)}(\gamma - z) = (k!)C(n, k)(\gamma - z)^{n-k},$$

we have

$$G(z) = \sum_{k=0}^{p} (k!)C(n, k)l_k(\gamma - z)^{n-k}.$$

Since $Z_f \subseteq A$ and $f \perp G$, Theorem (1.3) implies the existence of an element $\alpha \in A$ such that $G(\alpha) = 0$. Hence

$$G(\alpha) = \sum_{k=0}^{p} (k!)C(n, k)l_k(\gamma - \alpha)^{n-k} = F(\gamma - \alpha).$$

Consequently, there exists a zero β of $F(z)$ such that $\gamma - \alpha = \beta$. That is $\gamma = \alpha + \beta$ and we are done.

As applications of the above theorem, we establish the following corollaries concerning some interesting polynomials in $\mathscr{F}_n(K)$.

COROLLARY 2.4.   *Given the polynomials $f(z) = \sum_{k=0}^{n} a_k z^k$, $g(z) = \sum_{k=0}^{n} b_k z^k$ in $\mathscr{F}_n(K)$, let us define*

$$h(z) = \sum_{k=0}^{n} (n-k)! \, b_{n-k} \, f^{(k)}(z).$$

If $A \in D(K_\omega)$ such that $Z_f \subseteq A$, then every zero γ of $h(z)$ is of the form $\gamma = \alpha + \beta$ for some suitable choice of elements $\alpha \in A$ and $\beta \in Z_g$.

Proof.   In the notations of Theorem (2.3), if we choose $l_k = (n-k)! \, b_{n-k}$, we see that $F(z) = (n!)g(z)$. Now the proof immediately follows from Theorem (2.3).

For the case when $K = \mathbf{C}$, the above corollary gives an improved version of a result due to Walsh [7] (cf. also [4, Theorem (18.1)]), improvement in the sense that we use generalized circular regions of $\mathbf{C}_\omega$ instead of the classical ones.

COROLLARY 2.5.   *Let $f \in \mathscr{F}_n(K)$ and $A \in D(K_\omega)$ such that $Z_f \subseteq A$. Then every zero γ of the polynomial*

$$h(z) = f(z) - \lambda f'(z), \; \lambda \in K - \{0\}$$

is of the form $\gamma = \alpha$ or $\gamma = \alpha + n\lambda$ for some $\alpha \in A$.

Proof.   Using the notations of Theorem (2.3) and putting $p = 1$, $l_0 = 1$ and $l_1 = -\lambda$, we obtain $F(z) = z^n - n\lambda z^{n-1} = z^{n-1}(z - n\lambda)$, so that the zeros β of $F(z)$ are given by $\beta = 0$ and $\beta = n\lambda$. Now Theorem (2.3) establishes our claim.

For $K = \mathbf{C}$, the above result gives an improved version of a result due to Marden (cf. [4, Corollary (18.1)]).

COROLLARY 2.6.   *If $f, g \in \mathscr{F}_n(K)$ such that $Z_f \subseteq S(0, r)$ and $Z_g \subseteq D(0, s)$, then all the zeros of the polynomial*

$$h(z) = \sum_{k=0}^{n} l_k \, f^{(k)}(z), \quad l_0 \neq 0$$

lie in $S(0, r + s)$ (cf. notations following Proposition (1.2)).

Proof.   Let $\gamma \in Z_h$. Then (cf. Theorem 2.3), there exist $\alpha \in S(0, r)$ and $\beta \in Z_g$ such that $\gamma = \alpha + \beta$, hence $|\gamma| \leq |\alpha| + |\beta| < r + s$ and the result follows.
For $K = \mathbf{C}$, the Corollary (2.6) is a result due to Kakeya [3] (see also [4, p. 86]).
As an application of Theorem (2.3) in the complex-plane, we give the following.

COROLLARY 2.7.   *Let $f \in \mathscr{F}_n(\mathbf{C})$ such that $Z_f \subseteq S(0, r)$. Then all the zeros γ of the polynomial*

$$h(z) = \sum_{k=0}^{p} \mu^k (n - k)! f^{(k)}(z), \quad 0 \leq p \leq n, \; \mu \in \mathbf{C} - \{0\},$$

lie in $S(0, r + |\mu|)$.

Proof.   In Theorem (2.3), putting $l_k = \mu^k(n - k)!$, we get

$$F(z) = (n!)(z^n + \mu z^{n-1} + \mu^2 z^{n-2} + \cdots + \mu^p z^{n-p})$$

$$= (n!)z^{n-p}(z^p + \mu z^{p-1} + \mu^2 z^{p-2} + \cdots + \mu^p)$$

$$= (n!)z^{n-p} \cdot \frac{z^{p+1} - \mu^{p+1}}{z - \mu}, \quad \forall z \neq \mu.$$

Obviously (since $F(\mu) \neq 0$), the only zeros of F are the origin ($z = 0$ is a zero of $F(z)$ only if $p < n$) and the roots other than μ of the equation $z^{p+1} = \mu^{p+1}$. If ω_k ($k = 0, 1, \ldots, p$) are the roots of this equation and if $\mu = re^{i\theta}$, then

$$\omega_k = r \exp\left[i\left(\theta + \frac{2k\pi}{p+1}\right)\right], \quad k = 0, 1, \ldots, p,$$

with $\omega_0 = \mu$. Therefore, the zeros β_k ($k = 0, 1, \ldots, p$) of F are given by $\beta_0 = 0$ (since $p < n$), $\beta_k = \omega_k$ ($k = 1, 2, \ldots, p$). Then any zero γ of $h(z)$ (by Theorem (2.3)) is either $\gamma = \alpha$ or $\gamma = \alpha + \omega_k$ ($k = 1, 2, \ldots, p$) for some complex number α such that $|\alpha| < r$. Finally, since $|\omega_k| = r = |\mu|$ for every $k = 1, 2, \ldots, p$, we see that either $|\gamma| = |\alpha| < r$ or $|\gamma| < r + |\mu|$. That is, in either case $|\gamma| < r + |\mu|$ and the result follows.

3. Some Examples

In this section, we give some examples to show that our results in Section 2 cannot be further generalized in certain directions. Our first example shows that Theorem (2.3) and Corollaries (2.4)–(2.6) do not hold, in general, for nonalgebraically closed fields of characteristic zero.

Example (3.1).   Let K_0 be a maximal ordered field, $A = \{-1\}$, and let

$$f(z) = z^3 + z^2 + z + 1 = (z^2 + 1)(z + 1), \quad \forall z \in K_0.$$

Then [6, pp. 233, 250] K_0 is a nonalgebraically closed field of characteristic zero, $f \in \mathscr{F}_3(K_0)$ such that $Z_f \subseteq A$ (since $z^2 + 1$ cannot vanish [1, p. 36]), where A is a generalized circular region of K_0. Taking $p = 1$, $l_0 = 1$ and $l_1 = -1$, we see that the polynomials $h(z)$ and $F(z)$ of Theorem (2.3) are, respectively, given by $h(z) = z[(z-1)^2 - 2]$, $F(z) = z^2 \cdot (z - 3)$. Now all the hypotheses in Theorem (2.3) are satisfied, whereas no zero of $h(z)$ can be expressed in the form as claimed by Theorem (2.3). That is, Theorem (2.3) does not hold (in general) for the field K_0.

The next example shows that Theorem (2.3) cannot be further generalized in the following sense: The generalized circular region A (employed in this result) cannot be replaced, in general, by generalized circular region adjoined with arbitrary subset of its boundary.

Example (3.2).   Take $K = \mathbf{C}$, $B = \{z \in \mathbf{C}\colon \text{Im}(z) < 0\} \cup \{-1, 7/5\}$, and $f(z) = 5z^2 - 2z - 7$ for $z \in \mathbf{C}$. Then $f \in \mathscr{F}_2(\mathbf{C})$ such that $Z_f \subseteq B$, where $B \notin D(\mathbf{C}_\omega)$ but the interior of B does belong to $D(\mathbf{C}_\omega)$. For $p = 1$, $l_0 = 1$, $l_1 = 1/2$, the polynomials $h(z)$, $F(z)$ of Theorem (2.3) are, respectively, given by $h(z) = 5z^2 + 3z - 8$, $F(z) = z(z + 1)$. Now it is easy to verify that no zero of $h(z)$ is of the form as claimed by Theorem (2.3). That is, it no longer holds when A is replaced by the above set B.

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