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REFLECTION COMPOUNDS AND CERTAIN r-BLOCKS OF FINITE CHEVALLEY GROUPS

BY
DAVID B. SUROWSKI

1. Introduction

Let G be a finite Chevalley group, of twisted or normal type, and let T be a Coxeter torus of G as defined by G. Lusztig in [6]. If ζ is the character of a unipotent representation of G and if x is a regular element in T, then Lusztig [6] showed that $\zeta(x) = 0$, 1 or -1. Moreover he determined which of the values is assumed for each such character ζ; see [6; 6.16, 7.8]. His methods essentially entail the study of the action of G on the eigenspaces of the Frobenius map on the l-adic cohomology space of the "Coxeter orbit". Therefore we feel that a thorough understanding of his work entails a substantial set of prerequisites.

In this note we shall re-interpret Lusztig's results in such a way as to obtain information about the principal r-blocks of G, corresponding to certain prime divisors r of $|T|$. Moreover, in case ζ is a principal series unipotent character, we shall supplant the sophisticated arguments of Lusztig by completely elementary ones. The present proof is available since the degrees of these characters have finally been computed for each family of Chevalley groups. The last case, that of $E_8(q)$, was finished by C. T. Benson in [1]. Using these degrees, we shall show that certain principal series unipotent characters are of r-defect 0, relative to certain prime divisors r of $|T|$. The remaining characters are the compounds of the reflection character, as defined by R. Kilmoyer in his thesis [5], and turn out to be nonexceptional characters in the principal r-blocks of G.

2. Coxeter primes and regular elements of T

We shall gather together some well known facts concerning a Coxeter torus T of G. Assume for the moment that G is not a Suzuki or Ree group; these cases will be considered later. Define the *Coxeter number* h_0 of G by $h_0 = [N_G(T): T]$. Set $m = h_0 \cdot \rho$, where ρ is the order of the twist yielding G. Thus $\rho = 1$ if G is untwisted. Let $\phi_m(x)$ denote the m-th cyclotomic polynomial in x.

LEMMA 1. (i) *$\phi_m(q) \mid |T|$, as polynomials in q.*

(ii) *If r is a prime divisor of $\phi_m(q)$, $r \nmid m$, then T contains a cyclic r-Sylow subgroup of G, and $r \equiv 1 (\text{mod } m)$.*

(iii) *$N_G(T)/T \simeq Z_{h_0}$.*

(iv) *If $n \in N_G(T)$ generates $N_G(T)$ (mod T), then $t^n = t^{q^\rho}$, for all $t \in T$.*

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The above facts can be found in or deduced from results in [6] of [10]. The second statement in (ii) follows from Theorem 94 in [8]. We shall call the primes r occurring in (ii) above the *Coxeter primes* of G. For the Suzuki and Ree groups we define the Coxeter primes to be the prime divisors of $|T|$. By using Theorem 94 in [8] one can show that for each Coxeter prime r, T contains a cyclic r-Sylow subgroup. Moreover, conclusions (iii) and (iv) of Lemma 1 are valid for these groups as well.

If $x \in T$ we say that x if *regular* if $C_N(x) = T$, where $N = N_G(T)$. This notion of regularity is the same as that given in [9; 6.10] and implies that given in [10]; see [9; 6.11].

As a corollary to Lemma 1 (iv), we see that if r is a Coxeter prime, and if $x \in T$ with $r \mid o(x)$, then x is regular. Unfortunately, the converse need not hold, as there are counterexamples in the classical groups. In the other hand, it is easily seen that the nonregular elements of T are all contained in a subgroup of T of order $O(q^{n-1})$, where $n = \text{rank } G$, and where $O(q^{n-1})$ is a polynomial in q of degree less than or equal to $n-1$. Thus the regular elements of T whose orders are divisible by some Coxeter prime comprise "almost all" of the regular elements of T.

3. Character values on r-singular elements

We maintain the same assumptions as in Section 2. Let B be a Borel subgroup of G and let 1_B^G be the permutation character of the action of G on the cosets of B. Thus the principal series unipotent characters are precisely the irreducible character constituents of 1_B^G. For any such character ζ, $\zeta(1)$ is a polynomial in q and can be found in one of the sources [1], [2], [4], [11], [12]. Let π denote the reflection character of G, with compounds $\pi^{(0)} = 1_G$, $\pi^{(1)} = \pi$, $\ldots$, $\pi^{(n)}$, as defined by R. Kilmoyer in [5].

PROPOSITION 1. *Let ζ be a principal series unipotent character of G. Then for any prime power q, we have $\phi_m(q) \mid m \cdot \zeta(1)$ if and only if ζ is not a compound of the reflection character.*

For the exceptional groups, the above fact is easy to check. For the classical groups, one checks that ζ is a compound of π if and only if ζ is parametrized by a partition of the form $[k, 1^{n+1-k}]$ for type A_n, and by a double partition of the form $[k]$, $[1^{n-k}]$ for types B_n, C_n and D_n. However, from the degree formulas of P. Hoefsmit in [4], one checks that these are the only partitions that yield characters ζ with $\phi_m(q) \nmid m \cdot \zeta(1)$.

COROLLARY. *Let r be a Coxeter prime of G and let ζ be a principal series unipotent character. Then ζ is of r-defect 0 if and only if ζ is not a compound of the reflection character.*

THEOREM 1. *Let ζ be a principal series unipotent character, and let r be a Coxeter prime of G. Let $x \in T$ with $r \mid o(x)$. Then*

$$\zeta(x) = \begin{cases} 0 & \text{if } \zeta \text{ is not a reflection compound,} \\ (-1)^i & \text{if } \zeta = \pi^{(i)}. \end{cases}$$

The above result re-establishes part of Lusztig's result [6; 6.16], except that our treatment is valid only for "almost all" regular elements of T.

Proof of Theorem 1. If $\zeta \neq \pi^{(i)}$ then by the corollary to Proposition 1, we have that $\zeta(x) = 0$. Thus we assume that $\zeta = \pi^{(i)}$, and work by induction on i.

Let J be a subset of a set R of fundamental reflections in the Weyl group W of G, and let P_J be the corresponding parabolic subgroup of G. By [5, Theorem 9] we have

$$(\pi^{(i)}, 1_{P_J}^G) = \binom{|R - J|}{i}.$$

Now pick J such that $i = |R - J|$. Since x is not contained in any proper parabolic subgroup of G, we have

$$0 = 1_{P_J}^G(x) = \sum_{j=0}^{i} \binom{|R - J|}{j} \pi^{(j)}(x) = \pi^{(i)}(x) - (-1)^i,$$

and so the result follows.

Note that the above result is valid for the Suzuki groups ${}^2B_2(q^2)$ $(q^2 = 2^{2n+1})$, and for the Ree groups ${}^2G_2(q^2)$ $(q^2 = 3^{2n+1})$, since these groups have a doubly transitive permutation representation on the cosets of a Borel subgroup. Thus it remains to consider the Ree groups ${}^2F_4(q^2)$ $(q^2 = 2^{2n+1})$.

From [5; Section 4] one can obtain the degrees of the irreducible constituents of of 1_B^G, $G = {}^2F_4(q^2)$. They are as follows:

$$1, \quad \tfrac{1}{4}q^4(q^2+1)(q^2-\sqrt{2}q+1)^2(q^6+1)(q^4+\sqrt{2}q^3+q^2+\sqrt{2}q+1),$$

$$q^{24}, \quad q^{10}(q^8-q^4+1)(q^4-q^2+1), \quad q^2(q^8-q^4+1)(q^4-q^2+1),$$

$$\tfrac{1}{2}q^4(q^4+1)(q^{12}+1),$$

and

$$\tfrac{1}{4}q^4(q^2+1)(q^2+\sqrt{2}q+1)^2(q^6+1)(q^4-\sqrt{2}q^3+q^2-\sqrt{2}q+1).$$

The first three degrees are those of $\pi^{(0)}$, $\pi^{(1)}$ and $\pi^{(2)}$, respectively. Since $|T| = q^4 - \sqrt{2}q^3 + q^2 - \sqrt{2}q + 1$, we conclude that the corollary to Proposition 1 is valid for G of type ${}^2F_4$. But then the arguments in the proof of Theorem 1 work, proving our result in this case, as well.

4. Principal r-blocks

Let r be a Coxeter prime and let $B_0^{(r)}$ denote the principal r-block of G. Then Theorem 1 together with the work of E. C. Dade in [3] guarantee that each reflection compound is a non-exceptional character in $B_0^{(r)}$. Whenever the rank of G equals $h_0 - 1$, we conclude from Dade's work that the reflection compounds exhaust the non-exceptional characters in $B_0^{(r)}$. However, this happens only for the groups $A_n(q)$ and ${}^2D_n(q^2)$. In the remaining cases we must appeal to the full force of Lusztig's work. In [6] he constructed the G-module $\oplus H_c^i(X(\mathcal{O}))$, whose irreducible constituents are precisely those unipotent representations whose characters assume constant value 1 or -1 on the regular elements of a Coxeter torus T. Thus, each such character is a non-exceptional character in $B_0^{(r)}$. By [6; 7.9] these characters correspond to the compounds of the reflection representation for a suitable "generalized Hecke algebra"; see also [7; 3.25]. Moreover, Lusztig showed that $\oplus H_c^i(X(\mathcal{O}))$ affords exactly h_0 such characters. By Dade's work, this is exactly the number of non-exceptional characters in $B_0^{(r)}$.

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Kansas State University
Manhattan, Kansas