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THE SPAN OF ALMOST CHAINABLE HOMOGENEOUS CONTINUA

BY

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In this paper we prove that every almost chainable homogeneous continuum has span zero. We do this by modifying an argument given by C. L. Hagopian in [4] in which he proved that every almost chainable homogeneous continuum has the fixed point property.

A *continuum* is a nondegenerate compact connected metric space. A *mapping* is a continuous function. A continuum M is *homogeneous* if for each pair (x, y) of points of M, there is a homeomorphism h from M onto M such that $h(x) = y$. If ε is a positive number, an *ε-cover* of a continuum M is a cover of M by open subsets of M each having diameter less than ε. A continuum M is *almost chainable* if for each positive number ε there exists an ε-cover $\mathscr{D}$ of M and a chain $\mathscr{C} = \{C(i)\colon 1 \leq i \leq n\}$ of members of $\mathscr{D}$ such that no member of $\mathscr{D} - \mathscr{C}$ intersects a member of $\{C(i)\colon 2 \leq i \leq n\}$, and such that for each point p of M the distance from p to some member of $\mathscr{C}$ is less than ε. This definition was introduced by C. E. Burgess in [1].

If M is a set, the first and second projections of $M \times M$ onto M will be denoted by π_1 and π_2 respectively. If M is a continuum, the *surjective span* of M, $\sigma^*(M)$, is the least upper bound of the set of all numbers ε for which there exists a subcontinuum Z of $M \times M$ such that $\pi_1(Z) = \pi_2(Z) = M$ and $d(x, y) \geq \varepsilon$ for each point (x, y) in Z. The *surjective semispan* of M, $\sigma_0^*(M)$, is the least upper bound of the set of all numbers ε for which there exists a subcontinuum Z of $M \times M$ such that $\pi_1(Z) = M$ and $d(x, y) \geq \varepsilon$ for each point (x, y) in Z. The *span*, $\sigma(M)$, and *semispan*, $\sigma_0(M)$, of M are defined by the following formulae:

$$\sigma(M) = \text{l.u.b. } \{\sigma^*(A)\colon A \text{ is a subcontinuum of } M\}$$

and

$$\sigma_0(M) = \text{l.u.b. } \{\sigma_0^*(A)\colon A \text{ is a subcontinuum of } M\}.$$

The various notions of span were introduced by A. Lelek [6], [7], [8]. In [6, p. 210] Lelek observes that chainable continua have span zero.

If $\mathscr{G}$ is a collection of sets, $\mathscr{G}^*$ will denote the union of the members of $\mathscr{G}$.

In [3] Hagopian used a theorem of E. G. Effros [2, Theorem 2.1] to prove that if M is a homogeneous continuum, x is a point of M, and ε is a positive

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number, then x belongs to an open subset, G, of M which possesses what he terms the *ε-push property*, that is, if each of y and z is a point of G, there is a homeomorphism h from M onto M such that $h(y) = z$ and $d(t, h(t)) < \varepsilon$ for each point t of M.

THEOREM 1. *If M is an almost chainable homogeneous continuum then* $\sigma_0^*(M) = 0$.

Proof. Suppose M is an almost chainable homogeneous continuum and suppose that $\sigma_0^*(M) > 0$. Then there exist a subcontinuum Z of $M \times M$ such that $\pi_1(Z) = M$ and a positive number ε such that if (s, t) is a point in Z then $d(s, t) > 4\varepsilon$.

For each positive integer j, let $\mathscr{D}_j$ be a $1/j$-cover of M and let

$$\mathscr{C}_j = \{C(i, j)\colon 1 \leq i \leq n_j\}$$

be a chain of members of $\mathscr{D}_j$ such that no member of $\mathscr{D}_j - \mathscr{C}_j$ intersects a member of

$$\{C(i, j)\colon 2 \leq i \leq n_j\}$$

and such that for each point p of M the distance from p to some member of $\mathscr{C}_j$ is less than $1/j$. Since $\pi_1(Z) = M$, for each positive integer j there is a point in Z with first projection in $C(n_j, j) - (C(n_j, j) \cap C(n_j - 1, j))$. Let (p_j, q_j) be such a point, and let

$$V_j = \{C(i, j)\colon 2 \leq i \leq n_j\}^* \times M.$$

The set V_j is an open subset of $M \times M$ containing (p_j, q_j). For each positive integer j let K_j be the closure of the component of $Z \cap V_j$ containing (p_j, q_j). Since $\pi_1(Z) = M$, K_j contains a point, (x_j, y_j), on the boundary of V_j [10, Theorem 50, p. 18].

Since M is homogeneous, by the previously mentioned result of Hagopian [3, Lemma 4], there is a finite collection $\mathscr{G}$ of open subsets of M covering M such that each member of $\mathscr{G}$ has the ε-push property. For each set U in $\mathscr{G}$ let $s(U)$ be a point of U. There is a positive number δ such that if t is a point in M, U is a set in $\mathscr{G}$, and $d(s(U), t) < 4\delta$ then t is in U. Let j be a positive integer such that $1/j < \min\{\varepsilon, \delta\}$ and $\mathscr{C}_j$ has at least three links. Each member of $\mathscr{G}$ includes a member of $\{C(i, j)\colon 3 \leq i \leq n_j\}$.

Let U be a member of $\mathscr{G}$ containing y_j. Since U has the ε-push property there is a homeomorphism h from M onto M such that $h(y_j)$ is a point of some member of $\{C(i, j)\colon 3 \leq i \leq n_j\}$, and such that if t is a point in M then $d(t, h(t)) < \varepsilon$.

Let $L_j = \{(s, h(t))\colon (s, t) \in K_j\}$. Since $1_M \times h$ is continuous, L_j is a continuum (1_M denotes the identity on M). Furthermore if $(s, h(t))$ is in L_j then

$$(*) \qquad d(s, h(t)) \geq d(s, t) - d(t, h(t)) \geq 3\varepsilon.$$

We define the following open subsets of $M \times M$:

$$A = \{C(i_1, j) \times C(i_2, j) \colon 1 \le i_1 \le n_j,\ i_2 \le n_j \quad \text{and} \quad i_1 + 2 \le i_2\}^*,$$

$$B_1 = \{C(i_1, j) \times C(i_2, j) \colon 1 \le i_2 \le n_j,\ i_1 \le n_j \quad \text{and} \quad i_2 + 2 \le i_1\}^*,$$

$$B_2 = \{C(i, j) \times V \colon 1 \le i \le n_j \quad \text{and} \quad V \in \mathscr{D}_j - \mathscr{C}_j\}^*,$$

$$B = B_1 \cup B_2,$$

$$D = \{C(i_1, j) \times C(i_2, j) \colon 1 \le i_1 \le n_j,\ 1 \le i_2 \le n_j \quad \text{and} \quad |i_1 - i_2| \le 1\}^*.$$

Since $h(y_j)$ is in $\{C(i, j) \colon 3 \le i \le n_j\}^*$ and x_j is in $C(1, j)$, $(x_j, h(y_j))$ is in A. Since p_j is in $C(n_j, i)$ and, by (*), $d(p_j, h(q_j)) \ge 3\varepsilon$, $h(q_j)$ is not in

$$C(n_j, j) \cup C(n_j - 1, j).$$

Thus $(p_j, h(q_j))$ is in B. It follows directly from (*) that no point in L_j is in D. Thus $L_j \subset A \cup B$, and L_j intersects each of A and B. Moreover $\bar{V}_j \subset A \cup B \cup D$, $L_j \subset \bar{V}_j$ and A and B are mutually exclusive. This is a contradiction since L_j is a continuum. This proves Theorem 1.

A continuum M has the *incidence point property* provided that if T is a continuum, f is a mapping of T onto M, and g is a mapping of T onto a subset of M then there is a point x of T such that $f(x) = g(x)$. We have the following strengthening of Hagopian's Theorem [4]:

THEOREM 2. *If M is an almost chainable homogeneous continuum then M has the incidence point property.*

Proof. Suppose M is an almost chainable homogeneous continuum, T is a continuum, f is a mapping from T onto M and g is a mapping from T onto a subset of M. Let $Z = \{(f(x), g(x)) \mid x \in T\}$. The set Z is a subcontinuum of $M \times M$ and $\pi_1(Z) = M$. By Theorem 1, $\sigma_0^*(M) = 0$, and hence there exists a point x of T such that $d(f(x), g(x)) = 0$. Thus M has the incidence point property.

THEOREM 3. *If M is an almost chainable homogeneous continuum then $\sigma_0(M) = 0$, and thus $\sigma(M) = 0$.*

Proof. Suppose M is an almost chainable homogeneous continuum. Suppose A is a subcontinuum of M. If $A = M$ then $\sigma_0^*(A) = 0$ by Theorem 1. If A is a proper subcontinuum of M then A is a pseudo-arc by a theorem of Burgess [1, Theorem 5], and thus A is chainable. Hence, by a result of Lelek [7, p. 44], $\sigma_0^*(A) = 0$. Therefore $\sigma_0(M) = 0$, and thus $\sigma(M) = 0$.

Lelek has proved [8], [9] that if M is a continuum and $\sigma(M) = 0$ then M is tree-like. Thus we have the following:

COROLLARY 1. *If M is an almost chainable homogeneous continuum then M is tree-like.*

Burgess proved [1, Theorem 13] that every k-junctioned tree-like homogeneous continuum is almost chainable (for the definition of a k-junctioned tree-like continuum see [1] or [3]). Thus we have the following:

COROLLARY 2. *If M is a k-junctioned tree-like homogeneous continuum then $\sigma_0(M) = 0$, and M has the incidence point property.*

W. T. Ingram has constructed and studied an uncountable collection of atriodic 1-junctioned tree-like continua each having positive span and no two of which are homeomorphic. In [5] he proved that no member of this collection is homogeneous. Corollary 2 provides an alternate proof of this fact.

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