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# THE VARIETY OF POINTS WHICH ARE NOT SEMI-STABLE

BY

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## 1. Introduction

(1.1) **Background.** Let $k$ be an algebraically closed field and let $V$ be a finite-dimensional vector space over $k$. Let $G$ be a reductive algebraic subgroup of $GL(V)$.

Let $k[V]$ be the algebra of regular functions on $V$. The group $G$ acts on $k[V]$ as follows:

$$(g \cdot f)(v) = f(g^{-1}v)$$

for all $f \in k[V]$, $g \in G$, and $v \in V$. The ring of $G$-invariant functions on $V$ is

$$k[V]^G = \{f \in k[V]: g \cdot f = f \quad \text{for all } g \in G\}.$$

We define an algebraic subvariety $X$ of $V$ by

$$X = \{v \in V: f(v) = 0 \text{ for each non-constant homogeneous } f \in k[V]^G\}.$$

A point in $V$, not in $X$, is called *semi-stable*.

In order to describe the points in $X$, it is useful to introduce the concept of an orbit. Let $v$ be in $V$. The *orbit* of $v$ with respect to the action of $G$ is

$$G \cdot v = \{g \cdot v: g \in G\}$$

The Zariski-closure of $G \cdot v$ will be denoted by cl $(G \cdot v)$.

THEOREM. *Let $G$ be a connected reductive algebraic subgroup of $GL(V)$. Let $v \in V$. The following statements are equivalent:*

(a) *$v$ is not semi-stable*;

(b) $0 \in \text{cl}\,(G \cdot v)$;

(c) *there is a one-parameter subgroup $\lambda$ of $G$ so that $\lambda(\alpha) \cdot v \to 0$ as $\alpha \to 0$.*

The notation in (c) will be explained in (2.1). The equivalence of (a), (b), and (c) is proved in [10; Sections 1 and 2] taking into account [5].

(1.2) **Summary of results.** The purpose of this paper is to prove some results aimed at explicitly describing the set $X$. The basic theorem is proved in (2.2). As consequences of this theorem, the following corollaries are proved in (2.3) and (3.2).

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(1) Let $G$ be a connected reductive algebraic subgroup of $GL(V)$ and let $B = TU$ be a Borel subgroup of $G$. Let $v_0$ be a point in $V$ which is not semi-stable. Then there is a one-parameter subgroup $\lambda_0 \colon G_m \to B$ such that $\lambda_0(\alpha)v_0 \to 0$ as $\alpha \to 0$.

(2) Let $G$ and $B$ be as in (1). There exist subspaces $W_1, \dots, W_r$ of $V$ such that the following statements hold:
(a) each $W_i$ is $B$-invariant;
(b) $X = G \cdot W_1 \cup \cdots \cup G \cdot W_r$;
(c) each $G \cdot W_i$ is closed.

(3) Let $G$ be as in (1) and suppose that $G$ acts irreducibly on $V$. Let $v_0 \in V$ be a point which is not semi-stable. Then the highest weight vector of $G$ is in cl $(G \cdot v_0)$.

(1.3) **Existence of semi-stable points.** Semi-stable points exist and form an open set if dim $V >$ dim $G$ and $G$ is semisimple. This fact follows from several well-known theorems but does not seem to have been stated before. We give the proof now.

THEOREM. *Let $G$ be a connected semisimple algebraic subgroup of $GL(V)$. Let*

$$m = \max \{\dim (G \cdot v) \colon v \in V\}.$$

*If* dim $V > m$, *then* $V - X$ *is not empty.*

*Proof.* Let $k(V)$ be the field of rational functions on $V$. The group $G$ acts on $k(V)$ via

$$(g \cdot f)(v) = f(g^{-1}v)$$

for all $f \in k(V)$, $g \in G$, and $v \in V$. Let

$$k(V)^G = \{f \in k(V) \colon g \cdot f = f \quad \text{for all } g \in G\}.$$

We begin by showing that $k(V)^G$ is the quotient field of $k[V]^G$. Let $f = a/b$ be in $k(V)^G$ where $a, b \in k[V]$. Let $a = p_1 \cdots p_r$ and $b = q_1 \cdots q_s$ be the factorizations of $a$ and $b$ into prime elements where we shall assume $a$ and $b$ have no common factors. There is a finite-dimensional subspace $E$ of $k[V]$ which is $G$-invariant and contains each $p_i$ and $q_j$ [6; Proposition, p. 62]. For $h \in k[V]$, let

$$\langle h \rangle = \{ch \colon c \in k\}.$$

Now since $g \cdot f = f$, we see that $(gb)a = (ga)b$. Since $a$ and $b$ have no common factors, each $gp_i$ is a multiple of some $p_j$. Thus $G$ permutes $\langle p_1 \rangle \cdots \langle p_r \rangle$. Since $G$ is connected, $G\langle p_i \rangle = \langle p_i \rangle$ for all $i = 1, \dots, r$. Hence, there are constants $c_g \in k$ satisfying

$$g \cdot p_i = c_g p_i \quad \text{for all } g \in G.$$

The map $g \to c_g$ is a character of $G$. But $G$ is semisimple so each character is trivial. Therefore, $g \cdot p_i = p_i$ for all $g \in G$ and $G$ fixes $a$. It follows that $G$ fixes $b$.

Next, let $m = \max \{\dim G \cdot v \colon v \in V\}$. According to a theorem of M. Rosenlicht [8; pp. 406–407], $\dim k(V)^G = \dim V - m$. If $\dim V > m$, then $\dim k(V)^G > 0$. By what was proved above, there are non-constant functions in $k[V]^G$ and, so, $X \neq V$.

## 2. The theorem and its corollaries

(2.1) **Preliminaries.** We begin this section by reçalling some notation and definitions along with some of the concepts in [7].

(1) A *one-parameter subgroup* $\lambda$ of an algebraic group $G$ is a homomorphism $\lambda\colon G_m \to G$ (where $G_m$ is the multiplicative group $k^* = k - \{0\}$).

Let $f\colon G_m \to X$ be a morphism of algebraic varieties. If $f$ extends to a morphism $f_1\colon G_a \to X$, then $y = f_1(0)$ is called the specialization of $f(\alpha)$ as $\alpha$ specializes to 0. We shall denote this by $f(\alpha) \to y$ as $\alpha \to 0$.

(2) Let $\lambda\colon G_m \to GL(V)$ be a one-parameter subgroup. There is a basis $v_1, \ldots, v_n$ of $V$ and integers $e_1, \ldots, e_n$ so that

$$\lambda(\alpha)v_i = \alpha^{e_i}v_i$$

for $i = 1, \ldots, n$ [6; 16.1]. Let

$$V(e_i) = \{v \in V\colon \lambda(\alpha)v = \alpha^{e_i}v\}.$$

Next, we define a subspace $W(\lambda)$ of $V$ by

$$W(\lambda) = \{v \in V\colon \lambda(\alpha)v \to 0 \text{ as } \alpha \to 0\}.$$

Then it is easily verified that $W(\lambda)$ is the direct sum of those subspaces $V(e_i)$ where $e_i > 0$.

(3) [10; Lemma 3.1]. Let $G$ be a reductive algebraic group and let $\lambda\colon G_m \to G$ be a one-parameter subgroup of $G$. There is a unique algebraic subgroup $P(\lambda)$ in $G$ such that $p$ is in $P(\lambda)$ if and only if $\lambda(\alpha)p\lambda(\alpha^{-1})$ has a specialization in $G$ when $\alpha$ specializes to 0. Moreover, $P(\lambda)$ is a parabolic subgroup of $G$.

For $g \in G$, let $g\lambda g^{-1}$ denote the one-parameter subgroup of $G$ defined by

$$\alpha \to g\lambda(\alpha)g^{-1}.$$

It is not hard to check that $P(g\lambda g^{-1}) = gP(\lambda)g^{-1}$.

(4) Let $G$ be a reductive algebraic group and let $\rho\colon G \to GL(V)$ be a representation of $G$. Let $\lambda\colon G_m \to G$ be a one-parameter subgroup of $G$. Let $W(\lambda)$ and $P(\lambda)$ be as (2) and (3). Then $P(\lambda) \cdot W(\lambda) \subset W(\lambda)$.

*Proof.* Let $p \in P(\lambda)$ and $v \in W(\lambda)$. Then

$$\lambda(\alpha)pv = \lambda(\alpha)p\lambda(\alpha^{-1})\lambda(\alpha)v \to 0 \quad \text{as } \alpha \to 0.$$

(2.2) THEOREM. *Let $G$ be a connected reductive algebraic subgroup of $GL(V)$. Let $B = TU$ be a Borel subgroup of $G$ and let $W(T) = N(T)/T$ be the Weyl group of $T$. Let $v_0$ be a point in $V$ which is not semi-stable. There is a one-parameter subgroup $\lambda\colon G_m \to T$ such that the following statements hold:*
(a) *$B \subset P(\lambda)$, i.e., if $\mu$ is a root of $B$ relative to $T$, then $\langle \mu, \lambda \rangle \geq 0$;*
(b) *$B \cdot W(\lambda) \subset W(\lambda)$;*
(c) *there are elements $u \in U$, $sT \in W(T)$ such that $v_0 \in us \cdot W(\lambda)$.*

*Proof.* According to statement (c) of the theorem in 1.1, there is a one-parameter subgroup $\lambda_0$ of $G$ such that $v_0 \in W(\lambda_0)$. Let $B_0$ be a Borel subgroup in $P(\lambda_0)$ such that each $\lambda_0(\alpha)$ is in $B_0$. There is an element $g \in G$ such that $B = gB_0g^{-1}$ and an element $b \in B$ so that

$$b(g\lambda_0(\alpha)g^{-1})b^{-1} \in T$$

for all $\alpha \in G_m$. Let $\lambda = (bg)\lambda_0(bg)^{-1}$.

To prove statement (a), we use preliminary (3) above to see that

$$P(\lambda) = (bg)P(\lambda_0)(bg)^{-1} \supset (bg)B_0(bg)^{-1} = B.$$

Also, we recall that there is an isomorphism $\varepsilon_\mu$ from $G_a$ into $G$ such that for all $t \in T$, $x \in G_a$, we have

$$t\varepsilon_\mu(x)t^{-1} = \varepsilon_\mu(\mu(t)x) \qquad [6; \text{Theorem, p. } 161].$$

Hence,

$$\lambda(\alpha)\varepsilon_\mu(x)\lambda(\alpha^{-1}) = \varepsilon_\mu(\alpha^e x)$$

where, by definition, $e = \langle \mu, \lambda \rangle$. We now apply (3) again to see that $\langle \mu, \lambda \rangle \geq 0$.

Statement (b) follows from (a) and preliminary (4). To prove (c), we first note that $W(\lambda) = bgW(\lambda_0)$ so that $v_0 \in G \cdot W(\lambda)$. Now, according to the Bruhat decomposition of $G$, we have $G = \cup\, UsB$ where $sT$ ranges over all the distinct cosets of the Weyl group $W(T) = N(T)/T$. Hence,

$$G \cdot W(\lambda) = \cup\, UsB \cdot W(\lambda) = \cup\, Us \cdot W(\lambda)$$

according to $(b)$. This proves (c).

(2.3) **Consequences.** Throughout this section, we shall denote by $G$ a connected reductive algebraic subgroup of $GL(V)$ and by $B = TU$ a given Borel subgroup of $G$.

COROLLARY 1. *Let $v_0$ be a point in $V$ which is not semistable. There is a one-parameter subgroup $\lambda_0\colon G_m \to B$ such that $\lambda_0(\alpha)v_0 \to 0$ as $\alpha \to 0$.*

*Proof.* According to (2.2), there is a one-parameter subgroup $\lambda: G_m \to T$ and elements $u \in U$, $sT \in W(T)$ such that $v_0 \in us \cdot W(\lambda)$. Let

$$\lambda_0(\alpha) = (us)\lambda(\alpha)(us)^{-1}$$

for all $\alpha \in G_m$. Then $\lambda_0$ is a one-parameter subgroup of $B$ since

$$\lambda_0(\alpha) = (us)\lambda(\alpha)(us)^{-1} \subset uTu^{-1} \subset B.$$

Furthermore, $\lambda_0(\alpha)v_0 \to 0$ as $\alpha \to 0$. For if $v_0 = us \cdot w$ with $w \in W(\lambda)$, then

$$\lambda_0(\alpha)v_0 = us\lambda(\alpha)s^{-1}u^{-1}usw = us\lambda(\alpha)w.$$

LEMMA. *Let $X$ be a closed subset of $V$ and let $P$ be a parabolic subgroup of $G$. If $P \cdot X$ is closed, then $G \cdot X$ is closed.*

*Proof.* Let $P$ act on the right on $G \times V$ by $(g, v) \cdot p = (gp, v)$. The quotient variety $(G \times V)/P$ exists and is $(G/P) \times V$[4; 6.6, Corollary]. Let

$$\pi: G \times V \to (G/P) \times V$$

be the quotient morphism. Then $\pi$ is open. Let

$$A = \{(g, v) \in G \times V: g^{-1}v \in P \cdot X\}$$

Since $A$ is the inverse image of $P \cdot X$ under the morphism $G \times V \to V$ defined by $(g, v) \to g^{-1}v$, we see that $A$ is closed. It is easily verified that $\pi^{-1}(\pi(A)) = A$ and, so, $\pi(A)$ is closed in $(G/P) \times V$ (since $\pi$ is open). Now $G/P$ is complete so the image $G \cdot X$ of $\pi(A)$ under the projection map $(G/P) \times V \to V$ is closed in $V$.

*Note.* The proof above is a slight extension of one in [3; Lemma 6.3]. A short "transcendental" proof can be given when $K = \mathbf{C}$. For then, $G = KP$ where $K$ is compact [11; Theorem 1, p. 102] and, so, $G \cdot X = K \cdot (P \cdot X))$. But $K \cdot (P \cdot X)$ is closed since compact transformation groups send closed sets to closed sets.

COROLLARY 2. *Let $X$ be the set of points in $V$ which are not semi-stable. There are one-parameter subgroups $\lambda_1, \ldots, \lambda_r$ of $T$ such that the following statements hold:*

(a) *$B \subset P(\lambda_i)$ and $B \cdot W(\lambda_i) \subset W(\lambda_i)$ for all $i = 1, \ldots, r$;*

(b) *each $G \cdot W(\lambda_i)$ is closed;*

(c) *$X = G \cdot W(\lambda_1) \cup \cdots \cup G \cdot W(\lambda_r)$ and this is the unique decomposition of $X$ into irreducible components unless there exist $i, j$, $i \neq j$, $sT \in W(T)$ such that $W(\lambda_i) \subset s \cdot W(\lambda_j)$.*

*Proof.* Let $T$ have weights $\chi_1, \ldots, \chi_n$ on $V$ and let

$$V(\chi) = \{v \in V: tv = \chi(t)v \quad \text{for all } t \in T\}.$$

Next let $\lambda$ be a one-parameter subgroup of $T$. Let $\chi$ be one of the weights above and put $e = \langle \chi, \lambda \rangle$. Then $\lambda(\alpha)v = \alpha^e v$ for all $v \in V(\chi)$. Therefore, $V(\chi) \subset W(\lambda)$ if and only if $e > 0$. It follows that there are (finitely many) one-parameter subgroups $\lambda_1, \ldots, \lambda_r$ of $T$ such that (i) $B \subset P(\lambda_i)$ and (ii) if $\lambda: G_m \to T$ is any one-parameter subgroup such that $B \subset P(\lambda)$, then $W(\lambda) = W(\lambda_i)$ for some $i = 1, \ldots, r$.

Statements (a), (b), and (c) follow from the theorem and lemma above, except for the decomposition of $X$.

Let us write $W_i = W(\lambda_i)$ for $i = 1, \ldots, r$. Now suppose that

$$G \cdot W_i \subset G \cdot W_1 \cup \cdots \cup G \cdot W_{i-1} \cup G \cdot W_{i+1} \cup \cdots \cup G \cdot W_r.$$

Since $W_i$ is irreducible, there is a $j \neq i$ so that $W_i \subset G \cdot W_j$. Applying the Bruhat decomposition of $G$, we now see that

$$W_i \subset \cup\, UsB \cdot W_j = \cup\, UsW_j.$$

But $W_i$ is $U$-invariant so $W_i \subset \cup\, sW_j$. Since $W_i$ is irreducible, we obtain the desired result that $W_i \subset s \cdot W_j$ for some $sT \in W(T)$.

COROLLARY 3. *Suppose that there is an element $sT$ in the Weyl group of $G$ so that $s\chi = -\chi$ for all weights $\chi$ of $T$. Let $X$ be the set of points in $V$ which are not semi-stable. Then* $\dim X \leq \frac{1}{2} \dim V + \dim U$.

*Proof.* Let us use the notation for $V(\chi)$ introduced in the proof of Corollary 2. Let $\lambda_i: G_m \to T$ be as in Corollary 2. If $\chi$ is a weight of $T$ on $V$ and if $V(\chi) \subset W(\lambda_i)$, then $V(-\chi) \cap W(\lambda_i) = \{0\}$. Since $s \cdot V(\chi) = V(-\chi)$, we have $\dim V(\chi) = \dim V(-\chi)$. Thus $\dim W(\lambda_i) \leq \frac{1}{2} \dim V$. The statement about $\dim X$ now follows from the Bruhat decomposition of $G$ and the fact that $B \cdot W(\lambda_i)$ is contained in $W(\lambda_i)$. For $G \cdot W(\lambda_i) = \cup\, UsB \cdot W(\lambda_i) = \cup\, UsW(\lambda_i)$.

*Notes.* Let $G$ be a simple algebraic group, not of type $A_n$, $D_n$ ($n$ odd), or $E_6$. Then there is an element $sT$ in the Weyl group of $G$ satisfying $s\chi = -\chi$ for all weights $\chi$ of $T$ [11; p. 226].

(2.4) **Properly stable points.** Let $G$ be a reductive algebraic subgroup of $GL(V)$. A point $v$ in $V$ is called *properly stable* if the orbit $G \cdot v$ is closed and has dimension equal to that of $G$. A point $v$ in $V$ is not properly stable if and only if there is a one-parameter subgroup $\lambda: G_m \to G$ such that $\lambda(\alpha) \cdot v$ has a specialization in $V$ as $\alpha$ specializes to 0 [10; Section 2].

In case char $k = 0$, one may prove the following result analogous to the Theorem of (1.3): Let $G$ be a connected semsimple algebraic group and let $\rho: G \to GL(V)$ be a finite-dimensional representation of $G$. There is an integer $M$ so that if $\dim V > M$, then the set of properly stable points in $V$ contains a non-empty open set [12] and [1]—the first paper holds in any characteristic.

Let us assume char $k \geq 0$ and let $\lambda: G_m \to G$ be a one-parameter subgroup of $G$. Let

$$W'(\lambda) = \{v \in V: \lambda(\alpha) \cdot v \text{ has a specialization in } V \text{ as } \alpha \to 0\}.$$

Then we may show that $P(\lambda) \cdot W'(\lambda) \subset W'(\lambda)$ as in (2.1) and prove the following theorem and corollaries just as in (2.2) and (2.3).

THEOREM. *Let $G$ be a connected reductive algebraic subgroup of $GL(V)$. Let $B = TU$ be a Borel subgroup of $G$ and let $W(T) = N(T)/T$ be the Weyl group of $T$. Let $v_0$ be a point in $V$ which is not properly stable. There is a one-parameter subgroup $\lambda$: $G_m \to T$ such that the following statements hold*:

(a) *$B \subset P(\lambda)$, i.e., if $\mu$ is a root of $B$ relative to $T$, then $\langle \mu, \lambda \rangle \geq 0$;*

(b) *$B \cdot W'(\lambda) \subset W'(\lambda)$;*

(c) *there are elements $u \in U$, $sT \in W(T)$ such that $v_0 \in us \cdot W'(\lambda)$.*

COROLLARY 1. *There is a one-parameter subgroup $\lambda_0$: $G_m \to B$ such that $v_0 \in W'(\lambda_0)$.*

COROLLARY 2. *Let $X'$ be the set of points in $V$ which are not properly stable. There are one-parameter subgroups $\lambda_1, \ldots, \lambda_r$ of $T$ such that the following statements hold*:

(a) *$B \subset P(\lambda_i)$ and $B \cdot W'(\lambda_i) \subset W'(\lambda_i)$ for all $i = 1, \ldots, r$;*

(b) *each $G \cdot W'(\lambda_i)$ is closed;*

(c) *$X' = G \cdot W'(\lambda_1) \cup \cdots \cup G \cdot W'(\lambda_r)$ and this is the unique decomposition of $X'$ into irreducible components unless there exist $i, j$, $i \neq j$, $sT \in W(T)$ such that $W'(\lambda_i) \subset s \cdot W'(\lambda_j)$.*

## 3. Borel subgroups and semi-stable points

(3.1) THEOREM. *Let $B$ be a connected solvable algebraic group acting on an affine variety $X$. Let $x \in X$ and $Z = \mathrm{cl}\,(B \cdot x)$. Then either $B \cdot x$ is closed or there is an $f \in k[Z]$ such that*

$$Z - B \cdot x = \{z \in Z : f(z) = 0\}.$$

*In the latter case, there is an element $c$ in $k^*$ so that the mapping $\chi$: $B \to k$ given by $\chi(b) = cf(b \cdot x)$ is a character of $B$.*

*Proof.* The group $B$ operates on $k[Z]$ via $(b \cdot f)(z) = f(b^{-1} \cdot z)$ for all $f \in k[Z]$, $z \in Z$, and $b \in B$. Let $I$ be the ideal in $k[Z]$ vanishing on $Z - B \cdot x$. Then $I$ is $B$-invariant, i.e., $b \cdot I \subset I$ for all $b \in B$. Suppose now that $I \neq \{0\}$ and let $f$ be any non-zero element in $I$. There is a finite-dimensional $B$-invariant subspace $E \subset I$ such that $f \in E$ [6; Proposition, p. 62]. By the Lie–Kolchin theorem, there is a non-zero common eigenvector $h$ in $E$ for $B$ [6; 17.6, p. 113]. Let $b \cdot h = c_b h$. Then

$$h(b^{-1} \cdot x) = (b \cdot h)(x) = c_b h(x).$$

If $h(x) = 0$, then $h = 0$ on $B \cdot x$ and $h = 0$. Hence, $h(x) \neq 0$ and $h$ is non-zero on $B \cdot x$. Since $h$ is in $I$, $h$ is 0 on $Z - B \cdot x$.

The mapping $b \to h(b \cdot x)$ is non-zero on $B$ and, so is a character of $B$ if $h(e \cdot x) = 1$ [9; Proposition 3, p. 29]. Modifying $h$ by a constant, we obtain the theorem.

COROLLARY (Kostant, Rosenlicht). *Let $U$ be a unipotent group acting on an affine variety $X$. For every $x \in X$, the orbit $U \cdot x$ is closed.*

*Proof.* The corollary follows at once from the theorem since the only character of $U$ is trivial.

*Notes.* The corollary above was first proved by B. Kostant. A shorter proof was found by M. Rosenlicht. Another proof was found by A. Borel [3; Theorem 12.1]. A modification of Borel's proof gives the theorem above.

(3.2) THEOREM. *Let $G$ be a connected reductive algebraic subgroup of $GL(V)$ and let $B = TU$ be a Borel subgroup of $G$. Suppose that 0 is the only point in $V$ fixed by $G$. Let $v_0$ be a non-zero vector in $V$ which is not semi-stable. There is a non-zero vector $v \in \mathrm{cl}\,(B \cdot v_0)$ such that $U \cdot v = v$.*

*Proof.* According to Corollary 1 in Section 2.3, the point 0 is in $\mathrm{cl}\,(B \cdot v_0)$. Let $w \in \mathrm{cl}\,(B \cdot v_0)$ be chosen so that $B \cdot w$ has the smallest possible positive dimension. Then $\mathrm{cl}\,(B \cdot w) - B \cdot w$ consists of points fixed by $B$. Since $G/B$ is complete, each of these points is fixed by $G$. However, by our assumption, then, $\mathrm{cl}\,(B \cdot w) - B \cdot w$ is $\{0\}$. The theorem in (3.1) now implies that $\dim\,(B \cdot w) = 1$.

Now $U$ must fix $w$. For otherwise, $U \cdot w$ is a closed subset (by the corollary above) of $B \cdot w$ having dimension 1. This would imply that $U \cdot w = B \cdot w$ and $B \cdot w$ is closed.

COROLLARY. *Let $G$ be a connected reductive algebraic subgroup of $GL(V)$ which acts irreducibly on $V$. Let $B$ be a Borel subgroup of $G$. Let $v_0$ be a non-zero vector in $V$ which is not semi-stable. Then the highest weight vector of $G$ on $V$ (relative to $B$) is $\mathrm{cl}\,(G \cdot v_0)$.*

## 4. Examples

(4.1) **The adjoint representation.** Let $G$ be a connected reductive algebraic group and let $L(G)$ denote the Lie algebra of $G$. Then $G$ acts on $L(G)$ via the adjoint representation.

Let $B = T \cdot U$ be a Borel subgroup of $G$. Let $L(T)$, $L(U)$, and $L(B)$ be the Lie algebras of $T$, $U$, and $B$, respectively. We shall denote the roots of $T$ acting on $L(U)$ by $\alpha$, $\beta$, ... . Then there is a basis $\{e_\alpha\}$ of $L(U)$ so that $t \cdot e_\alpha = \alpha(t)e_\alpha$ for all $t \in T$.

Next, let $W$ be a subspace of $L(G)$ which is $B$-invariant. If $W$ contains $e_{-\beta}$ (where $e_\beta \in L(U)$), then $w = [e_\beta, e_{-\beta}]$ is a non-zero element in $W$ which is fixed by $T$.

Let $\lambda\colon G_m \to T$ be a one-parameter subgroup of $T$ such that $W(\lambda)$ is $B$-invariant. Then $W(\lambda) \subset L(U)$ by the argument just given. Also, there is a one-parameter subgroup $\lambda$ of $T$ so that $\langle \lambda, \alpha \rangle > 0$ if $\alpha > 0$ [4; Theorem, p. 317]. For this one-parameter subgroup, we have $W(\lambda) = L(U)$ and $P(\lambda) = B$.

Finally, let $X$ be the set of points in $L(G)$ which are not semi-stable. According to the remarks above and Corollary 2 in (2.3), we have

$$X = G \cdot L(U).$$

It is known that $G \cdot L(U)$ is precisely the set of nilpotent elements in $L(G)$. Hence, we obtain a result of B. Kostant: *a point $v$ in $L(G)$ is not semi-stable if and only if $v$ is nilpotent.*

(4.2) **Certain actions of** $SL_n$. Let $SL_n$ be the group of all $n \times n$ matrices with entries in $k$ and having determinant 1. Let

$$T = \{t = (t_{ij}) \in SL_n\colon t_{ij} = 0 \quad \text{for } i \neq j\}.$$

We shall denote a typical matrix $t = (t_{ij})$ in $T$ by $t = [t_{11}, \ldots, t_{nn}]$. Let us define characters $\chi_1, \ldots, \chi_n$ of $T$ by

$$\chi_i[t_{11}, \ldots, t_{nn}] = t_{ii} \quad \text{for each } i = 1, \ldots, n$$

(so $\chi_1 + \cdots + \chi_n = 0$). Let

$$B = \{(b_{ij}) \in SL_n\colon b_{ij} = 0 \quad \text{for } i > j\}.$$

Then $B$ is a Borel subgroup with maximal torus $T$. A simple system of roots for $T$ on $B$ is $\{\mu_1, \ldots, \mu_{n-1}\}$ where $\mu_i = \chi_i - \chi_{i+1}$. If $\lambda$ is a one-parameter subgroup of $T$, then there are integers $u_1, \ldots, u_n$ so that

$$\lambda(\alpha) = [\alpha^{u_1}, \ldots, \alpha^{u_n}]$$

and $u_1 + \cdots + u_n = 0$. The subgroup $B$ is contained in $P(\lambda)$ if and only if each $\langle \mu, \lambda \rangle \geq 0$, that is, if and only if

$$u_i \geq u_{i+1} \text{ for } i = 1, \ldots, n-2 \quad \text{and} \quad 2u_{n-1} + u_1 + \cdots + u_{n-2} \geq 0.$$

The group $SL_n$ acts on the vector space $k^n$ of all $n \times 1$ column matrices in the natural way, namely, $g \cdot v = gv$ for all $g \in SL_n$, $v \in k^n$. This action gives rise to an action on $k[x_1, \ldots, x_n]$, the algebra of regular functions on $k^n$, via

$$(g \cdot f)(v) = f(g^{-1} \cdot v) \quad \text{for all } g \in SL_n,\ v \in k^n, f \in k[x_1, \ldots, x_n].$$

Let $S_m$ be the vector space consisting of all those polynomials in $k[x_1, \ldots, x_n]$ which are homogeneous of degree $m$. Then $S_m$ is a finite-dimensional subspace of $k[x_1, \ldots, x_n]$ which is stable under the action of $SL_n$. We shall study the variety $X$ in $S_m$.

Let $v = x_1^{e_1} \cdots x_n^{e_n}$, $e_1 + \cdots + e_n = m$, be in $S_m$ and let

$$\lambda(\alpha) = [\alpha^{u_1}, \ldots, \alpha^{u_n}]$$

be a one-parameter subgroup of $T$. Then

$$\lambda(\alpha) \cdot v = \alpha^e v \quad \text{where } e = u_1(e_n - e_1) + \cdots + u_{n-1}(e_n - e_{n-1}).$$

To summarize, we have seen that:

(1) a one-parameter subgroup $\lambda$ of $T$ may be identified with a point $(u_1, \ldots, u_{n-1})$ where each $u_i$ is an integer;

(2) $B \subset P(\lambda)$ if and only if

$$u_1 - u_2 \geq 0, \ldots, u_{n-2} - u_{n-1} \geq 0, \quad \text{and} \quad 2u_{n-1} + u_1 + \cdots + u_{n-2} \geq 0;$$

(3) $x_1^{e_1} \cdots x_n^{e_n} \in W(\lambda)$ if and only if $u_1(e_n - e_1) + \cdots + u_{n-1}(e_n - e_{n-1}) > 0$.

We turn now to the cases $n = 2$ and $n = 3$.

$SL_2$. Let us put $\lambda(\alpha) = [\alpha^u, \alpha^{-u}]$ where we may assume that $u > 0$ (by (2)). Then, by (3), $x_1^e x_2^{m-e}$ is in $W(\lambda)$ if and only if $u(m - 2e) > 0$, i.e., $e < m/2$.

If $m = 2s$, then $W(\lambda)$ is spanned by $x_2^m, x_1 x_2^{m-1}, \ldots, x_1^{s-1} x_2^{m-s+1}$. In each of these monomials, the multiplicity of $x_2$ is $\geq s + 1$. Hence, $G \cdot W(\lambda)$ consists of all those polynomials in $S_m$ having a linear factor whose multiplicity is $\geq s + 1$.

If $m = 2s + 1$, we arrive at a conclusion just like the one just given: $G \cdot W(\lambda)$ consists of all those polynomials in $S_m$ having a linear factor whose multiplicity is $\geq s + 1$.

In both cases above, $X$ has only one component and $P(\lambda) = B$.

$SL_3$. Let us change notation here and write $u$, $t$ instead of $u_1$, $u_2$ and $a$, $b$, $c$ instead of $e_1$, $e_2$, $e_3$. According to (2) and (3) above, we should study pairs $u$, $t$ so that $u \geq t$ and $u + 2t \geq 0$. (If $\lambda$ is to be non-trivial, we should take $u > 0$.) Then $x_1^a x_2^b x_3^c$ is in $W(\lambda)$ if and only if $u(c - a) + t(c - b) > 0$. Let us distinguish two types of one-parameter subgroups of $T$, namely:

(I) $u > 0, u \geq t \geq 0$;
(II) $u > 0, t \leq 0, u + 2t \geq 0$.

The chart below summarizes the conditions $u$, $t$ must satisfy for $x_1^a x_2^b x_3^c$ to be in $W(\lambda)$.

| | I | II |
|---|---|---|
| $a = b = c$ | impossible | impossible |
| $a = b \neq c$ | $\begin{cases} c > a & \text{all } u, t \\ c < a & \text{impossible} \end{cases}$ | $\begin{cases} c > a & \text{all } u, t \\ c < a & \text{impossible} \end{cases}$ |
| $a = c \neq b$ | $\begin{cases} c > b & \text{all } t \neq 0 \\ c < b & \text{impossible} \end{cases}$ | $\begin{cases} c > b & \text{impossible} \\ c < b & \text{all } t \neq 0 \end{cases}$ |
| $a \neq b = c$ | $\begin{cases} c > a & \text{all } u, t \\ c < a & \text{impossible} \end{cases}$ | $\begin{cases} c > a & \text{all } u, t \\ c < a & \text{impossible} \end{cases}$ |
| $a > b > c$ | impossible | impossible |
| $a > c > b$ | $t/u > (a - c)/(c - b)$ | impossible |
| $b > a > c$ | impossible | $-t/u > (a - c)/(b - c)$ |
| $b > c > a$ | $t/u < (c - a)/(b - c)$ | all $u$, $t$ |
| $c > b > a$ | all $u$, $t$ | all $u$, $t$ |
| $c > a > b$ | all $u$, $t$ | $-t/u < (c - a)/(c - b)$ |

To illustrate how this chart may be used, let us look at the case $m = 8$. Using Corollary 2c, (2.3), one may show that

$$X = G \cdot W(\lambda_1) \cup G \cdot W(\lambda_2) \cup G \cdot W(\lambda_3) \cup G \cdot W(\lambda_4)$$

is the unique decomposition of $X$ into irreducible components where

$\lambda_1$ is of type I with $0 < t/u < 1/6$;

$\lambda_2$ is of type I with $2/3 < t/u < 1$;

$\lambda_3$ is of type II with $1/4 < -t/u < 1/3$;

$\lambda_4$ is of type II with $1/3 < -t/u < 1/2$.

In each case, $P(\lambda_i) = B$.

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