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THE ISOMORPHISM PROBLEM FOR INCIDENCE ALGEBRAS OF MÖBIUS CATEGORIES

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Introduction

Möbius cagegories were introduced in [7] to provide a unified setting for Möbius inversion. They include locally finite partially ordered sets (posets), Cartier and Foata monoids (see [2]), free categories [9], and a wide class of so-called "triangular categories" which are intimately related with the familiar objects of enumerative combinatories (subsets, subspaces, partitions, permutations, etc. [8]). One obtains combinatorial applications by studying inversion relations in the incidence algebra I($\mathscr{C}$) of the category $\mathscr{C}$, over a field K (see [3]). This approach was introduced by G. C. Rota in 1964 for locally finite posets and has been fruitfully developed since then (see [4] for example).

In 1970, R. P. Stanley proved that if P and Q are finite posets with isomorphic incidence algebras, then they are isomorphic as posets [12]; this result was later extended to arbitrary locally finite posets. The introduction of Möbius categories, and the possibility that different ones play the same role with respect to Möbius inversion, motivated an attempt to solve the analogous "isomorphism problem": Does $I(\mathscr{C}) \cong I(\mathscr{D})$ imply $\mathscr{C} \cong \mathscr{D}$? Striking examples (the first one is due to R. P. Stanley) show that the answer to this question is negative for general Möbius categories $\mathscr{C}$ and $\mathscr{D}$ (see Examples 1.5, 1.6, and 2.9).

In this paper, we try to determine how much structure of the category $\mathscr{C}$ is determined by its incidence algebra $I(\mathscr{C})$. First we show that the set $\mathscr{C}_0$ of vertices (or objects) of $\mathscr{C}$ can be recovered from $I(\mathscr{C})$ upon division by its Jacobson radical (corollary 1.2). However to get more information about $\mathscr{C}$, one has to make some further finiteness assumptions. The main result is the following: If $\mathscr{C}$ and $\mathscr{D}$ are finitely generated (Definition 2.3) Möbius categories with isomorphic incidence algebras, then $\mathscr{C}$ and $\mathscr{D}$ have isomorphic length n graphs: $(\mathscr{C}_0, \mathscr{C}_n) \cong (\mathscr{D}_0, \mathscr{D}_n)$, $\forall n \geq 1$. This implies Stanley's theorem, i.e. the positive answer to the isomorphism problem for locally finite posets and a similar theorem for finitely generated free categories (Applications 2.8).

A fundamental tool here is the so-called "standard topology" of $I(\mathscr{C})$, which, when $\mathscr{C}$ is finitely generated, is essentially determined by the decreasing sequence of product ideals J^n of the Jacobson radical J of $I(\mathscr{C})$. This implies, for

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instance, that if $\phi\colon I(\mathscr{C}) \to I(\mathscr{D})$ is an algebra homomorphism and if ϕ restricts to a bijection $\phi_0\colon \mathscr{C}_0 \to \mathscr{D}_0$ on vertices or if $\mathscr{C}_0$ is finite, then ϕ is continuous, and our main result follows. Note that all the continuous homomorphisms $\phi\colon I(\mathscr{C}) \to I(\mathscr{D})$ for Möbius categories have been characterized in [3, Section 5] as maps of the form

$$\phi\left(\sum_{f \in \mathscr{C}} \alpha(f)f\right) = \sum_{f \in \mathscr{C}} \alpha(f)\phi(f),$$

i.e. substitution of an element $\phi(f) \in I(\mathscr{D})$ for each $f \in \mathscr{C}$, where $\phi\colon \mathscr{C} \to I(\mathscr{D})$ is an "admissible map".

It also follows that any automorphism of $I(\mathscr{C})$ is continuous if $\mathscr{C}$ is a finitely generated Möbius category. Because of the analogy with the algebra of formal power series, we define the *Lagrange inversion problem* for such categories to be that of the explicit inversion of the automorphisms of $I(\mathscr{C})$ (Section 3).

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1. The incidence algebra

We consider categories as sets of morphisms, here called *arrows*, equipped with a partial composition law which satisfies the usual rules concerning associativity and identity arrows. In a category $\mathscr{C}$, we identify the objects, here called *vertices*, with their associated identity arrows; this set $\mathscr{C}_0$ of vertices is then a subset of $\mathscr{C}$. We say that $\mathscr{C}$ is *decomposition-finite* if for each arrow f of $\mathscr{C}$ the set $(f) = \{(f', f'') \mid f' \circ f'' = f\}$ is finite.

$$\begin{tikzcd} & \cdot \arrow[dr, "f''"] & \\ \cdot \arrow[ur, "f'"] \arrow[rr, "f=f'\circ f''"'] & & \cdot \end{tikzcd}$$

The *incidence algebra* $I(\mathscr{C})$ of $\mathscr{C}$ over an arbitrary field K is then defined to be the vector space $K^{\mathscr{C}}$ of all functions α from $\mathscr{C}$ to K together with a multiplication $*$ defined by

$$(\alpha * \beta)(f) = \sum_{(f)} \alpha(f')\beta(f''), \quad \alpha, \beta \in I(\mathscr{C}), f \in \mathscr{C}.$$

$I(\mathscr{C})$ is easily seen to be an associative K-algebra with identity element δ defined by

$$\delta(f) = \chi(f \in \mathscr{C}_0), \quad f \in \mathscr{C},$$

where χ denotes the truth function on statements.

Note that $\mathscr{C}$ is naturally embedded in $I(\mathscr{C})$ by identifying $f \in \mathscr{C}$ with the function $f\colon \mathscr{C} \to K$ defined by $f(h) = \chi(f = h)$, $h \in \mathscr{C}$. With this identification, for $f, g \in \mathscr{C}$, we have $f * g = f \circ g$ if the composite $f \circ g$ is defined in $\mathscr{C}$, and 0 otherwise. Also for $c, d \in \mathscr{C}_0$, i.e. vertices of $\mathscr{C}$, and $\alpha \in I(\mathscr{C})$,

$$(c * \alpha * d)(h) = \alpha(h) \cdot \chi(h \in \mathscr{C}(c, d)), \quad h \in \mathscr{C},$$

where $\mathscr{C}(c, d)$ denotes the set of arrows of $\mathscr{C}$ with domain c and codomain d. We sometimes use a formal sum notation: $\alpha = \sum_{f \in \mathscr{C}} \alpha(f)f$. In the above case, we have

$$c * \alpha * d = \sum_{f \in \mathscr{C}(c, d)} \alpha(f)f.$$

These possibly infinite sums can be given a rigorous setting in terms of "summable families" of elements of $I(\mathscr{C})$, or equivalently in terms of limits in $I(\mathscr{C})$, which indeed is a topological algebra with respect to the so-called "standard topology"; this is simply the product topology, on $K^{\mathscr{C}}$, of the discrete topology on K. See [3] for more details.

Examples of incidence algebras are the algebras of formal power series on a set of (commuting or not) variables, the incidence algebra of a locally finite poset, and the group algebra of a finite group.

$\mathscr{C}$ is called a *Möbius category* if it is a decomposition-finite category for which the incidence algebra $I(\mathscr{C})$ has the property that $\forall \alpha \in I(\mathscr{C})$, α is invertible iff $\alpha(c) \neq 0$, $\forall c \in \mathscr{C}_0$. Möbius categories can also be characterized by the fact that each arrow f of $\mathscr{C}$ admits only a finite number of strict decompositions $f = f_1 \circ f_2 \circ \cdots \circ f_n$ or equivalently, for decomposition-finite categories, has finite length $l(f)$ in the sense of Mitchell [10 p. 80]. We refer the reader to [3] for more details on Möbius categories and their incidence algebras.

We use the following definition for the *Jacobson radical* $J(A)$ of a ring A:

$$J(A) = \{a \in A \mid \forall b, c \in A, 1 - \text{bac is invertible}\}.$$

Propositions 1.1 and 1.4 which follow are straightforward generalizations of results of R. P. Stanley [12], [4].

PROPOSITION 1.1. *Let $\mathscr{C}$ be a Möbius category. The Jacobson radical $J(\mathscr{C}) = J(I(\mathscr{C}))$ is the set*

$$J(\mathscr{C}) = \{\alpha \in I(\mathscr{C}) \mid \alpha(c) = 0, \forall c \in \mathscr{C}_0\}.$$

Proof. Indeed, in $I(\mathscr{C})$, $\delta - \beta * \alpha * \gamma$ is invertible iff

$$1 \neq (\beta * \alpha * \gamma)(c) = \beta(c)\alpha(c)\gamma(c), \quad \forall c \in \mathscr{C}_0;$$

so α is in $J(I(\mathscr{C}))$ iff $\alpha(c) = 0$, $\forall c \in \mathscr{C}_0$. ∎

COROLLARY 1.2. *If $\mathscr{C}$ is a Möbius category, there is a canonical isomorphism $I(\mathscr{C})/J(\mathscr{C}) \cong K^{\mathscr{C}_0}$ which is induced by the restriction to $\mathscr{C}_0 \subseteq \mathscr{C}$ of functions $\alpha \in I(\mathscr{C})$.* ∎

COROLLARY 1.3. *Let $\mathscr{C}$ and $\mathscr{D}$ be Möbius categories and let $\phi\colon I(\mathscr{C}) \to I(\mathscr{D})$ be an algebra homomorphism. Then $\phi(J(\mathscr{C})) \subseteq J(\mathscr{D})$.*

Proof Let α be in $J(\mathscr{C})$, with $\phi(\alpha) \notin J(\mathscr{D})$. Then there exists $d \in \mathscr{D}_0$ such that

$$\phi(\alpha)(d) = x \neq 0;$$

so $\delta - x^{-1} \cdot \alpha$ is invertible in $I(\mathscr{C})$ but $\phi(\delta - x^{-1} \cdot \alpha) = \delta - x^{-1}\phi(\alpha)$ is not since it has value 0 on $d \in \mathscr{D}_0$. This is a contradiction. ▮

PROPOSITION 1.4. *Let $\mathscr{C}$ and $\mathscr{D}$ be Möbius categories and let $\Psi\colon I(\mathscr{C}) \to I(\mathscr{D})$ be an algebra isomorphism. Then there is an inner automorphism ι of $I(\mathscr{D})$ such that the composite isomorphism*

$$\phi = I(\mathscr{C}) \xrightarrow{\ \Psi\ } I(\mathscr{D}) \xrightarrow{\ \iota\ } I(\mathscr{D})$$

can be restricted to a bijection $\phi_0\colon \mathscr{C}_0 \to \mathscr{D}_0$ on vertices.

Proof. The isomorphism $\psi\colon I(\mathscr{C}) \to I(\mathscr{D})$ must preserve the Jacobson radicals so it induces an isomorphism $\bar{\psi}$ of algebras

$$K^{\mathscr{C}_0} \cong I(\mathscr{C})/J(\mathscr{C}) \xrightarrow{\ \bar{\Psi}\ } I(\mathscr{D})/J(\mathscr{D}) \cong K^{\mathscr{D}_0}.$$

As it lies in $K^{\mathscr{C}_0}$, the set $\mathscr{C}_0$ is the unique, up to a bijection, maximal family of orthogonal primitive idempotents of $K^{\mathscr{C}_0}$. This must be preserved by $\bar{\Psi}$ which consequently restricts to a bijection $\Psi_0\colon \mathscr{C}_0 \to \mathscr{D}_0$. Then

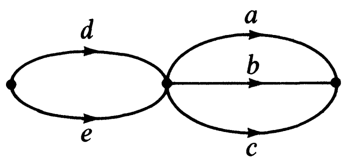
$$\{\alpha_d = \Psi\Psi_0^{-1}(d)\}_{d \in \mathscr{D}_0}$$

is a family of orthogonal idempotents of $I(\mathscr{D})$ such that $\alpha_d - d \in J(\mathscr{D})$, $\forall d \in \mathscr{D}_0$. Define $\beta \in I(\mathscr{D})$ by $\beta(f) = \alpha_{d(f)}(f)$ where $d(f)$ is the codomain of $f \in \mathscr{D}$, or equivalently $\beta = \sum_{d \in \mathscr{D}_0} \alpha_d * d$. It is easily seen that β is invertible and that $\alpha_d * \beta = \beta * d$, $\forall d \in \mathscr{D}_0$, so that β induces the desired inner automorphism ι of $I(\mathscr{D})$. ▮

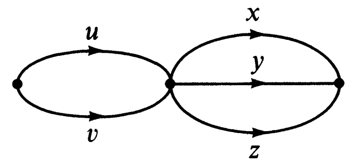
Hence if $I(\mathscr{C}) \cong I(\mathscr{D})$, $\mathscr{C}$ and $\mathscr{D}$ have the same set of vertices, up to a bijection. We cannot conclude however that, as in the case of locally finite posets, $\mathscr{C}$ and $\mathscr{D}$ are isomorphic Möbius categories, as the two following examples, and others given later, show.

Example 1.5 (Due to Richard P. Stanley). Let L and M be commutative monoids generated respectively by $\{a, b, c\}$ and $\{x, y, z\}$, with relations $ab = ac$ and $x^2 = xy$ respectively. Then L and M are non isomorphic Möbius monoids, i.e. Möbius categories with only one vertex, but the mapping $x \mapsto a$, $y \mapsto a - b + c$, $z \mapsto c$ extends to an algebra isomorphism $I(M) \xrightarrow{\sim} I(L)$.

Example 1.6. This example is similar to the preceding one. The categories involved this time are *deltas*, i.e. their only circuits are the identity arrows: Let $\mathscr{L}$ and $\mathscr{M}$ be the categories generated by the graphs



and



with relations $da = ea$, $db = dc$ and $ux = vx$, $ux = uy$, respectively. Then again, $I(\mathscr{L}) \cong I(\mathscr{M})$, but $\mathscr{L}$ and $\mathscr{M}$ are non isomorphic Möbius categories.

However, L and M, as well as $\mathscr{L}$ and $\mathscr{M}$, although not isomorphic, have "isomorphic presentations"; roughly speaking, they have the same number of arrows of each length. We make this precise in the next section.

2. Finitely generated Möbius categories

Let $\mathscr{C}$ be a fixed Möbius category. For each element n in the set N of natural numbers, let $\mathscr{C}_n = \{f \in \mathscr{C} \mid l(f) = n\}$ be the set of arrows of $\mathscr{C}$ of length n. Then $\mathscr{C}_0$ is the set of identity arrows, or vertices, of $\mathscr{C}$, as before, and $\mathscr{C}_1$ is the set of length 1 arrows, called *elementary* or *indecomposable* arrows of $\mathscr{C}$. The pair $(\mathscr{C}_0, \mathscr{C}_n)$ forms a (directed multi-) graph called the *length n graph* of $\mathscr{C}$; $(\mathscr{C}_0, \mathscr{C}_1)$ is also called the *elementary graph* of $\mathscr{C}$. For $c, d \in \mathscr{C}_0$, we define $\mathscr{C}_n(c, d) = \mathscr{C}_n \cap \mathscr{C}(c, d)$, the set of length n narrow "from c to d" in $\mathscr{C}$. In a Möbius category $\mathscr{C}$, $\mathscr{C}_1$ is a minimal set of generators since any arrow is the composite of elementary arrows.

Definition 2.1. We will say that $\mathscr{C}$ is *countably generated* (respectively *countably presented*) if $\forall c, d \in \mathscr{C}_0$, the set $\mathscr{C}_1(c, d)$(resp. the set $\mathscr{C}_n(c, d)$, $\forall n \in N$) is countable.

For $n \in N$, we define also the sets

$$J_n = J_n(\mathscr{C}) = \{\alpha \in I(\mathscr{C}) \mid l(f) < n \Rightarrow \alpha(f) = 0\}.$$

For instance, $J_0 = I(\mathscr{C})$ and $J_1 = J$, the Jacobison radical of $I(\mathscr{C})$. J_n is easily seen to be a closed ideal of $I(\mathscr{C})$, with $J_{n+1} \subseteq J_n$, and $J_n/J_{n+1} \cong K^{\mathscr{C}_n}$ as vector spaces. Note also that for $n \geq 1$, the ideal product J^n is contained in J_n and that in fact J_n is the topological closure $\overline{J^n}$ of J^n. We are now in a position to prove the following basic theorem:

Theorem 2.2. *Let $\mathscr{C}$ and $\mathscr{D}$ be countably generated Möbius categories with homeomorphically isomorphic incidence algebras. Then the elementary graphs $(\mathscr{C}_0, \mathscr{C}_1)$ and $(\mathscr{D}_0, \mathscr{D}_1)$ are isomorphic graphs. Moreover if $\mathscr{C}$ and $\mathscr{D}$ are countably presented, they have isomorphic length n graphs, $\forall n \in N$.*

Proof. Let $\phi: I(\mathscr{C}) \to I(\mathscr{D})$ be a bicontinuous algebra isomorphism. By proposition 1.4 we can assume that ϕ restricts to a bijection $\phi_0: \mathscr{C}_0 \to \mathscr{D}_0$ on vertices. For $n \geq 1$, we have

$$\phi(J_n(\mathscr{C})) = \phi(\overline{J^n(\mathscr{C})}) = \overline{\phi(J^n(\mathscr{C}))} = \overline{J^n(\mathscr{D})} = J_n(\mathscr{D}),$$

and for $c, d \in \mathscr{C}_0$, ϕ induces linear isomorphisms

$$K^{\mathscr{C}_n(c,d)} \cong \frac{c * J_n(\mathscr{C}) * d}{c * J_{n+1}(\mathscr{C}) * d} \cong \frac{\phi_0(c) * J_n(\mathscr{D}) * \phi_0(d)}{\phi_0(c) * J_{n+1}(\mathscr{D}) * \phi_0(d)} \cong K^{\mathscr{D}_n(\phi_0(c),\, \phi_0(d))}.$$

Hence under the hypothesis of countability, a dimension argument shows that there is a bijection $\mathscr{C}_n(c, d) \xrightarrow{\sim} \mathscr{D}_n(\phi_0(c), \phi_0(d))$ yielding the desired graph isomorphism. ▌

DEFINITION 2.3. $\mathscr{C}$ is said to the *finitely generated* if $\forall c, d \in \mathscr{C}_0$,
(1) the set $\mathscr{C}_1(c, d)$ is finite,
(2) the set $[c, d] = \{e \in \mathscr{C}_0 \mid \mathscr{C}(c, e) \neq \phi \neq \mathscr{C}(e, d)\}$ is finite.

Example 2.4. (a) Any locally finite poset P is a finitely generated Möbius category. Indeed in this case, $\mathscr{C}_0$ is the set of elements of P; elementary arrows correspond to "covers" in P so $|\mathscr{C}_1(c, d)| \leq 1$, and condition (2) is exactly the local finiteness property of P. The graph $(\mathscr{C}_0, \mathscr{C}_1)$ is usually called the Hasse diagram of P.

(b) For a Möbius monoid M, condition (2) is trivially satisfied and condition (1) says that M has a finite number of generators.

(c) Given a (directed multi-) graph Γ, the free category $\mathscr{C}(\Gamma)$ generated by Γ is the category whose vertices are those of Γ and whose arrows are paths using arcs of Γ. $\mathscr{C}(\Gamma)$ is always a Möbius category and its elementary graph is exactly Γ. Hence $\mathscr{C}(\Gamma)$ is finitely generated iff for all vertices c, d of Γ,
(1) the set $\Gamma(c, d)$ of arcs from c to d is finite,
(2) there is only a finite number of vertices e of Γ "between" c and d, i.e. with a path from a to e and one from e to d.

PROPOSITION 2.5. *Let $\mathscr{C}$ be a finitely generated Möbius category and choose $c, d \in \mathscr{C}_0$. Then, $\forall n \geq 1$,*
(1) *$\mathscr{C}_n(c, d)$ is finite ($\mathscr{C}$ is finitely presented),*
(2) $c * J_n * d = c * J^n * d$.
Moreover, if $\mathscr{C}_0$ is finite, $\mathscr{C}_n$ is finite and $J_n = J^n$.

Proof. For $n \geq 1$ and for each $f \in \mathscr{C}_n(c, d)$, we choose arrows x_f and y_f such that $f = x_f \circ y_f$, $l(x_f) = 1$, $l(y_f) = n - 1$. This shows that $\mathscr{C}_n(c, d)$ is contained in the image of the following map which is induced by composition:

$$\bigcup_{e \in [c, d]} \mathscr{C}_1(c, e) \times \mathscr{C}_{n-1}(e, d) \to \mathscr{C}(c, d).$$

Hence a simple induction on n gives the first part of the proposition.

To prove part (2), we define, for $e \in [c, d]$,

$$S_e = \{x \in \mathscr{C}_1(c, e) \mid x = x_f \text{ for some } f \in \mathscr{C}(c, d)\}$$

and for $x \in S_e$,

$$V_{e,x} = \{h \in \mathscr{C}(e, d) \mid x_{x \circ h} = x \text{ and } y_{x \circ h} = h\}$$

and proceed by induction on n, assuming that the statement is true for $n - 1$, $\forall c, d \in \mathscr{C}_0$. Suppose $\alpha \in c * J_n * d$. Define, for $e \in [c, d]$ and $x \in S_e$, $\beta_{e,x} \in I(\mathscr{C})$ by

$$\beta_{e,x}(h) = \alpha(x \circ h) \cdot \chi(h \in V_{e,x}), \quad h \in \mathscr{C}.$$

Then $\beta_{e,x} \in e * J_{n-1} * d = e * J^{n-1} * d$, i.e. $\beta_{e,x} = e * \beta'_{e,x} * d$, with $\beta'_{e,x} \in J^{n-1}$, and α can be expressed as a finite sum where e and x range in the finite sets $[c, d]$ and S_e respectively:

$$\begin{aligned}\alpha &= \sum_{e,x} x * \beta_{e,x} = \sum_{e,x} x * e * \beta'_{e,x} * d \\ &= \sum_{e,x} c * x * \beta'_{e,x} * d = c * \left(\sum_{e,x} x * \beta'_{e,x}\right) * d.\end{aligned}$$

Thus $\alpha \in c * J^n * d$. This completes the proof since we already know that $J^n \subseteq J_n$. Finally, if $\mathscr{C}_0$ is finite, $\mathscr{C}_n$ is obviously finite and we have

$$J_n = \sum_{c,d} c * J_n * d = \sum_{c,d} c * J^n * d \subseteq J^n,$$

since J^n is an ideal and the sum is over the finite set $\mathscr{C}_0 \times \mathscr{C}_0$. ∎

Proposition 2.5 is central, as we will see, because the standard topology of $I(\mathscr{C})$ for a Möbius category $\mathscr{C}$ is closely related to the decreasing sequence of ideals $J_n(\mathscr{C})$.

THEOREM 2.6. *Let $\mathscr{C}$ and $\mathscr{D}$ be Möbius categories, $\mathscr{C}$ finitely generated, and let $\phi: I(\mathscr{C}) \to I(\mathscr{D})$ be an algebra homomorphism. If either $\mathscr{C}_0$ is finite or ϕ restricts to a bijection $\mathscr{C}_0 \to \mathscr{D}_0$, then ϕ is continuous.*

Proof. Let $\{\alpha_h\}_{h \in H}$ be a net in $I(\mathscr{C})$ over a directed set $(H, \leq)$ such that $\lim_{h \in H} \alpha_h = 0$. We wish to show that $\lim_{h \in H} \phi(\alpha_h) = 0$ in $I(\mathscr{D})$ also.

If $\mathscr{C}_0$ is finite, then $U_n = \bigcup_{k=0}^{n-1} \mathscr{C}_k$ is finite so there exists $h_0 \in H$ such that $h \geq h_0$ implies $\alpha_h(f) = 0$, $\forall f \in U_n$, i.e. $l(f) < n$, that is $a_h \in J_n(\mathscr{C})$. Then

$$\phi(\alpha_h) \in \phi(J_n(\mathscr{C})) = \phi(J^n(\mathscr{C})) = [\phi(J(\mathscr{C}))]^n \subseteq J^n(\mathscr{D}) \subseteq J_n(\mathscr{D}),$$

by Corollary 1.3. So $\phi(\alpha_h)(g) = 0$, $\forall g \in \mathscr{D}$ with $l(g) < n$, and we are done.

In the case where ϕ restricts to a bijection $\mathscr{C}_0 \to \mathscr{D}_0$, one shows in a similar manner that $d * \phi(\alpha_h) * e$ has 0 as a limit in $I(\mathscr{D})$, $\forall d, e \in \mathscr{D}_0$, which is sufficient. ∎

COROLLARY 2.7. *Let $\mathscr{C}$ and $\mathscr{D}$ be finitely generated Möbius categories with isomorphic incidence algebras. Then $\mathscr{C}$ and $\mathscr{D}$ have isomorphic presentations, i.e. $\forall n \geq 1$ the length n graphs $(\mathscr{C}_0, \mathscr{C}_n)$ and $(\mathscr{D}_0, \mathscr{D}_n)$ are isomorphic.*

Proof. By Proposition 1.4, there is an isomorphism $\phi: I(\mathscr{C}) \to I(\mathscr{D})$ which induces a bijection $\mathscr{C}_0 \to \mathscr{D}_0$. By Theorem 2.6, ϕ is bicontinuous and Theorem 2.2 applies. ▌

Applications 2.8. (1) *Posets.* If P and Q are locally finite posets with isomorphic incidence algebras, they are isomorphic as posets. Indeed, by Corollary 2.7, they have isomorphic Hasse diagrams, which determine the order relations. This was first proved by R. P. Stanley in [12, 1970] for finite posets.

(2) *Free categories.* If $\mathscr{C}$ and $\mathscr{D}$ are freely and finitely generated by the graphs Γ and Δ respectively and have isomorphic incidence algebras, then, by Corollary 2.7, $\Gamma = (\mathscr{C}_0, \mathscr{C}_1) \cong (\mathscr{D}_0, \mathscr{D}_1) = \Delta$ and hence the categories $\mathscr{C}$ and $\mathscr{D}$ themselves are isomorphic.

The following example shows that the hypothesis of finite generation is necessary in Theorem 2.6 and Corollary 2.7.

Example 2.9. Let $\mathscr{C}$ and $\mathscr{D}$ be the free categories generated by the indicated graphs

$$\begin{tikzcd} \bullet \arrow[r, "\{x_n\}_{n\in N}"'] & \bullet \arrow[r, "\{y_n\}_{n\in N}"'] & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r, "\{x_n\}_{n\in N}"'] \arrow[rr, bend left, "t"] & \bullet \arrow[r, "\{y_n\}_{n\in N}"'] & \bullet \end{tikzcd}$$

respectively. Then $I(\mathscr{C})$ and $I(\mathscr{D})$ are isomorphic algebras, although the graphs themselves, and the categories, are not isomorphic. This is because there is an infinite dimensional complement H of $J^2(\mathscr{C})$ in $J_2(\mathscr{C})$. Elements of H, like $\sum_{n\in N} x_n y_n$, behave like indeterminates and one of them can be mapped into t to give the isomorphism.

3. The Lagrange inversion problem for Möbius categories

Combining Proposition 1.4 and Theorem 2.6 one can easily prove the following useful result:

PROPOSITION 3.1. *Let $\mathscr{C}$ be a finitely generated Möbius category. Then every automorphism of $I(\mathscr{C})$ is continuous. If $\mathscr{C}_0$ is finite. every endomorphism of $I(\mathscr{C})$ is continuous.* ▌

The special case where $\mathscr{C}$ is a locally finite poset P was obtained by K. Baclawski in [1] as a by-product of his characterisation of the automorphisms of $I(P)$.

Again the hypothesis of finite generation cannot be relaxed as the category $\mathscr{C}$ of Example 2.9 shows. Any non identity linear automorphism or endomap of H will extend to a non-continuous automorphism or endomorphism of $I(\mathscr{C})$.

If $\mathscr{C}$ and $\mathscr{D}$ are Möbius categories then any continuous algebra homomorphism $\phi\colon I(\mathscr{C}) \to I(\mathscr{D})$ is of the form

$$\phi\left(\sum_{f \in \mathscr{C}} \alpha(f) f\right) = \sum_{f \in \mathscr{C}} \alpha(f)\phi(f), \quad \alpha \in I(\mathscr{C}),$$

that is, the extension by substitution of an *admissible map* $\phi\colon \mathscr{C} \to I(\mathscr{D})$; this means (see [3], Section 5) that

(1) the family $\{\phi(f)\}_{f \in \mathscr{C}}$ is summable in $I(\mathscr{D})$,
(2) $\sum_{c \in \mathscr{C}_0} \phi(c) = \delta$, the identity element of $I(\mathscr{D})$,
(3) $\phi(f) * \phi(g) = \phi(fg)$ if fg is defined in $\mathscr{C}$, 0 otherwise.

Example 3.2. Let $\mathscr{C}$ be the free commutative monoïd generated by n letters, say $X_1, \ldots, X_n$. Then $I(\mathscr{C})$ is isomorphic to $K[\![X_1, \ldots, X_n]\!]$ the formal power series algebra. By Proposition 3.1, any endomorphism of $K[\![X_1, \ldots, X_n]\!]$ is continuous, hence is obtained by substitutions $X_1 \mapsto F_1, \ldots, X_n \mapsto F_n$ in the formal power series, where $F_1, \ldots, F_n$ are themselves elements of $K[\![X_1, \ldots, X_n]\!]$ with zero constant terms, as follows easily from the notion of admissible maps; furthermore the endomorphism is an automorphism iff the Jacobian determinant of $F_1, \ldots, F_n$ is non-zero. For the case $n = 1$, this characterization of the automorphisms of $K[\![X]\!]$ is given by E. Spiegel in [11, Theorem 2.1].

Note then that the Lagrange inversion formula in n variables gives explicitly the inverse of any automorphism of $K[\![X_1, \ldots, X_n]\!]$ (see [5] for example). It is then natural to try to extend this calculus to other finitely generated Möbius monoïds or categories. Consequently we state the following problem.

Problem 3.3. The *Lagrange inversion problem* for a finitely generated Möbius category $\mathscr{C}$ consists in characterizing and explicitely inverting the automorphisms of its incidence algebra $I(\mathscr{C})$.

For instance, if $\mathscr{C} = \mathscr{C}_n$ is the totally ordered set with n elements, $I(\mathscr{C}_n)$ is the algebra of $n \times n$ triangular matrices. It is then easy to see, using results of R. P. Stanley [12], that the automorphism group of $I(\mathscr{C}_n)$ is isomorphic to the group of invertible affine transformations of the form

$$\begin{array}{ll}
x_1 \mapsto a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n-1}x_{n-1} + a_{1,n} \\
x_2 \mapsto \qquad\quad\; a_{2,2}x_2 + \cdots + a_{2,n-1}x_{n-1} + a_{2,n} \\
\cdots \qquad\qquad\qquad\qquad\qquad \cdots \\
x_{n-1} \mapsto \qquad\qquad\qquad\qquad\; a_{n-1,n-1}x_{n-1} + a_{n-1,n},
\end{array}$$

the Lagrange inversion formula being that which gives explicitely the inverse of the triangular matrix

$$\begin{bmatrix}
a_{1,1} & a_{1,2} & \cdots & a_{1,n-1} & a_{1,n} \\
 & a_{2,2} & \cdots & a_{2,n-1} & a_{2,n} \\
 & & & \vdots & \\
 & 0 & & a_{n-1,n-1} & a_{n-1,n} \\
 & & & & 1
\end{bmatrix}.$$

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