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A REPRESENTATION THEOREM IN STRICTLY PSEUDOCONVEX DOMAINS

BY

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Introduction

Let Ω be a smooth bounded strictly pseudoconvex domain contained in $\mathbf{C}^n$. In this note, the dual space of $H^\infty(\bar{\Omega})$, the space of holomorphic functions on Ω which are smooth up to the boundary, is characterized as a space of holomorphic functions, $H^{-\infty}(\Omega)$. The duality is exhibited via an extension of the usual L^2 inner product and this allows a strong converse to a theorem in [1] to be proved concerning the linear span of the Bergman kernel function in $H^\infty(\bar{\Omega})$ and sets of determinacy.

In [1], it is shown that every function in $H^\infty(\bar{\Omega})$ is the Bergman projection of a function in $C^\infty(\bar{\Omega})$ which vanishes to arbitrarily high order on $b\Omega$. In the present work, this result is improved: every u in $H^\infty(\bar{\Omega})$ is the Bergman projection of a function in $C^\infty(\bar{\Omega})$ which vanishes to infinite order on $b\Omega$.

The methods and results have applications in the theory of boundary behavior of biholomorphic mappings.

1. Preliminaries

Throughout this note, Ω will denote a smooth bounded strictly pseudoconvex domain contained in $\mathbf{C}^n$. $\rho\colon \mathbf{C}^n \to \mathbf{R}$ will be a C^∞ defining function for Ω, i.e., $\Omega = \{z\colon \rho(z) < 0\}$, $b\Omega = \{z\colon \rho(z) = 0\}$, and $d\rho \neq 0$ on $b\Omega$.

Let s be positive integer.

$W^s(\Omega)$ is the Sobolev space of complex valued functions on Ω with inner product given by

$$\langle u, v\rangle_s = \sum_{|\alpha| \leq s} 2^{-|\alpha|} \int_\Omega D^\alpha u \overline{D^\alpha v}.$$

$H^s(\Omega)$ is the subspace of $W^s(\Omega)$ consisting of holomorphic functions. For u and v in $H^s(\Omega)$, the inner product becomes

$$\langle u, v\rangle_s = \sum_{|\alpha| \leq s} \int_\Omega \frac{\partial^\alpha u}{\partial z^\alpha} \overline{\frac{\partial^\alpha v}{\partial z^\alpha}}$$

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where, as usual,

$$\frac{\partial}{\partial z_k} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} - i\frac{\partial}{\partial y_k}\right).$$

$W_0^s(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^s(\Omega)$.

$W^{-s}(\Omega)$ is the dual space of $W_0^s(\Omega)$ and is identified with the space of distributions λ such that

$$\|\lambda\|_{-s} = \operatorname{Sup}\{|\lambda(\phi)| : \phi \in C_0^\infty(\Omega);\ \|\phi\|_s = 1\}$$

is finite. $W^{-s}(\Omega)$ is a Banach space with this norm. We write $\langle \lambda, \phi \rangle_0$ for $\lambda(\phi)$.

$H^{-s}(\Omega)$ will denote the subspace of $W^{-s}(\Omega)$ consisting of holomorphic functions. For f in $H^{-s}(\Omega)$,

$$\|f\|_{-s} = \operatorname{Sup}\left\{\left|\int_\Omega f\phi\right| : \phi \in C_0^\infty(\Omega);\ \|\phi\|_s = 1\right\}$$

$H^\infty(\bar{\Omega}) = \bigcap_{s\ge 0} H^s(\Omega)$ is a Frechet space of holomorphic functions on Ω which are smooth up to the boundary. The family of $H^s(\Omega)$ norms define the Frechet topology on $H^\infty(\bar{\Omega})$.

$H^{-\infty}(\Omega) = \bigcup_{s\ge 0} H^{-s}(\Omega)$ is a topological vector space under the usual inductive limit topology.

P will denote the Bergman orthogonal projection of $L^2(\Omega)$ onto its subspace $H(\Omega)$ of holomorphic functions. $K(w, z)$ is the Bergman kernel function. K and P are related via

$$Pf(z) = \langle f, K(\cdot, z)\rangle_0 = \int_\Omega K(z, w) f(w)\, dV_w \quad \text{for } f \in L^2(\Omega).$$

When Ω is smooth bounded and strictly pseudoconvex, it is known that P is bounded from $W^s(\Omega)$ to $H^s(\Omega)$ for each s (see Kohn [5]), and that $K(w, z) \in C^\infty(\Omega \times \bar{\Omega})$ (Kerzman [4]).

A set $D \subset \Omega$ will be called a set of determinacy for $H^\beta(\Omega)$ $(-\infty \le \beta \le \infty)$ if the only function in $H^\beta(\Omega)$ which vanishes on D is the zero function.

2. Results

For Ω a smooth bounded strictly pseudoconvex domain contained in $\mathbf{C}^n$ and for s a positive integer, we obtain:

THEOREM 1. *There is a bounded operator* Φ^s: $H^s(\Omega) \to W_0^s(\Omega)$ *such that* $P\Phi^s u = u$ *for all* $u \in H^s(\Omega)$.

THEOREM 2. *The operator* Λ^s *defined via*

$$\Lambda^s f(z) = \langle f, K(\cdot, z)\rangle_s = \sum_{|\alpha|\le s} \int_\Omega \frac{\partial^\alpha}{\partial \bar{w}^\alpha} K(z, w) \frac{\partial^\alpha f}{\partial w^\alpha}(w)\, dV_w$$

is a Banach space isomorphism of $H^s(\Omega)$ *onto* $H^{-s}(\Omega)$.

THEOREM 3. $H^\infty(\bar{\Omega})$ *is dense in* $H^{-s}(\Omega)$.

THEOREM 4. *There is a non-degenerate sesquilinear pairing*

$$\langle\ ,\ \rangle_0 : H^\infty(\bar{\Omega}) \times H^{-\infty}(\Omega) \to \mathbf{C}$$

which exhibits $H^\infty(\bar{\Omega})$ and $H^{-\infty}(\Omega)$ as being mutually dual. Furthermore,

$$\langle f, g\rangle_0 = \int_\Omega f\bar{g}$$

whenever f and g are in $L^2(\Omega)$. In general, the pairing is given by $\langle \Phi^s f, g\rangle_0$ when $g \in H^{-s}(\Omega)$.

THEOREM 5. *The norm $\|f\|_{H^s(\Omega)}$ is equivalent to the norm*

$$\|f\|_s = \operatorname{Sup}\left\{\left|\int_\Omega f\bar{g}\right| : g \in H^\infty(\bar{\Omega});\ \|g\|_{-s} = 1\right\}.$$

Furthermore, $u \in H(\Omega)$ is in $H^s(\Omega)$ if and only if $\|u\|_s < \infty$.

THEOREM 6. *The linear span of $\{K(\cdot, z) : z \in D\}$ is dense in $H^\infty(\bar{\Omega})$ if and only if D is a set of determinacy for $H^{-\infty}(\Omega)$.*

THEOREM 7. *If u is in $H^\infty(\bar{\Omega})$, then $u = P\phi$ for some ϕ in $C^\infty(\bar{\Omega})$ which vanishes to infinite order on $b\Omega$.*

3. The proofs

3.1 *Proof of Theorem 1.* The construction of Φ^s is discussed in [1] and [2]. Let $\delta > 0$ be small enough so that $d\rho \neq 0$ on $\{z : |\rho(z)| \leq \delta\} = A_\delta$. If $\{\phi_i\}_{i=1}^m$ is a C^∞ partition of unity of $A_{\delta/2}$ supported in A_δ, and $\{z_i\}_{i=1}^m$ are complex directions such that $\partial\rho/\partial z_i \neq 0$ on Supp ϕ_i, then a suitable Φ^s can be written as

$$\Phi^s u = u - \sum_{i=1}^m \frac{\partial}{\partial z_i}\left(\sum_{k=0}^{s-1} \theta_k^i \rho^{k+1}\right)$$

where the θ_k^i are defined inductively via

$$\theta_0^i = \frac{\phi_i u}{\left(\dfrac{\partial\rho}{\partial z_i}\right)},$$

$$U_l^i = \phi_i u - \frac{\partial}{\partial z_i}\left(\sum_{k=0}^{l} \theta_k^i \rho^{k+1}\right),$$

$$\theta_{l+1}^i = \frac{\left(\dfrac{\partial}{\partial v}\right)^{l+1} U_l^i}{(l+2)!\,\dfrac{\partial\rho}{\partial z_i}}.$$

Here, $(\partial/\partial v)^l$ stands for differentiation in the $\nabla\rho/|\nabla\rho|^2$ direction l times.

$\Phi^s u$ can be written as

$$\Phi^s u = \sum_{|\alpha| \leq k \leq N_s} b_{\alpha, k} D^\alpha u \rho^k = \sum_{|\alpha| \leq k \leq N_s} \tilde{b}_{\alpha, k} \frac{\partial^\alpha u}{\partial z^\alpha} \rho^k$$

where the b's are in $C^\infty(\bar{\Omega})$, and $N_s = (1/2)(s+1)s$.

If $u \in H^\infty(\bar{\Omega})$, then $\Phi^s u$ vanishes to order $s-1$ on $b\Omega$, i.e., $D^\alpha \Phi^s u(\xi) = 0$ for all $\xi \in b\Omega$, $|\alpha| \leq s-1$ (see [1], [2]). Hence $\Phi^s u \in W_0^s(\Omega)$. Furthermore,

$$\langle \Phi^s u, \Phi^s u \rangle_s = \sum_{\substack{k, m \leq N_s, \\ |\alpha| - k \leq s, \\ |\beta| - m \leq s}} \int_\Omega b_{\beta, m}^{\alpha, k} \frac{\partial^\alpha u}{\partial z^\alpha} \rho^k \overline{\frac{\partial^\beta u}{\partial z^\beta}} \rho^m = \sum_{\substack{|\alpha| \leq s, \\ |\beta| \leq s}} \int_\Omega b_{\alpha, \beta} \frac{\partial^\alpha u}{\partial z^\alpha} \overline{\frac{\partial^\beta u}{\partial z^\beta}}.$$

(Here, we have used integration by parts and the fact that u is holomorphic.)

Hence,

$$\|\Phi^s u\|_s \leq C\|u\|_s \quad \text{for } u \in H^\infty(\bar{\Omega}).$$

$H^\infty(\bar{\Omega})$ is dense in $H^s(\Omega)$ because $C^\infty(\bar{\Omega})$ is dense in $W^s(\Omega)$ and P is bounded from $W^s(\Omega)$ to $H^s(\Omega)$. Finally, $P\Phi^s u = u$ because

$$\Phi^s u = u - \sum \frac{\partial}{\partial z_i} v_i \quad \text{where } v_i = 0 \text{ on } b\Omega.$$

3.2 *Proof of Theorem 2.* Λ^s is a bounded operator from $H^s(\Omega)$ to $H^{-s}(\Omega)$ because for $u \in H^s(\Omega)$ and $\phi \in C_0^\infty(\Omega)$, we obtain

$$\langle \Lambda^s u, \phi \rangle_0 = \iint_{\Omega\Omega} \sum_{|\alpha| \leq s} \frac{\partial^\alpha}{\partial \bar{w}^\alpha} K(z, w) \frac{\partial^\alpha u}{\partial w^\alpha}(w)\, dV_w \overline{\phi(z)}\, dV_z = \langle u, P\phi \rangle_s$$

using Fubini's theorem and the fact that $K(z, w) \in C^\infty(\Omega \times \bar{\Omega})$ (Kerzman [4]).

Hence,

$$\|\Lambda^s u\|_{H^{-s}(\Omega)} = \operatorname{Sup}\{|\langle u, P\phi \rangle_s| : \phi \in C_0^\infty(\Omega);\ \|\phi\|_s = 1\},$$

because P is bounded from $W^s(\Omega)$ to $H^s(\Omega)$.

The relation $\langle \Lambda^s u, \phi \rangle_0 = \langle u, P\phi \rangle_s$ extends to hold for all $u \in H^s(\Omega)$ and $\phi \in W_0^s(\Omega)$. From this, it follows easily that Λ^s is one to one because if $\Lambda^s u = 0$, then

$$0 = \langle \Lambda^s u, \Phi^s v \rangle_0 = \langle u, P\Phi^s v \rangle_s = \langle u, v \rangle_s$$

for all $v \in H^s(\Omega)$. Hence $u = 0$.

To see that Λ^s is surjective, notice that for $f \in H^{-s}(\Omega)$,

$$u \mapsto \langle f, \Phi^s u \rangle_0$$

is a continuous anti-linear functional on $H^s(\Omega)$. Hence, there is a function $F \in H^s(\Omega)$ such that $\langle f, \Phi^s u \rangle_0 = \langle F, u \rangle_s$ for all $u \in H^s(\Omega)$. Let $u = K(\cdot, z)$ to obtain $\Lambda^s F(z) = \langle f, \Phi^s K(\cdot, z) \rangle_0$.

The proof of Theorem 2 will be finished if we show that

$$f(z) = \langle f, \Phi^s K(\cdot, z)\rangle_0.$$

To do this, we will need to use a stability property of the Bergman kernel function proved by Greene and Krantz [7]. Let

$$\Omega_\varepsilon = \{z: \rho_\varepsilon(z) < 0\} \quad \text{where } \rho_\varepsilon = \rho + \varepsilon,$$

and let $K_\varepsilon(w, z)$ be the Bergman kernel function associated to Ω_ε. Greene and Krantz prove that

$$\|K_\varepsilon(\cdot, z) - K(\cdot, z)\|_{H^t(\Omega_\varepsilon)} \to 0$$

as $\varepsilon \to 0$ for all t. Using the same partition of unity and complex directions as in the construction of Φ^s in the proof of Theorem 1, we can define operators

$$\Phi^s_\varepsilon \colon H^s(\Omega_\varepsilon) \to W^s_0(\Omega_\varepsilon) \subseteq W^s_0(\Omega)$$

for small $\varepsilon > 0$. It is easy to check that $\Phi^s_\varepsilon K_\varepsilon(\cdot, z) \to \Phi^s K(\cdot, z)$ in $W^s_0(\Omega)$ as $\varepsilon \to 0$. Hence

$$\begin{aligned} \langle f, \Phi^s K(\cdot, z)\rangle_0 &= \lim_{\varepsilon \to 0} \langle f, \Phi^s_\varepsilon K_\varepsilon(\cdot, z)\rangle_0 \\ &= \lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} f(w)\overline{K_\varepsilon(w, z)}\, dV_w = f(z). \end{aligned}$$

Hence $f(z) = \langle f, K(\cdot, z)\rangle_0 = \Lambda^s F(z)$, and the proof of Theorem 2 is complete.

3.3 *Proof of Theorem 3.* It suffices to prove that Λ^s maps $H^\infty(\bar{\Omega})$ into $H^\infty(\bar{\Omega})$ because $H^\infty(\bar{\Omega})$ is dense in $H^s(\Omega)$. If $u \in H^\infty(\bar{\Omega})$, then

$$\begin{aligned} \Lambda^s u(z) &= \sum_{|\alpha| \le s} \left\langle \frac{\partial^\alpha u}{\partial w^\alpha}, \frac{\partial^\alpha}{\partial w^\alpha} K(\cdot, z)\right\rangle_0 \\ &= \sum_{|\alpha| \le s} \left\langle \Phi^s \frac{\partial^\alpha u}{\partial w^\alpha}, \frac{\partial^\alpha}{\partial w^\alpha} K(\cdot, z)\right\rangle_0 \\ &= \sum_{|\alpha| \le s} (-1)^{|\alpha|} \left\langle \frac{\partial^\alpha}{\partial \bar{w}^\alpha} \Phi^s \left(\frac{\partial^\alpha u}{\partial w^\alpha}\right), K(\cdot, z)\right\rangle_0 \\ &= P\left(\sum_{|\alpha| \le s} (-1)^{|\alpha|} \frac{\partial^\alpha}{\partial \bar{w}^\alpha} \Phi^s \frac{\partial^\alpha u}{\partial w^\alpha}\right)(z). \end{aligned}$$

The function inside the brackets is in $C^\infty(\bar{\Omega})$; hence, $\Lambda^s u \in H^\infty(\bar{\Omega})$.

3.4 *Proof of Theorem* 4. If $u \in H^\infty(\bar{\Omega})$ and $v \in H^{-\infty}(\Omega)$, then the pairing $\langle u, v\rangle_0$ is defined to be $\langle \Phi^s u, v\rangle_0$ where s is any integer such that $v \in H^{-s}(\Omega)$. The pairing is well defined because $\langle \Phi^{s_1} u - \Phi^{s_2} u, f\rangle_0 = 0$ for all f in $H^\infty(\bar{\Omega})$, and $H^\infty(\bar{\Omega})$ is dense in $H^{-\infty}(\Omega)$ by Theorem 3.

If λ is a continuous linear functional on $H^{\infty}(\bar{\Omega})$, then λ satisfies

$$|\lambda(u)| \leq (\text{const})\|u\|_{H^s(\Omega)}$$

for some s. Therefore, there is a function $f \in H^s(\Omega)$ such that

$$\lambda(u) = \langle u, f\rangle_s = \langle P\Phi^s u, f\rangle_s = \langle \Phi^s u, \Lambda^s f\rangle_0 = \langle u, \Lambda^s f\rangle_0.$$

Hence, λ is represented by $\Lambda^s f \in H^{-s}(\Omega)$.

If η is a continuous linear functional on $H^{-\infty}(\Omega)$, then for each s, there is a function $\phi_s \in W_0^s(\Omega)$ such that $\eta(v) = \langle v, \phi_s\rangle_0$ for all $v \in H^{-s}(\Omega)$. Restricting v to be in $H(\Omega)$ reveals that $P\phi_1 = P\phi_2 = P\phi_3 = \cdots$. Write $u = P\phi_1 \in H^{\infty}(\bar{\Omega})$. Now $\eta(v) = \langle v, u\rangle_0$, because if $v \in H^{-s}(\Omega)$, then $\eta(v) - \langle v, u\rangle_0 = \langle v, \phi_s - \Phi^s u\rangle_0$, and this is zero because $\langle f, \phi_s - \Phi^s u\rangle_0 = 0$ for all $f \in H^{\infty}(\bar{\Omega})$.

The non-degeneracy of $\langle\ ,\ \rangle_0$ follows from the relation

$$\langle u, v\rangle_s = \langle u, \Lambda^s v\rangle_0 \quad \text{for } u \text{ and } v \text{ in } H^s(\Omega)$$

and the isomorphism and density theorems.

3.5 *Proof of Theorem 5.* Given $u \in H(\Omega)$ such that $\|u\|_s < \infty$, we wish to show that $u \in H^s(\Omega)$ and that $c_1\|u\|_{H^s(\Omega)} \leq \|u\|_s \leq c_2\|u\|_{H^s(\Omega)}$.

Let Ω_ε and Φ_ε^s be defined as in the proof of Theorem 2. Write

$$\psi_\varepsilon = \sum_{|\alpha|\leq s} (-1)^{|\alpha|} \frac{\partial^\alpha}{\partial \bar{w}^\alpha} \Phi_\varepsilon^s\left(\frac{\partial^\alpha u}{\partial w^\alpha}\right).$$

Then $P\psi_\varepsilon \in H^{\infty}(\bar{\Omega})$ because ψ_ε has compact support in Ω, and

$$\langle u, P\psi_\varepsilon\rangle_0 = \langle u, \psi_\varepsilon\rangle_0 = \|u\|^2_{H^s(\Omega_\varepsilon)}.$$

Hence,

$$\|u\|^2_{\Omega^s(\Omega_\varepsilon)} \leq \|u\|_s\|P\psi_\varepsilon\|_{H^{-s}(H)}.$$

Now,

$$\begin{aligned}\|P\psi_\varepsilon\|_{H^{-s}(\Omega)} &= \text{Sup}\,\{|\langle P\psi_\varepsilon, \phi\rangle_0| : \phi \in C_0^{\infty}(\Omega);\ \|\phi\|_s = 1\} \\ &= \text{Sup}\ |\langle u, P\phi\rangle_{H^s(\Omega_\varepsilon)}| \leq \text{Sup}\ \|u\|_{H^s(\Omega_\varepsilon)}\|P\phi\|_{H^s(\Omega_\varepsilon)} \\ &\leq C\|u\|_{H^s(\Omega_\varepsilon)}.\end{aligned}$$

So $\|u\|_{H^s(\Omega_\varepsilon)} \leq C\|u\|_s$. If $\varepsilon \to 0$ then $u \in H^s(\Omega)$ and $\|u\|_{H^s(\Omega)} \leq C\|u\|_s$.

The second inequality $\|u\|_s \leq c_2\|u\|_{H^s(\Omega)}$ follows from

$$\begin{aligned}|\langle u, v\rangle_0| &= |\langle \Phi^s u, v\rangle_0| \leq \|\Phi^s u\|_{W_0{}^s(\Omega)}\|v\|_{H^{-s}(\Omega)} \\ &\leq (\text{const})\|u\|_{H^s(\Omega)}\|v\|_{H^{-s}(\Omega)},\end{aligned}$$

for u and v in $H^{\infty}(\bar{\Omega})$.

3.6 *Proof of Theorem 6.* Given $D \subset \Omega$, the linear span of $\{K(\cdot, z) : z \in D\}$ is not dense in $H^{\infty}(\bar{\Omega})$ if and only if there is a function $g \neq 0$ in $H^{-\infty}(\Omega)$ such that

$0 = \langle K(\cdot, z), g\rangle_0 = \overline{g(z)}$ for all $z \in D$ if and only if D is not a set of determinacy for holomorphic functions in $H^{-\infty}(\Omega)$.

3.7 *Proof of Theorem* 7. The proof is inspired by a construction due to L. Hörmander.

Suppose that u is in $H^\infty(\bar{\Omega})$. We construct ϕ_s in $W^s_0(\Omega)$ such that $P\phi_s = u$ and $\|\phi_s - \phi_{s-1}\|_{s-1} \leq 2^{-s}$. Then

$$\phi = \phi_1 + \sum_{s=1}^{\infty} (\phi_{s+1} - \phi_s)$$

is the desired C^∞ function which vanishes to infinite order on $b\Omega$ with $P\phi = u$.

Suppose $\phi_1, \phi_2, \ldots, \phi_{k-1}$ have been constructed. Let $\psi_k = \Phi^k u$. We construct V_k in $W^k_0(\Omega)$ with $PV_k = 0$ and

$$\|\psi_k + V_k - \phi_{k-1}\|_{k-1} \leq 2^{-k}.$$

$\phi_k = \psi_k + V_k$ then satisfies the induction hypothesis. Choose ω_k in $C^\infty_0(\Omega)$ with

$$\|\psi_k + \omega_k - \phi_{k-1}\|_{k-1}$$

so small that

$$\|\psi_k + \omega_k - \Phi^k P\omega_k - \phi_{k-1}\|_{k-1} < 2^{-k}.$$

This is possible because

$$\|\Phi^k P\omega_k\|_{k-1} \leq C_1 \|P\omega_k\|_{k-1}$$

and

$$\|P\omega_k\|_{k-1} = \|P(\psi_k + \omega_k - \phi_{k-1})\|_{k-1} \leq C_2 \|\psi_k + \omega_k - \phi_{k-1}\|_{k-1}.$$

$V_k = \omega_k - \Phi^k P\omega_k$ is in $W^k_0(\Omega)$, projects to zero, and

$$\|\psi_k + V_k - \phi_{k-1}\|_{k-1} \leq 2^{-k}.$$

This completes the proof of Theorem 7.

4. Remarks

(1) Theorem 6 has applications in the theory of boundary behavior of biholomorphic mappings. If Ω_1 and Ω_2 are two smooth bounded domains in $\mathbf{C}^n$ such that span $\{K_{\Omega_i}(\cdot, z): z \in \Omega_i\}$ is dense in $H^\infty(\overline{\Omega_i})$, $i = 1, 2$, then biholomorphic mappings between Ω_1 and Ω_2 extend smoothly to the boundary (see [2]).

(2) The inverse to the operator Λ^s can be written down explicitly. The orthogonal projection of $W^s(\Omega)$ onto $H^s(\Omega)$ defines an $H^s(\Omega)$ Bergman kernel function $K_s(w, z)$ which satisfies $P_s f(z) = \langle f, K_s(\cdot, z)\rangle_s$ for all f in $W^s(\Omega)$. K_s has many properties in common with the usual kernel function.

We now define $L^s\colon H^s(\Omega) \to H^s(\Omega)$ via

$$L^s f(z) = \int_\Omega K_s(z, w) f(w)\, dV_w = \langle f, K_s(\cdot, z)\rangle_0.$$

It is easy to check that

(a) $\|L^s u\|_{H^s(\Omega)} \leq \|u\|_{L^2(\Omega)}$

(b) $\Lambda^s L^s = I$

(c) L^s is compact self adjoint on $H^s(\Omega)$ and satisfies

$$\langle L^s u, v\rangle_s = \langle u, v\rangle_0 = \langle u, L^s v\rangle_s.$$

L^s extends to be a bounded operator from $H^{-s}(\Omega)$ to $H^s(\Omega)$ via

$$L^s f(z) = \langle f, \Phi^s K_s(\cdot, z)\rangle_0,$$

and this is the inverse of Λ^s.

(3) The operator Φ^s can be made canonical. Let V be the $W_0^s(\Omega)$ orthogonal complement of the closed subspace of functions ϕ in $W_0^s(\Omega)$ such that $\langle \phi, f\rangle_0 = 0$ for all f in $H(\Omega)$. Let Q be the orthogonal projection of $W_0^s(\Omega)$ onto V. The canonical Φ^s is given by $\Phi^s_{\text{canonical}} = Q\Phi^s$.

(4) $H^{-\infty}(\Omega)$ is the space of holomorphic functions which satisfy finite growth conditions at the boundary, i.e., u is in $H^{-\infty}(\Omega)$ if and only if $u\rho^k$ is in $L^\infty(\Omega)$ for some integer k.

(5) The special Sobolev inequality $|\langle f, g\rangle_0| \leq C\|f\|_{H^s(\Omega)}\|g\|_{H^{-s}(\Omega)}$ holds for f and g in $H^\infty(\bar{\Omega})$ in any smooth bounded domain.

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