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VALUES OF CERTAIN WHITTAKER FUNCTIONS ON A p-ADIC REDUCTIVE GROUP

BY

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Introduction

It is our purpose here to calculate explicitly the values of Whittaker functions on the p-adic points of a reductive algebraic group G of type E_7 and defined over $\mathbf{Q}$, the field of rational numbers. These values appear as Euler factors $a_s(T)_p$ of the Fourier coefficients $a_s(T)$ for the Eisenstein series of weight s constructed by Baily on the exceptional tube domain of dimension 27. Here, the index T ranges through a certain lattice Λ in the exceptional simple Jordan algebra of dimension 27.

Our calculation is based on W. Casselman's idea for exploiting the functional equations of Whittaker functions on the p-adic group $GL(n, \mathbf{Q}_p)$ to obtain a formula of J. Shalika for their values; see [5]. Casselman's argument works for split groups, at least, with the Whittaker functions that one attaches to a minimal parabolic subgroup. However, we deal here with Whittaker functions attached to a *maximal* parabolic $\mathbf{Q}$-subgroup, and we have available only a single functional equation, whose existence we established in [9]. This makes it necessary for us to calculate some complicated p-adic integrals, unfortunately.

For a large class of discrete groups one can define Eisenstein series; see L.-C. Tsao's paper [14]. The exceptional modular group is distinguished among these by the lack of theta functions as an arithmetic tool at present, making it seem likely that a better understanding of Eisenstein series for this group will be essential eventually.

Before summarizing our results, we sketch some history. Although the computability of the Fourier expansion of Eisenstein series is fundamental for the arithmetic of elliptic modular forms, little is known for forms in several variables. Among Siegel modular forms, the Eisenstein series have Fourier coefficients that can be interpreted in terms of representations of quadratic forms, but even for Siegel's standard Eisenstein series in the rank 2 case the Fourier expansion is not easy to calculate explicitly. Igusa found several coefficients and used them in determining the structure of the graded ring of rank 2 Siegel modular forms; see [7]. Later, H. Maass computed all the coefficients before discovering, as he kindly informed the author, that one can derive the results

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easily from calculations that Kaufhold made in his derivation of the functional equations of Eisenstein series; see [10] and [12], also [13]. In the case of the exceptional modular group, the author used a p-adic analogue of Siegel's Babylonian reduction process to find a formula for the Fourier coefficients $a_s(T)$ with T of rank 2 in the Jordan algebra; see [8]. In terms of the p-adic order invariants of T, $\tau(1)$ and $\tau(2)$, one has

$$a_s(T)_p = f_T(p^{5-s})(1 - p^{-s})(1 - p^{4-s}),$$

where

$$f_T(X) = \sum_{j=0}^{\tau(1)} (p^4X)^j \left[\sum_{k=j}^{d-j} p^k \right], \tag{1}$$

where $d = \tau(1) + \tau(2)$. This may be rewritten in the form

$$f_T(X) = \sum_{j=0}^{d} X^j \left(\sum_{k=0}^{j*} p^k \right) \quad (j^* = \min(j, d-j, \tau(2))). \tag{1'}$$

Of course, the above formulas are local analogues of classical divisor sum formulas. There are similar, though slightly more complicated, formulas in the Siegel rank 2 case.

We summarize our results. One attaches "elementary divisors" d_1, d_2, d_3 to each T in the lattice Λ. Fix a prime p. Then the *p-th* Euler factor $a_s(T)_p$ is determined by the p-adic orders of the d_i's, say $\tau(1) \le \tau(2) \le \tau(3)$. If $\tau(1) = 0$, then $a_s(T)_p$ can be found easily from results in [8]. We have, in this case,

$$a_s(T)_p = (1 - p^{-s})(1 - p^{4-s})(1 - p^{8-s})F_T(p^{9-s})$$

where

$$F_T(X) = \sum_{k=0}^{\tau(2)} (p^4X)^k(1 - X^{\tau(3)+\tau(2)+1-2k})/(1 - X).$$

Let $q = p^4$ and $X = p^{9-s}$. Then for T as above,

$$a_s(p^mT)_p/a_s(T)_p = C_0(X^{-1})X^{3m} + C_1(X^{-1})q^{2m}X^{2m} + C_1(X)q^{2m}X^m + C_0(X),$$

where .

$$C_0(X) = 1/(1 - X)(1 - qX)(1 - q^2X)F_T(X),$$

$$C_1(X) = [-(1 + (q + 1)X)/(1 - q^2X) + F_T(X) - X^2F_{T/p}(X)]/[(1 - X)(1 - X/q)F_T(X)].$$

Notice that, since it depends only on $\tau(2)$ and $\tau(3)$, the polynomial is defined even for T such that $\tau(1)$ is negative (i.e., T is not in the lattice Λ) provided we use the convention that $F_T(X)$ vanishes identically whenever $\tau(2)$ is negative.

There is strong evidence that if T satisfies $\tau(1) = 0$, then

$$a_s(p^m T)_p = (1 - p^{-s})(1 - p^{4-s})(1 - p^{8-s}) \sum_{i=0}^{m} q^i \left(\sum_{j=1}^{S(i)} q^j \left(\sum_{k=j}^{m''-j} X^k \right) \right),$$

where $m' = m + \tau(2)$, $m'' = m' + \tau(3)$ and $S(i) = \min(m' - i, [(m'' - i)/2])$, $[x]$ the largest integer not greater than x. One can restate the conjectured formula as

$$a_s(p^m T)_p = a_s(E)_p \sum_{i=0}^{m''} X^i \left(\sum_{j=0}^{i^*} q^j \left(\sum_{k=0}^{j^{**}} q^k \right) \right),$$

where $i^* = \min(i, m'' - i, m')$ and $j^{**} = \min(j, m' - j, m'' - 2j, m)$.

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1. Some general results on reductive groups

We require some facts about admissible representations, for which the main reference is Casselman's forthcoming book [4].

1.1. Let G be the group of rational points of a reductive algebraic group $\mathbf{G}$ over a p-adic field Ω, let $\mathbf{o}$ be the ring of integers of Ω and let $\mathbf{p}$ be the maximal ideal of $\mathbf{o}$. We fix an embedding of G into some $GL(n)$ over Ω. A representation π of G on a complex vector space V is *smooth* if each $v \in V$ is fixed by some open subgroup of G. For any subgroup K of G let V^K be the space of K-fixed vectors in V. Then π is *admissible* if it is smooth and if V^K has finite dimension for each open subgroup of G.

By a *character* on a closed subgroup of G, we will understand a homomorphism into $\mathbf{C}^\times$, the multiplicative group of complex numbers, with open kernel. If χ is a character on the closed subgroup P of G, then we let

(1) $\operatorname{Ind}(\chi \mid P, G)$

$= \{\text{locally constant } f\colon G \to \mathbf{C} \mid f(bg) = \chi(b) f(g) \text{ for each } b \in P \text{ and } g \in G\}.$

Then G acts by right translation on $\operatorname{Ind}(\chi \mid P, G)$. If $P = \mathbf{P}(\Omega)$ is a parabolic subgroup of G, then we refer to the resulting representation of G as an *unnormalized principal series representation.* These are admissible, [4].

Let H be a locally compact group with left Haar measure dx, and let a be an automorphism of H. Define the modulus $\delta_H(a)$ by

(2) $$d(a \cdot x) = \delta_H(a)\, dx.$$

In particular, if $g \in G$ normalizes a subgroup H, let $\delta_H(g) = \delta_H(\operatorname{Int}(g))$, where $\operatorname{Int}(g)\colon x \mapsto gxg^{-1}$.

We shall use as convention the notation $H = \mathbf{H}(Q_p)$ for the $\mathbf{Q}_p$-rational points of an algebraic group defined over $\mathbf{Q}_p$. Often we will omit mention of $\mathbf{H}$ entirely when speaking of parabolic subgroups and their unipotent radicals.

1.2. Suppose that (V, π) is admissible for G. Let P be a parabolic subgroup of G, let N be the unipotent radical of P, and for any compact subgroup N_0 of N, define

$$V(N_0) = \left\{ v \in V \,\middle|\, \int_{N_0} \pi(n)v \, dn = 0 \right\}.$$

Let P^- be a parabolic subgroup of G opposed to P, so $P^- \cap P = M$ is a Levi complement to N, and let N^- be the unipotent radical of P^-. Suppose that B is a compact open subgroup of G. We define a projection Π_B from V to V^B by

$$\Pi_B(v) = \int_B \pi(b)v \, db \qquad (v \in V),$$

where db is a left Haar measure on B.

The subgroup B is said to have *an Iwahori decomposition with respect to* (N, M, N^-) if there are subgroups $N_0^- \subset N^-$, $M_0 \subset M$, $N_0 \subset N$ such that multiplication

$$N \times M \times N^- \to G$$

maps $N_0 \times M_0 \times N_0^-$ bijectively onto B.

The following important technical lemma is due to Casselman, [4].

1.2.1. LEMMA. *Suppose that B has Iwahori decomposition $B = N_0 M_0 N_0^-$. If $v \in V^{M_0 N_0^-}$, then $\Pi_B(v) = \Pi_{N_0}(v)$ and $v - \Pi_{N_0}(v) \in V(N_0)$.*

1.2.2. Let A be the split component in the central torus of M and let

$$A^- = \{a \in A \colon \operatorname{Int}(a) N_0^- = N_0^-\}.$$

Then, for $v \in V^B$ and $a \in A^-$, we have $\pi(a)v \in V^{M_0 N_0^-}$. Thus,

$$\pi(a)v - \Pi_B(\pi(a)v) \in V(N_0). \tag{1}$$

1.3. Continue with the same notation, but now take (V, π) to be $\operatorname{Ind}(\chi \mid P, G)$. Suppose that for $w \in G$,

(i) $wAw^{-1} \subset P$,

and,

(ii) for each $f \in V$,

$$\langle \lambda_w, f \rangle = \int_{N \cap w^{-1}Pw \backslash N} f(wn) d_w n$$

converges absolutely.

We normalize the Haar measure $d_w n$ on $N_w = N \cap w^{-1}Pw\backslash N$ so that the image of $\mathbf{N}(\mathbf{o})$ has measure $= 1$.

If $a \in A$, let $\delta_w(a)$ be the modulus of $x \mapsto axa^{-1}$ on N_w. Then

$$\langle \lambda_w, \pi(a)f \rangle = \chi(waw^{-1})\delta_w(a)\langle \lambda_w, f \rangle \tag{1}$$

for all $a \in A$ and $f \in V$. We let $w^{-1}\chi(a) = \chi(waw^{-1})$.

Suppose now that B is an open subgroup of $\mathbf{G}(\mathbf{o})$ with Iwahori decomposition $\mathbf{N}(\mathbf{o})M_0 N_0^-$ with respect to (N, M, N^-). Assume that the double coset decomposition $P\backslash G/P$ has a family W of representatives with the following properties:

(i) *W is also a set of representatives for $P\backslash G/B$*;

(ii) $wAw^{-1} \subset P$ for each $w \in W$;

(iii) The λ_w's are absolutely convergent and form a basis of the dual space of V^B.

Then we can consider the dual basis $\{f_w\}$ of V^B. Since each λ_w annihilates $V(\mathbf{N}(\mathbf{o}))$, we see from 1.2.2 (1) and 1.3 (1) that, if $a \in A^-$,

$$\Pi_B(\pi(a)f_w) = w^{-1}\chi(a)\delta_w(a)f_w. \tag{2}$$

1.4. Let w_l be the element of W that interchanges each positive root with a negative root. Suppose that ψ is a character on $N/\mathbf{N}(\mathbf{o})$. According to [9, §3], we can define a linear functional $\Lambda_\psi = \Lambda_\psi(\chi)$ on Ind $(\chi | P, G)$ by

$$\langle \Lambda_\psi, f \rangle = \lim_H \int_H f(w_l n)\psi(n)^{-1}\, dn, \tag{1}$$

where the limit is taken over a sequence of compact subgroups H_m of N with $\bigcup_{m=1}^{\infty} H_m = N$. We refer to such limits as *Cauchy principal value integrals.* Note that Λ_ψ annihilates $V(\mathbf{N}(\mathbf{o}))$.

We are interested in the case $\chi | \mathbf{P}(\mathbf{o}) = 1$, $G = P \cdot \mathbf{G}(\mathbf{o})$. Then $V^{\mathbf{G}(\mathbf{o})}$ is one-dimensional. Let the *spherical function* $\phi = \phi_\chi$ be the unique function in $V^{\mathbf{G}(\mathbf{o})}$ with $\phi(e) = 1$. We must evaluate $\langle \Lambda_\psi, \phi \rangle$ for certain characters ψ. To do this, first notice that

$$\langle \Lambda_{\psi \circ \mathrm{Int}(a)}, \phi \rangle = \delta_P(a)\chi^{-1}(a)\langle \Lambda_\psi, \phi(a) \rangle, \tag{2}$$

for each $a \in A^-$. In each orbit of A acting on the characters of N, there is a character ψ for which the integral $\langle \Lambda_\psi, \pi(a)\phi \rangle$ is not so difficult to evaluate. The idea is then to write $\phi = \sum_{PwP} \langle \lambda_w, \phi \rangle f_w$, so

$$\langle \Lambda_\psi, \pi(a)\phi \rangle = \sum_{PwP} w^{-1}\chi(a)\delta_w(a)\langle \lambda_w, \phi \rangle \langle \Lambda_\psi, f_w \rangle. \tag{3}$$

Take ϕ_w to be the restriction of ϕ to PwB. To evaluate $\langle \Lambda_\psi, f_w \rangle$ we write

$$\phi_w = \sum_{PxP} \langle \lambda_x, \phi_w \rangle f_x$$

and invert the linear operator with matrix entries $\langle \lambda_x, \phi_w \rangle$. By virtue of the functional equation proved in [9] we can avoid calculating half of the terms $\langle \Lambda_\psi, f_w \rangle$. However, to use the functional equation, we must extend each λ_w to all but finitely many spaces Ind $(\chi \mid P, G)$ with $\chi \mid \mathbf{P(o)} \equiv 1$. Such χ are called *unramified.*

1.5. We now extend the definition of $\lambda_w = \lambda_w(\chi)$ to almost all unramified χ. Let $V = \text{Ind}\,(\chi \mid P, G)$ as before, and given a subset X of G with $PX = X$, let $V[X]$ be the space of locally constant functions $f\colon X \to \mathbf{C}$ such that

(1) $f(bg) = \chi(b)f(g)$ for all $b \in P$ and $g \in G$,
(2) $P\backslash\text{supp}\,(f)$ is compact.

Let $l(w) = \dim\,(P\backslash PwP)$ and arrange a family $w_1, w_2, \ldots, w_t$ of representatives for $P\backslash G/P$ with $l(w_1) \le l(w_2) \le \cdots \le l(w_t)$. Then one has a sequence of open subsets X^i of G such that $X^t = Pw_tP$ and each X^i is the disjoint union of X^{i+1} with Pw_iP. As in [9, Lemma 1.4.3], one sees that the sequence

$$(1) \qquad 0 \to V[X^{i+1}] \xrightarrow{\alpha^i} V[X^i] \xrightarrow{\beta^i} V[Pw_iP] \to 0$$

is exact, with α^i inclusion and β^i restriction.

For any smooth N-module V', let $V'(N) = \bigcap_H V'(H)$, where H runs through the compact subgroups of N; see 1.2. Let $V'_N = V'/V'(N)$. The functor $V' \mapsto V'_N$ is easily seen to be exact, hence the sequence

$$(2) \qquad 0 \to V[X^{i+1}]_N \xrightarrow{\alpha^i_N} V[X^i]_N \xrightarrow{\beta^i_N} V[Pw_iP]_N \to 0$$

is exact for each i. Let v_N be the image of $v \in V'$ in V'_N.

Each space $V[PwP]_N$ is naturally an A-module, and we denote the representation by π_N. For $a \in A$ and $v \in V[PwP]_N$ we find after a brief calculation

$$(3) \qquad \pi_N(a)v = w^{-1}\chi(a)\delta_w(a)v.$$

If the characters $w^{-1}\chi \cdot \delta_w$ are distinct for the different double cosets PwP, then χ is called *regular.* In this case, $V_N \cong \bigoplus_{PwP} V[PwP]_N$, and, to make the isomorphism explicit, we construct projections $\Pi^k\colon V_N \to V[Pw_kP]_N$ for $1 \le k \le t$.

Fix k with $1 \le k \le t$. For $i = 1, 2, \ldots, t$ let $w = w_i$ and $\chi_i = w^{-1}\chi \cdot \delta_w$. Choose $a_i \in A$ such that $\chi_i(a_i) \ne \chi_k(a_i)$ and define $S^i\colon V \to V$ by $S^i(v) = \pi(a_i)v - \chi_i(a_i)v$. Note that if $v \in V[X^i]$, then $S^i(v)_N \in V[X^{i+1}]_N$ because $\beta^i_N(S^i(v)_N) = 0$, so there are induced maps $S_i\colon V[X^i]_N \to V[X^{i+1}]_N$. Define $\Pi^k\colon V_N \to V[X^k]_N$ by

$$(4) \qquad \Pi^k = \left(\prod_{i=1}^{k-1} (\chi_k(a_i) - \chi_i(a_i))\right)^{-1} \beta^k_N \circ S_{k-1} \circ S_{k-2} \circ \cdots \circ S_1.$$

By 1.3 (1) we have, for $w = w_k$ and $j < k$,

$$\langle \lambda_w, S^j(v)\rangle = (\chi_k(a_j) - \chi_j(a_j))\langle \lambda_w, v\rangle$$

whenever the right side exists, e.g., if $n \mapsto f(wn)$ is compactly supported. Therefore,

$$\langle \lambda_w, v\rangle = \int_{N \cap w^{-1}Pw\backslash N} f(wn)d_w n, \tag{5}$$

for any $f \in V[PwP]$ such that $f_N = \Pi^k(v)$. We can find such f for any v, however, so formula (5) allows us to extend the definition of λ_w to all of V, provided χ is regular.

2. The arithmetic group

2.1. We recall Freudenthal's description of a simply connected algebraic $\mathbf{Q}$-group G such that $\mathscr{G}(\mathbf{R})$ is isogenous to Aut (T); cf. the introduction. For details, see [1, Section 3]. Let $\det_J$ be the generic norm and tr_J be the generic trace on J. The Jordan product of X and Y is $X \circ Y$ and the identity element is E. There is a bilinear map $(X, Y) \mapsto X \times Y$ from $J \times J$ to J, with the property that $X \circ (X \times X) = \det_J(X)E$, and we let $X^* = X \times X$.

For any field K let V and V' be copies of the underlying vector space of the Jordan algebra $J(K)$ of K-rational points in J. Let Ξ and Ξ' be copies of K, and let the K-vector space $\mathbf{W}(K)$ be defined by

$$\mathbf{W}(K) = V \oplus \Xi \oplus V' \oplus \Xi'.$$

The group $\mathscr{G}(K)$ is then the group of elements in $GL(\mathbf{W}, K)$ that preserve both a certain quartic form $\mathscr{Q}$ on $\mathbf{W}(K)$ and a certain alternating bilinear form $\{\ ,\ \}$ on $\mathbf{W}(K)$. Let $\Lambda = J(\mathbf{Z})$ be the lattice of matrices in J whose coefficients lie in the maximal order $\mathfrak{o}$ of the Cayley algebra $\mathfrak{c}$, as defined in [1, Section 1]. The identifications of V and V' with J give lattices $V(\mathbf{Z})$ in V and $V'(\mathbf{Z})$ in V'. Define a lattice $\mathbf{W}(\mathbf{Z})$ in $\mathbf{W}$ by

$$\mathbf{W}(\mathbf{Z}) = V(\mathbf{Z}) \oplus \mathbf{Z}\mathfrak{e} \oplus V'(\mathbf{Z}) \oplus \mathbf{Z}\mathfrak{e}',$$

where $\mathfrak{e} = (0, 1, 0, 0)$ and $\mathfrak{e}' = (0, 0, 0, 1)$. Then the stabilizer $\Gamma = \mathscr{G}(\mathbf{Z})$ of $\mathbf{W}(\mathbf{Z})$ in $\mathscr{G}(\mathbf{Q})$ is an arithmetic group considered by Baily in [1], where it is proved that Γ is maximal among arithmetic subgroups of $G(\mathbf{Q})$. As a consequence of strong approximation, [11], one finds that for each rational prime p, the group $G(\mathbf{Z}_p)$ is a maximal compact subgroup of $\mathscr{G}(\mathbf{Q}_p)$.

Let $\mathscr{P}(K)$ be the stabilizer, in $\mathscr{G}(K)$, of $K\mathfrak{e}'$ and let $\mathscr{P}^-(K)$ be the stabilizer, in $\mathscr{G}(K)$, of $K\mathfrak{e}$. For each element $B \in J(K)$ we define an element u_B^- in the unipotent radical $\mathscr{N}^-$ of $\mathscr{P}^-$ by

$$u_B^- \begin{pmatrix} X \\ \xi \\ X' \\ \xi' \end{pmatrix} = \begin{pmatrix} X + \xi' B \\ \xi + (B, X') + (B^*, X) + \xi' \det B \\ X' + 2B \times X + \xi' B^* \\ \xi' \end{pmatrix} \tag{1}$$

Note that u_B^- maps the additive group of J isomorphically onto $\mathcal{N}^-$. As a Levi complement to $\mathcal{N}^-$ we take $\mathcal{M} = \mathcal{P} \cap \mathcal{P}^-$.

2.2. It is necessary to consider an extension $\mathcal{G}^\#$ of $\mathcal{G}$, namely, the group of similitudes of the forms $\mathcal{Q}$ and $\{\ ,\ \}$. That is, an element $g \in GL(\mathbf{W}, K)$ lies in $\mathcal{G}^\#$ if there exists $\lambda \in K$ such that $\mathcal{Q}(g \cdot x) = \lambda^2\mathcal{Q}(x)$ and $\{g \cdot x, g \cdot y\} = \lambda\{x, y\}$ for all $x, y \in W$. In particular $G^\#$ contains the elements

$$\mu(t)\colon (X, \xi, X', \xi') \longrightarrow (tX, t^{-1}\xi, X', t^2\xi') \quad (t \in GL(1)).$$

By a short calculation $\mu(t)^{-1}u_B^-\mu(t) = u_{tB}^-$. The element $\iota \in \mathcal{G}(\mathbf{Z})$ defined by

$$\iota(X, \xi, X', \xi') = (-X, -\xi', X, \xi)$$

satisfies Int $(\iota)\mathcal{N} = \mathcal{N}'$ and Int $(\iota) \cdot \mu(t) = \mu(t)^{-1}c(-t)$, where $c(-t)$ acts on $\mathbf{W}$ as multiplication by $-t$ and clearly lies in the center of $\mathcal{G}^\#$. It follows that Int $(\mu(t))$ acts on $\mathcal{N}$ as scalar multiplication by $t \in GL(1)$. Let S_μ be the image of μ in $\mathcal{G}^\#$ and, for each algebraic subgroup H of $\mathcal{G}$, normalized by S_μ, let $H^\# = H \cdot S_\mu$. Note that for each rational prime p, $\mathcal{G}^\#(\mathbf{Z}_p) = \mathcal{G}(\mathbf{Z}_p) \cdot S_\mu(\mathbf{Z}_p)$ is a maximal compact subgroup of $\mathcal{G}^\#(\mathbf{Q}_p)$.Let $u_B = $ Int $(\iota)u^-_B$.

Define a rational character $\chi_J\colon P^\# \to GL(1)$ by

$$\chi_J(b) = \det_J(\operatorname{Int}(b)E).$$

Note that $\chi_J(\mu(t)) = t^3$ for $t \in GL(1)$, while $\det_{\mathfrak{n}}(\operatorname{Ad}(\mu(t)) = t^{27}$, where $\mathfrak{n}$ is the Lie algebra of $\mathcal{N}$. Since both χ_J and $\det_{\mathfrak{n}} \circ \operatorname{Ad}$ are trivial on the center of $\mathcal{G}^\#$, and since $\mathcal{P}^\#$ is a rank-1 parabolic subgroup of $\mathcal{G}^\#$, we have $\det_{\mathfrak{n}} \circ \operatorname{Ad} = (\chi_J)^9$. Note that the modulus character of $\mathcal{P}^\#(\mathbf{Q}_p)$ for any prime p is given by

$$\delta(b) = |\det_{\mathfrak{n}}(\operatorname{Ad}(b))|_p \quad \text{for } b \in \mathcal{P}^\#(\mathbf{Q}_p).$$

For each prime p and for each complex number s we let $\chi(b) = |\chi_J(b)|_p^{s/2}$, $b \in \mathcal{P}^\#(\mathbf{Q}_p)$. We sometimes write $\chi = \chi_s$.

2.3. LEMMA. *For each rational prime p, $\mathcal{G}(\mathbf{Q}_p) = \mathcal{G}(\mathbf{Z}_p)\mathcal{P}(\mathbf{Q}_p)$.*

Proof. For each $g \in \mathcal{G}(\mathbf{Q}_p)$ we wish to find $\gamma \in \mathcal{G}(\mathbf{Z}_p)$ such that $\gamma g(\mathrm{e}') \in \mathbf{Q}_p \cdot \mathrm{e}'$. Our proof is adapted from [1, 5.2]. In [6], Freudenthal defined an algebraic cone $\mathfrak{M}$, in $\mathbf{W}$ invariant under $\mathcal{G}$ and satisfying, among other conditions,

$$X \times X = \xi' X', \quad X' \times X' = \xi X, \quad X \circ X' = \xi\xi' E$$

for $(X, \xi, X', \xi') \in \mathfrak{M}$. Since $\mathrm{e}' \in \mathfrak{M}$ and since $(X, \xi, X', \xi') \in \mathfrak{M}$ with $X = 0$ but $\xi' \neq 0$ implies $X' = 0$ and $\xi = 0$, it suffices to find $\gamma \in \mathcal{G}(\mathbf{Z}_p)$ such that $\gamma g(\mathrm{e}') = (0, \xi, X', \xi')$. We begin by choosing γ_0 in $\mathcal{G}(\mathbf{Z}_p)$ so that $g(\mathrm{e}') = (X, \xi, X', \xi')$ has $|\xi'|_p$ maximal. We claim that $(\xi')^{-1}X \in J(\mathbf{Z}_p)$. Suppose not. We may assume that X is in elementary divisor form by [1, 3.4]. Now the first diagonal entry of X, say d, must have p-adic order $\operatorname{ord}_p(d) < \operatorname{ord}_p(\xi')$. Let e_i denote the idempotent of J with *ith* diagonal entry 1 and all other entries 0, and set

$B = -e_1$. Then $u_B \gamma_0 g(e') = (*, *, *, \xi' - d)$, and the p-adic order of $\xi' - d$ is strictly smaller than that of ξ', contrary to assumption.

Let $Y = (\xi')^{-1}X$ and note that $u_Y^- \gamma_0 g(e') = (0, *, *, \xi')$. Since $\xi' \neq 0$, $u_Y^- \gamma_0 g(e') = \xi' e'$, as required.

2.4. Let B_p be the subgroup of all $\gamma \in \mathscr{G}(\mathbf{Z}_p)$ such that $\gamma(e') = \alpha e' + pw$ with $w \in \mathbf{W}(\mathbf{Z}_p)$. We shall need an Iwahori decomposition for B_p. Let $N^-(p)$ be the image of $pJ(\mathbf{Z}_p)$ under $X \mapsto u_X^-$.

2.4.1. Lemma. $B_p = N^-(p)\mathscr{M}(\mathbf{Z}_p)\mathscr{N}(\mathbf{Z}_p)$.

Proof. We first show that $B_p = N^-(p)\mathscr{P}(\mathbf{Z}_p)$. Suppose $g \in B_p$, so $g(e') = ae' + pw$, with $w \in \mathbf{W}(\mathbf{Z}_p)$ and $a \in \mathbf{Z}_p^\times$. Then it suffices to show that we can find $Y \in pJ(\mathbf{Z}_p)$ with $u_Y^- g(e') \in \mathbf{Q}_p \cdot e'$. However, since $g(e') \in \mathfrak{M}$, as in 2.3 above, we need only show that $u_Y^- g(e') = (0, *, *, \xi')$ with $\xi' \neq 0$. Take $Y = X$, where $g(e') = (X, \xi, X', \xi')$. Then $u_Y^-(X, \xi, X', \xi') = (0, *, *, \xi')$, and ξ' is a p-adic unit. Therefore, $u_Y^- g(e') = \xi'(e')$.

To finish, we must show that $\mathscr{P}(\mathbf{Z}_p) = \mathscr{M}(\mathbf{Z}_p)\mathscr{N}(\mathbf{Z}_p)$. By [2, 3.14] each $b \in \mathscr{P}(\mathbf{Q}_p)$ can be written mn with $m \in \mathscr{M}(\mathbf{Q}_p)$, $n \in \mathscr{N}(\mathbf{Q}_p)$. Clearly, it suffices to show that $m \in \mathscr{M}(\mathbf{Z}_p)$ whenever $b \in \mathscr{P}(\mathbf{Z}_p)$. We have

$$m(X, \xi, X', \xi') = (\pi(m) \cdot X, \nu(m)\xi, \pi^*(m) \cdot X, \nu^*(m)\xi')$$

where $\pi^* = \pi \circ \mathrm{Int}\,(\iota)$ and $\nu^* = \nu \circ \mathrm{Int}\,(\iota)$. Since $\iota \in \mathscr{G}(\mathbf{Z})$ and since $\nu(m) = \det\,(\pi(m) \cdot E)$ is in $\mathbf{Z}_p^\times$ whenever $\pi(m) \in GL(J, \mathbf{Z}_p)$, it suffices to check that $\pi(m) \cdot X \in J(\mathbf{Z}_p)$ whenever $X \in J(\mathbf{Z}_p)$. However, if $X \in J(\mathbf{Z}_p)$ and $n = u_Y$, then

$$mu_Y(X, 0, 0, 0) = (\pi(m) \cdot X, 0, 0, *) \in \mathbf{W}(\mathbf{Z}_p),$$

so $\pi(m) \cdot X \in J(\mathbf{Z}_p)$ also, as required.

2.5. In [1, 4.1] there appears a maximal $\mathbf{Q}$-split torus S_1 of $\mathscr{G}$ such that the components of J in its Peirce decomposition (as an algebra of matrices over the Cayley algebra) are the weight spaces of S_1 in $\mathscr{N}$. The relative $\mathbf{Q}$-root system of $\mathscr{G}$ with respect to S_1 is of type C_3. Also, Baily defined representatives $\iota_{(j)} \in \mathscr{G}(\mathbf{Z})$, $j = 0, 1, 2, 3$, for the double cosets $\mathscr{P}(\mathbf{Q})x\mathscr{P}(\mathbf{Q})$ with $x \in \mathscr{G}(\mathbf{Q})$. It is easy to describe the effect of $\iota_{(j)}$, on the roots of S_1 in $\mathscr{N}$. Namely, the roots are of the form $\varepsilon_i + \varepsilon_{i'}$, where $\mathrm{Int}\,(t)e_i = \varepsilon_i(t)^2 e_i$ if $t \in S_1$, and where we write the product additively. We have

$$\iota_{(j)}(\varepsilon_i + \varepsilon_{i'}) = \iota_{(j)}(\varepsilon_i) + \iota_{(j)}(\varepsilon_{i'})$$

and

$$\iota_{(j)}(\varepsilon_k) = \begin{cases} \varepsilon_k & \text{if } k > j, \\ -\varepsilon_k & \text{if } k \leq j. \end{cases}$$

An important fact about the double coset decomposition with respect to $\mathscr{P}$

is that each double coset $\mathscr{P}(\mathbf{Q}_p)g\mathscr{P}(\mathbf{Q}_p)$ with $g \in \mathscr{G}(\mathbf{Q}_p)$ contains a point of $\mathscr{G}(\mathbf{Q})$, see [9, 4.5.3]. Thus, the $\iota_{(j)}$'s are a complete family of representatives for the double cosets $\mathscr{P}(\mathbf{Q}_p)g\mathscr{P}(\mathbf{Q}_p)$, $g \in \mathscr{G}(\mathbf{Q}_p)$.

We let $S_0^{\#}$ be the largest subtorus of $S_1^{\#}(= S_1 S_\mu)$ on which $\varepsilon_i = \varepsilon_j$ for all i and j. Let $A^{\#} = S_0^{\#}(\mathbf{Q}_p)$.

2.6. Recall that $u_B = \iota^{-1}u_{-B}^{-}\iota \in N$; see 2.1 (1). Then the action of $\mathscr{M}^{\#}$ on $\mathscr{N}$ via conjugation leads to an action of $\mathscr{M}^{\#}$ on J. Namely, for each $m \in \mathscr{M}^{\#}$, define $m \cdot B \in J$ by

$$u_{m \cdot B} = m \cdot u_B \cdot m^{-1}. \tag{1}$$

Fix a prime p. According to [1, 3.4], each element X of $J(\mathbf{Q}_p)$ can be brought into *elementary divisor form* (that is, *reduced form*) by the action of $\mathscr{M}(\mathbf{Z}_p)$. Namely, for some $m \in \mathscr{M}(\mathbf{Z}_p)$,

$$m \cdot X = \begin{pmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{pmatrix} \quad \text{(diagonal matrix)},$$

and d_i divides d_{i+1} for $i = 1, 2$. We let $\nu_i(X) = \operatorname{ord}_p(d_i)$ be *ith p-adic order invariant* of X, for $i = 1, 2$, or 3, and we let $\nu_0(X) = 1$. Let $G = \mathscr{G}(\mathbf{Q}_p)$, $B = B_p$, $P = \mathscr{P}(\mathbf{Q}_p)$ and $w_i = \iota_{(i)}$ for $0 \le i \le 3$.

Let N_i be the product of the root spaces for $\varepsilon_j + \varepsilon_k$ with $j \le i$ and $k \le i$, and let $J^{(i)}$ be the corresponding Jordan subalgebra. The proof of Lemma 3.4 in [1] shows that each $X \in J^{(i)}(\mathbf{Q}_p)$ can be brought into reduced form by elements of

$$\mathscr{M}(\mathbf{Z}_p) \cap w_i^{-1}\mathscr{M}(\mathbf{Z}_p)w_i.$$

2.6.1. LEMMA. *If $X \in J^{(i)}(\mathbf{Q}_p)$ and $\nu_{i-j+1}(X) \ge 0 \ge \nu_{i-j}(X)$, then $w_i u_X \in Pw_j B$. Moreover, if $w_i u_X = bw_j\beta$ with $b \in P$ and $\beta \in B$, then*

$$\delta_P(b) = \prod_{k=1}^{i-j} p^{18\nu_k(X)}.$$

Proof. First observe that a similar statement holds for $G_0 = SL(2, \mathbf{Q}_p)$ with P_0 the upper triangular Borel subgroup, B_0 the Iwahori subgroup obtained as the inverse image of the reduction of P_0 mod p in $SL(2, \mathbf{Z}/p\mathbf{Z})$, $\sigma_0 = e$ and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad v_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

Namely, $\sigma_0 v_x \in P_0 B_0$ for all $x \in \mathbf{Q}_p$, $\sigma_1 v_x \in P_0 \sigma_1 B_0$ if $x \in \mathbf{Z}_p$, and $\sigma_1 v_x \in P_0 B_0$ if $x \notin \mathbf{Z}_p$.

We begin by reducing X by elements of $\mathscr{M}(\mathbf{Z}_p) \cap w_i \mathscr{M}(\mathbf{Z}_p)w_i^{-1}$, so we may assume that $X = \sum_{k=1}^{i} a_k e_k$. Suppose that $j = i - 1$, so $a_1 \notin \mathbf{Z}_p$, but $a_k \in \mathbf{Z}_p$ for $k \geq 2$. By [1, 7.7], there are isomorphic injections $\mathbf{i}_k$ of $SL(2)$ into $\mathscr{G}$ that take σ_1 to ι_{e_k} and v_x to u_{xe_k}. We have $\mathbf{i}_k(SL(2, \mathbf{Z}_p)) \subset \mathscr{G}(\mathbf{Z}_p)$, $\mathbf{i}_k(B_0) \subset B$ and $\mathbf{i}_k(P_0) \subset P$. Let $B_1 = \mathbf{i}_1(B_0)$, $P_1 = \mathbf{i}_1(P_0)$. Let $x = a_1 e_1$. Then $w_1 u_x \in P_1 B_1$, so $w_1 u_x \in P_1 B$ and

$$w_i u_X \in w_i w_1^{-1} P_1 B = P_1 w_i w_1^{-1} B$$

because $w_i w_1^{-1}$ is a product of ι_{e_k}'s, each of which centralizes P_1. Now, by [1, 2.4], both P and B contain representative for the Weyl group of $\mathscr{M}$, which acts as the full symmetric group on the ε_i's. Therefore $P w_i w_1^{-1} B = P w_{i-1} B$, and we have $w_i u_X \in P w_{i-1} B$.

If $w_1 u_X \in b_1 B$ with $b_1 \in P_1$, then $w_i u_X \in b_1 m w_{i-1} B$ for some $m \in \mathscr{M}(\mathbf{Z}_p)$ and $\delta_P(b_1 m) = \delta_P(b_1) = p^{18\nu_1(X)}$, as required.

Cases where $j \neq i - 1$ are handled similarly.

2.7. By Lemma 2.6.1, the double cosets $P w_i B$ are distinct. However, in order to apply the method described in 1.4, we would require that $G = \bigcup_{i=0}^{3} P w_i B$. In fact, this is not difficult to prove if $p > 3$. We sketch a proof. Let $\mathscr{G}_p$ be the reduction of $\mathscr{G}$ modulo p. It is a split group over $\mathbf{F}_p$, the field of p elements, and is of type E_7. Corresponding to the algebraic subgroups $\mathscr{P}$, $\mathscr{P}^-$, $\mathscr{N}$, etc. in $\mathscr{G}$ are algebraic subgroups $\mathscr{P}_p$, $\mathscr{P}_p^-$, $\mathscr{N}_p$ in $\mathscr{G}_p$. Let $\rho\colon \mathscr{G}(\mathbf{Z}_p) \to G_p = \mathscr{G}_p(\mathbf{F}_p)$ be the mod p reduction map, and note that $\rho(\mathscr{P}(\mathbf{Z}_p)) \subseteq P_p = \mathscr{P}_p(\mathbf{F}_p)$, etc. By [3, Theorem 5], N_p and N_p^- generate G_p, but clearly $N_p = \rho(\mathscr{N}(\mathbf{Z}_p))$, and similarly with N_p^-, so $\rho(\mathscr{G}(\mathbf{Z}_p)) = G_p$. Now, one can deduce from root theoretic manipulations plus [9, 4.5.3] that the $\rho(w_i)$'s are representatives for the P_p double cosets in G_p. Since ρ is surjective, it follows that

$$G(\mathbf{Z}_p) = \bigcup_{i=0}^{3} \rho^{-1}(P_p \rho(w_i) P_p) = \bigcup_{i=0}^{3} B w_i B.$$

By Iwahori decomposition for B we have $B w_i B = \mathscr{P}(\mathbf{Z}_p) w_i B$, so $P \cdot B w_i B = P w_i B$, but $P \cdot \mathscr{G}(\mathbf{Z}_p) = G$, so $\bigcup_{i=0}^{3} P w_i B = G$.

For $p \leq 3$, however, we do not know that $G = \bigcup_{i=0}^{3} P w_i B$. To avoid this, we modify the procedure described in 1.4.

2.7.1. Let ϕ_w be the restriction to PwB of the spherical function ϕ, and let $\phi_i = \phi_{w_i}$ for $i = 0, 1, 2, 3$. Let $V_{\mathrm{reg}}^B = \bigoplus_{i=0}^{3} \mathbf{C}\phi_i$ and let V_{sing}^B be the space generated by ϕ_w's corresponding to double cosets PwB not among the $P w_i B$'s. Then $V^B = V_{\mathrm{reg}}^B \oplus V_{\mathrm{sing}}^B$, and we let Π_{reg} be the projection onto V_{reg}^B.

Let ψ be a character of $J(\mathbf{Q}_p)/J(\mathbf{Z}_p)$ and note that, for any non-trivial character τ_p of $\mathbf{Q}_p$, there exists $T \in J(\mathbf{Q}_p)$ such that

$$\psi(x) = \tau_p((T, x)) \quad (x \in J(\mathbf{Q}_p)),$$

where $(T, x) = \mathrm{tr}_J (T \circ x)$. We take τ_p to be trivial on $\mathbf{Z}_p$ but non-trivial on $p^{-1}\mathbf{Z}_p$. Then $T \in J(\mathbf{Z}_p)$ because $J(\mathbf{Z}_p)$ is self-adjoint with respect to the bilinear form $(\ ,\ )$ on J; see [1, 1.5].

2.7.2. We wish to calculate the linear functionals $\Lambda_\psi = \Lambda_\psi(\chi)$ defined by the Cauchy principal value integral (see 1.4)

$$\langle \Lambda_\psi, f \rangle = \int_N f(w_3 n)\psi(n)^{-1}\, dn$$

for each $f \in \operatorname{Ind}(\chi \mid P, G)$. Let $w = w_i$. Recall from 1.5 that the linear functional λ_i satisfies

$$\langle \lambda_i, f \rangle = \int_{N \cap w^{-1}Pw \backslash N} f(wn) d_w n$$

for $f \in \operatorname{Ind}(\chi \mid P, G)$ whenever the right side converges. Since, by Lemma 2.6.1, $w_i N \cap PxB$ is empty unless $PxB = Pw_j B$ for some $j \leq i$, we see that Λ_ψ annihilates V^B_{sing}, as do the λ_i's. Furthermore, $\langle \lambda_i, \phi_j \rangle = 0$ if $i < j$, and $\langle \lambda_i, \phi_i \rangle = 1$ because $PwN \cap PwB = Pw\mathcal{N}(\mathbf{Z}_p)$ by Lemma 2.6.1. Clearly, the matrix $(\langle \lambda_i, \phi_j \rangle)_{ij}$ is lower triangular unipotent, hence the λ_i's form a basis for the dual of V^B_{reg}. Let $\{f_i\}$ be the dual basis to the basis $\{\lambda_i\}_{0 \leq i \leq 3}$ of V^B_{reg}. Then $f_3 = \phi_3$ and $f_2 = \phi_2 - \langle \lambda_3, \phi_2 \rangle \phi_3$.

2.7.3. Let $A^- = \{a \in A^\# : aN^-(p)a^{-1} \subseteq N^-(p)\}$. Suppose $a \in A^-$. Then, by 1.2.2 (1) and 1.3 (1),

$$\Pi_{\text{reg}} \circ \Pi_B(\pi(a)f_j) = w_j^{-1}\chi(a)\delta_j(a)f_j, \tag{1}$$

where $\delta_j = \delta_{w_j}$ is the modulus of $\operatorname{Int}(a)$ on $N \cap w_j^{-1}Pw_j\backslash N$. It follows that

$$\langle \Lambda_\psi, \pi(a)f_j \rangle = w_j^{-1}\chi(a)\delta_j(a)\langle \Lambda_\psi, f_j \rangle. \tag{2}$$

Since $\pi(a)\phi = \sum_{j=0}^{3} \langle \lambda_j, \phi \rangle \pi(a)f_j$, we have the important formula

$$\langle \Lambda_\psi, \pi(a)\phi \rangle = \sum_{j=0}^{3} \langle \lambda_j, \phi \rangle w_j^{-1}\chi(a)\delta_j(a)\langle \Lambda_\psi, f_j \rangle. \tag{3}$$

Let

$$\zeta_\psi(\chi, a) = \langle \Lambda_\psi, \pi(a)\phi \rangle / \langle \Lambda_\psi, \phi \rangle, \tag{4}$$

$$C_j(\chi) = \langle \lambda_j, \phi \rangle \langle \Lambda_\psi, f_j \rangle / \langle \Lambda_\psi, \phi \rangle. \tag{5}$$

By [9, 4.7 (1)] we have the functional equation $\zeta_\psi(\chi, a) = \zeta_\psi(\delta\chi^{-1}, a)$.

2.7.4. LEMMA. *For* $a \in A^\#(=S_0^\#(\mathbf{Q}_p))$ *we have* $w_i\delta(a)\delta_i(a) = \delta_{3-i}(a)$, $0 \leq i \leq 3$.

Proof. This is obvious for $i = 0$ or $i = 3$, so suppose $i = 1$. Recall that the maximal Q-split torus $S_1^\#$ in $\mathcal{M}^\#$ leaves invariant under conjugation the lines spanned by the orthogonal idempotents e_i $(i = 1, 2, 3)$ in J $(= \mathcal{N})$. The character α_i with $\operatorname{Int}(t)e_i = \alpha_i(t)e_i$, for $t \in S_1$, corresponds to the $\mathbf{Q}$-root $2\varepsilon_i$, and we find that the modulus of $\operatorname{Int}(t)$ on the 8-dimensional Peirce components J_{1j} (with $j \neq 1$) is $|\alpha_1(t)\alpha_j(t)|_p^4$. Let $t_j = \alpha_j(t)$. Then $\delta(t) = |t_1 t_2 t_3|_p^9$ and, noting that

$N \cap w_j^{-1}Pw_j \backslash N$ can be identified with the subalgebra $J^{(j)} = \bigoplus J_{kk'}$ where $0 \leq k, k' \leq j$, we see that $\delta_1(t) = |t_1|_p$, $\delta_2(t) = |t_1t_2|_p^5$, and $w_1\delta(t) = |t_1^{-1}t_2t_3|_p^9$. Since $S_0^\#$ is the subtorus of $S_1^\#$ defined by $t_1 = t_2 = t_3$, if we let $t = \alpha_i(a)$, then

$$w_1\delta(a)\delta_1(a) = |t^9|_p|t|_p = |t|_p^{10} = \delta_2(a).$$

The case $i = 2$ is similiar.

2.7.5. Let $\chi = \chi_s = |\chi_J|_p^{s/2}$ as in 2.2, $s \in \mathbf{C}$. Then $\chi(t) = |t_1t_2t_3|_p^{s/2}$ and $(w_i(t))_j = t_j^\varepsilon$ with $\varepsilon = 1$ if $i < j$ and $\varepsilon = -1$ if $i \geq j$. Therefore,

$$\chi(w_1a) = \chi(a)^{1/3},\ \chi(w_2a) = \chi(a)^{-1/3} \text{ and } \chi(w_3a) = \chi(a)^{-1} \quad \text{for } a \in A^\#.$$

Let $\alpha = \delta_1$, so $\chi = \alpha^{3s}$, $\delta_2 = \alpha^{10}$, $\delta = \delta_3 = \alpha^{27}$. We have

$$w_1^{-1}\chi \cdot \delta_1 = \alpha^{1+s},\ w_2^{-1}\chi \cdot \delta_2 = \alpha^{10-s} \quad \text{and} \quad w_3^{-1}\chi \cdot \delta_3 = \alpha^{27-3s}.$$

For $s \neq 1/2$, $5/2$, $9/2$, $13/2$ or $17/2$, the characters $w_i^{-1}\chi \cdot \delta_1$ are distinct and therefore linearly independent, i.e., χ is a *regular character.* The functional equation $\zeta_\psi(\chi, a) = \zeta_\psi(\delta\chi^{-1}, a)$ therefore implies that, if χ is regular, then

$$\text{(1)} \qquad C_i(\delta\chi^{-1}) = C_{3-i}(\chi).$$

Moreover, since $s \mapsto \zeta_\psi(\chi_s, a)$ is an analytic function of s and since our formulas will be analytic, there is no harm in restricting our attention to the regular characters.

To compute the value of $\zeta_\psi(\chi, a)$, it suffices to compute $C_3(\chi)$ and $C_2(\chi)$, which requires the calculation of $\langle\Lambda_\psi, \phi\rangle$, $\langle\lambda_3, \phi\rangle$, $\langle\Lambda_\psi, \phi_3\rangle$, $\langle\lambda_3, \phi_2\rangle$ and $\langle\Lambda_\psi, \phi_2\rangle$, but only for a restricted class of characters ψ. This will be done in the next section.

3. Values of some p-adic integrals

3.1 We first evaluate $\langle\Lambda_\psi(\chi_s), \phi\rangle$, where $\psi(X) = \tau_p((T, X))$ with τ_p the standard character on $\mathbf{Q}_p/\mathbf{Z}_p$ and $T \in J(\mathbf{Z}_p)$ but $T \notin pJ(\mathbf{Z}_p)$. Call such ψ and T *primitive.* Observe that each orbit of $A^\#$ on $J(\mathbf{Q}_p)$ contains a primitive T. Observe also that $\langle\Lambda_\psi, \phi\rangle$ coincides with $\mathbf{S}_p(T)$ calculated in [8, Section 12] except that we must replace g there by s. In particular, we may assume that $T = \mathbf{D}(t, t', t'')$, the diagonal matrix with diagonal (t, t', t''), and that t is a p-adic unit. From formula (12.11) of [8],

$$\text{(1)} \qquad \langle\Lambda_\psi(\chi_s), \phi\rangle = (1 - p^{-s})(1 - p^{4-s})(1 - p^{8-s})r_p(T),$$

$$r_p(T) = \sum_{m=0}^{\infty} p^{(8-s)m}\alpha_m''(T),$$

and since $\operatorname{ord}_p(t) = 0$ we have, from (12.10) of [8],

$$\text{(2)} \qquad \alpha_m''(T) = \phi_m(T),$$

where $\phi_m(T)$ is defined by [8, 12.3]. Let T' be the 2 by 2 lower right corner of

T. Then $\phi_m(T)$ coincides with $\alpha'_m(T')$ of [8, Section 6]. By (6.5) and (6.6) of [8], then

$$(3) \qquad \sum_{m=0}^{\infty} \alpha'_m(T')p^{m(4-g)} = \sum_{k=0}^{\tau'} p^{4k} \sum_{m=k}^{d-k} p^{m(5-g)},$$

where $\tau' = \operatorname{ord}_p(t')$ and $d = \operatorname{ord}_p(t't'')$. It follows from (1) that

$$(4) \qquad \langle \Lambda_\psi(\chi_s), \phi\rangle = (1-p^{-s})(1-p^{4-s})(1-p^{8-s})\sum_{k=0}^{\tau'} p^{4k} \sum_{m=k}^{d-k} p^{m(9-s)}.$$

3.2. We let $J^{\{2,3\}}$ be the subalgebra of all matrices in J with first row equal zero. *Henceforth Z will always denote an element of $J^{\{2,3\}}(\mathbf{Z})$.*

To finish computing $C_3(\chi_s)$ we must evaluate $\langle \Lambda_\psi, f_3\rangle$ and $\langle \lambda_3, \phi\rangle$. Recall from 2.7.2 that $f_3 = \phi_3$ is the restriction of ϕ to Pw_3B. By Lemma 2.6.1 we have

$$w_3N \cap Pw_3B = w_3\mathcal{N}(\mathbf{Z}_p);$$

hence,

$$(1) \qquad \langle \Lambda_\psi, f_3\rangle = \int_{\mathcal{N}(\mathbf{Z}_p)} \phi(w_3n)\psi(n)^{-1}dn = 1$$

because $\phi_3(w_3n) = 1 = \psi(n)$ if $n \in \mathcal{N}(\mathbf{Z}_p)$.

The integral $\langle \lambda_3, \phi\rangle = \int_N \phi(w_3n)dn$ can be evaluated by using results from [8], since it is $S_p(0)$. Namely, from Section 12 of [8],

$$(2) \qquad \langle \lambda_3, \phi\rangle = (1-p^{-s})(1-p^{4-s})(1-p^{8-s})\sum_{m=0}^{\infty} p^{m(8-s)}\alpha''_m(0),$$

where

$$\alpha''_m(0) = \sum_{k=0}^{m} C_k\psi'_{m-k}, \quad C_k = p^k\sum_{i=0}^{k} p^{3i},$$

and ψ'_m is the number of $Z \in J^{\{2,3\}}(\mathbf{Z})/p^mJ^{\{2,3\}}(\mathbf{Z})$ satisfying $Q(\mathbf{Z}) \equiv 0 \pmod{p^m}$ with Q the quadratic form

$$Q(Z) = bc - z\bar{z} \quad \text{if } Z = \begin{pmatrix} b & z \\ \bar{z} & c\end{pmatrix}.$$

However, ψ'_{m-k} is the same as $\alpha_{m-k}(0)$ in Section 6 of [8], so its value can be extracted from the results there:

$$(3) \qquad \psi'_{m-k} = p^{4(m-k)}\left\{p^{m-k}\sum_{j=0}^{m-k} p^{4j} - p^{m-1-k}\sum_{j=0}^{m-1-k} p^{4j}\right\}.$$

Therefore, we find that

$$(4) \qquad \alpha''_m(0) = p^m \sum_{k=0}^{m} p^{3k}\left(\sum_{i=0}^{m-k} p^{3i}\right)\{d_k - d_{k-1}\}, \quad \text{where } d_k = p^k\sum_{j=0}^{k} p^{4j}.$$

By letting $l = i + k$ and changing the order of summation we have

(5) $$\sum_{m=0}^{\infty} p^{m(8-s)}\alpha''_m(0) = (1-p^{9-s})^{-1} \sum_{k=0}^{\infty} (d_k - d_{k-1}) \sum_{l=k}^{\infty} p^{l(12-s)},$$

and the right side reduces to $(1-p^{9-s})^{-1} \sum_{k=0}^{\infty} d_k p^{k(12-s)}$. Substituting for d_k and summing yields the value $(1-p^{9-s})^{-1}(1-p^{13-s})^{-1}(1-p^{17-s})^{-1}$ for the right side, hence

(6) $$\langle \lambda_3, \phi \rangle = (1-p^{-s})(1-p^{4-s})(1-p^{8-s})/(1-p^{9-s})(1-p^{13-s})(1-p^{17-s}).$$

Theretore,

(7) $$1/C_3(\chi_s) = (1-p^{9-s})(1-p^{13-s})(1-p^{17-s}) \sum_{k=0}^{\tau'} p^{4k} \sum_{m=k}^{d-k} p^{m(9-s)}.$$

3.3. Recall from 2.7.2 that $f_2 = \phi_2 - \langle \lambda_3, \phi_2 \rangle \phi_3$. We now calculate

$$\langle \lambda_3, \phi_2 \rangle = \int_N \phi_2(w_3 n) dn \quad \text{for } \chi = \chi_s$$

with Re (s) sufficiently large to insure convergence.

From Lemma 2.6.1 we find that $\phi_2(w_3 u_X) \neq 0$ if and only if X has exactly one elementary divisor d_1 not in $\mathbf{Z}_p$, in which case, if $\operatorname{ord}_p(d_1) = -m$, then $\phi_2(w_3 u_X) = p^{-ms}$. For such an X with elementary divisors d_1, d_2, and d_3, we have $p^m X$ lies in $J(\mathbf{Z}_p)$ but not in $pJ(\mathbf{Z}_p)$; $(p^m X)^* = (p^m X) \times p^m X$ has elementary divisors

$$p^{2m} d_1 d_2, \quad p^{2m} d_1 d_3 \quad \text{and} \quad p^{2m} d_2 d_3,$$

which are in $p^m \mathbf{Z}_p$, so $(p^m X)^* \in p^m J(\mathbf{Z}_p)$; $\det(p^m X) = p^{3m} d_1 d_2 d_3 \in p^{2m}\mathbf{Z}_p$. Conversely, if $X \in J(\mathbf{Z}_p)$ satisfies $X \not\equiv 0 \pmod{p}$, $X^* \equiv 0 \pmod{p^m}$ and $\det X \equiv 0 \pmod{p^{2m}}$, then $Y = p^{-m}X$ has $\phi_2(w_3 u_Y) = p^{-ms}$. Let $\mathbf{a}_m$ be the number of such $X \in J(\mathbf{Z}_p)/p^m J(\mathbf{Z}_p)$. Then

(1) $$\langle \lambda_3, \phi_2 \rangle = \sum_{m=0}^{\infty} \mathbf{a}_m p^{-ms}.$$

Let A_m be the number of integral X mod p^m with $X^* \equiv 0 \pmod{p^m}$ and $\det X \equiv 0 \pmod{p^{2m}}$. Let A'_m be the number of X mod p^m with $X \equiv 0 \pmod{p}$, $X^* \equiv 0 \pmod{p^m}$ and $\det X \equiv 0 \pmod{p^{2m}}$. Then $\mathbf{a}_m = A_m - A'_m$. By [8, 4.7] we have

(2) $$(1-p^{-s}) \sum_{m=0}^{\infty} A_m p^{-ms} = \mathbf{S}_p(0),$$

which is $\langle \lambda_3, \phi \rangle$ and has been calculated in 3.2 (6). The next few subsections are devoted to evaluating A'_m by the techniques of [8].

3.3.1. Let $q = p^m$. As in [8, Section 8]

(1) $$A'_m = \sum \beta_m(Z).$$

where the sum ranges over $Z \in J^{\{2,3\}}(\mathbf{Z})$ modulo q with $Z \equiv 0 \pmod{p}$, $Q(Z) \equiv 0 \pmod{q}$ and where $\beta_m(Z)$ is the number of pairs (a, W) mod q with

$$W = \begin{pmatrix} 0 & x & y \\ * & 0 & 0 \\ * & 0 & 0 \end{pmatrix} \in J_{12} \oplus J_{13}$$

satisfying:

(i) $a \equiv 0 \pmod{p}$ and $W \equiv 0 \pmod{p}$,
(ii) $W^* + 2ae_1 \times Z \equiv 0 \pmod{q}$,
(iii) $W(2e_1 \times Z) \equiv 0 \pmod{q}$,
(iv) $aQ(Z) + (W^*, Z) \equiv 0 \pmod{q^2}$.

By the argument of [8, p. 192] we may restrict our attention to Z's that are in reduced form, i.e., $Z = \mathbf{D}(0, b, c)$ with $b \mid c$, because the p-adic order invariants of Z determine $\beta_m(Z)$.

We distinguish three cases: (I) $c \not\equiv 0 \pmod{q}$, (II) $b \not\equiv 0 \pmod{q}$ but $c \equiv 0 \pmod{q}$, (III) $b \equiv 0 \pmod{q}$. Case I reduces to calculation done in [8] because $c \not\equiv 0 \pmod{q}$ implies that $a \equiv 0 \pmod{p}$ and $W \equiv 0 \pmod{p}$ whenever conditions (ii), (iii), and (iv) hold. To see this, note that one has $ab \equiv N_{\mathfrak{c}}(x)$ and $ac \equiv N_{\mathfrak{c}}(y)$, where $N_{\mathfrak{c}}$ denotes the reduced norm in the Cayley algebra $\mathfrak{c}$. This gives $abc \equiv cN_{\mathfrak{c}}(x) \pmod{qc}$, $abc \equiv bN_{\mathfrak{c}}(y) \pmod{qb}$ and we have also, from (iv), $abc \equiv cN_{\mathfrak{c}}(x) + bN_{\mathfrak{c}}(y) \pmod{q^2}$. Thus, $abc \equiv 0 \pmod{qb}$, from which it follows that $ac \equiv 0 \pmod{q}$, hence $a \equiv 0 \pmod{p}$. From (iii) we have $cx \equiv by \equiv 0 \pmod{q}$ hence $x \equiv y \equiv 0 \pmod{p}$. It follows that in Case I,

$$\beta_m(Z) = p^{4m}\{\beta'_m(Z) - \beta'_{m-1}(p^{-1}Z)\}, \tag{1}$$

as in [8, (10.9)] with $\beta'_m(Z) = |b|_p^{-4}|c|_p^{-4}\sum_{i=0}^{k} p^{-3i}$, and

$$k = k_m(Z) = \min\{m, \operatorname{ord}_p(b), \operatorname{ord}_p(bc) - m\}.$$

Thus, in Case I we have

$$\beta_m(Z) = p^{8m+k}\sum_{i=0}^{k} p^{3i} - p^{8m+k-5}\sum_{i=0}^{k-1} p^{3i}. \tag{2}$$

3.3.2. We now turn to Case II, where we may take $\operatorname{ord}_p(b) = k < m$ and $\operatorname{ord}_p(c) = m$. Conditions (i)–(iv) of 3.3.1 can be rewritten:

$$x \equiv 0 \pmod{p}, \quad y \equiv 0 \pmod{p^{m-k}}, \quad \bar{x}(y/p^{m-k}) \equiv 0 \pmod{p^k},$$
$$N_{\mathfrak{c}}(y/p^{m-k}) \equiv 0 \pmod{p^k} \quad \text{and} \quad N_{\mathfrak{c}}(x) \equiv 0 \pmod{p^{k+1}}.$$

If x is in the maximal order $\mathfrak{o}$ of the Cayley algebra, let $A(x)$ denote the number of $y \in \mathfrak{o}/p^{m+k}\mathfrak{o}$ with $\bar{x}y \equiv N_{\mathfrak{c}}(y) \equiv 0 \pmod{p^k}$. Then

$$\beta_m(Z) = p^{k-16m}\sum A(x), \tag{1}$$

where the sum runs over $x \in \mathfrak{o}/q^2\mathfrak{o}$ with $x \equiv 0 \pmod{p}$, $N_{\mathfrak{c}}(x) \equiv 0 \pmod{p^{k+1}}$.

Let

$$f(x) = \min \{k, \operatorname{ord}_p(x), \operatorname{ord}_p(N_c(x)) - k\},$$

where $\operatorname{ord}_p(x)$ is the integer n with $p^{-n}x \in \mathfrak{o}$, $p^{-1-n}x \notin \mathfrak{o}$. By results from [8, Section 10] we have

$$\beta_m(Z) = p^{8m-8-3k} \sum_x \left\{ \sum_{n=0}^{f(x)} p^{3n} - \sum_{n=1}^{f(x)} p^{3n-4} \right\}, \tag{2}$$

where x ranges over $\mathfrak{o}/p^{k+1}\mathfrak{o}$ subject to the restrictions $x \equiv 0 \pmod p$ and $N_c(x) \equiv 0 \pmod{p^{k+1}}$; these restrictions are equivalent to $f(x) > 0$.

Let $\sigma_k(F)$ be the number of $x \in \mathfrak{o}/p^{k+1}\mathfrak{o}$ with $f(x) \geq F$ and $f(x) \geq 1$, and note that $\sigma_k(F) = p^8\sigma'(k-F)$ whenever $k \geq F \geq 1$, where $\sigma'(n)$ is defined to be the number of $x \in \mathfrak{o}/p^n\mathfrak{o}$ with $\operatorname{ord}_p(N_c(x)) \geq n$. Also, $\sigma_k(0) = p^8\sigma'(k-1)$ for $k \geq 1$. From [8, Section 10],

$$\sigma'(k) = p^{4k} \sum_{i=0}^{k} p^{3i} - p^{4k-1} \sum_{i=0}^{k-1} p^{3i}.$$

Clearly then,

$$\beta_m(Z) = p^{8m-3k} \sum_{n=1}^{k} p^{3n}\sigma'(k-n) + \sigma'(k-1) - \sum_{n=1}^{k} p^{3n-4}\sigma'(k-n). \tag{3}$$

Since $\sum_{n=0}^{k} p^{3n}\sigma'(k-n) = p^{4k} \sum_{i=0}^{k} p^{3i}$, we have

$$\beta_m(Z) = p^{8m-3k} \left\{ (1-p^{-4}) \left(p^{4k} \sum_{i=0}^{k} p^{3i} - \sigma'(k) \right) + \sigma'(k-1) \right\} \tag{4}$$

and finally in Case II, we have

$$\beta_m(Z) = p^{8m+k-5} \left\{ (p^4 + p - 1) \sum_{i=0}^{k-1} p^{3i} - \sum_{i=0}^{k-2} p^{3i} \right\}. \tag{5}$$

A straightforward calculation shows that in Case II we can write

$$\beta_m(Z) = p^{8m+k} \sum_{i=0}^{k} p^{3i} - p^{8m+k-5} \sum_{i=0}^{k-1} p^{3i} + d_m(k), \tag{6}$$

$$d_m(k) = p^{8m+k-5}(p^4-1)(p-1)/(p^3-1) - p^{8m+4k-8}(p^4-1)(p^7-1)/(p^3-1).$$

3.3.3. For later use we need the number of $Z \in J^{\{2,3\}}(\mathbf{Z})/p^m J^{\{2,3\}}(\mathbf{Z})$ that fall into Case II with $k_m(Z) = k$, $1 \leq k < m$. The number is the same as the number of $Z \pmod{p^{m-k}}$ with $Z \not\equiv 0 \pmod p$ and $Q(Z) \equiv 0 \pmod{p^{m-k}}$, call it d'_{m-k}. Let d''_n be the number of $Z \pmod{p^n}$ with $Q(Z) \equiv 0 \pmod{p^n}$. If $n > 1$, then

$d'_n = d''_n - p^{10}d''_{n-2}$, while $d'_1 = d''_1 - 1$. We have $d''_n = \alpha_n(0)$ in the notation of [8, Section 6], so it follows that

(1) $$d''_n = p^{5n}\sum_{i=0}^{n} p^{4i} - p^{5n-1}\sum_{i=0}^{n-1} p^{4i}.$$

A brief calculation yields

(2) $$d'_n = (1 + p^{-4})(p^{9n} - p^{4+9(n-1)}).$$

Let $d = \sum_{m=2}^{\infty} p^{-ms}\{\sum_Z d_m(Z)\}$, where for each m the inner sum ranges over Z (mod p^m) that fall into Case II, and $d_m(Z) = d_m(k_m(Z))$. One verifies easily that

$$\sum_{m=2}^{\infty} p^{(8-s)m-8}\sum_{k=1}^{m-1} p^{4k}d'_{m-k} = p^{-8}(1 + p^{-4})(p^5 - 1)p^{24-2s}/(1 - p^{17-s})(1 - p^{12-s})$$

and that

$$\sum_{m=2}^{\infty} p^{(8-s)m}\sum_{k=1}^{m-1} p^{k}d'_{m-k} = p^{-1}(p^4 + 1)(p^5 - 1)p^{18-2s}/(1 - p^{17-s})(1 - p^{9-s}).$$

It follows that

(3) $$d = (p^5 - 1)(p^8 - 1)p^{12-2s}[(p - 1)/(1 - p^{9-s}) \\ - (p^7 - 1)/(1 - p^{12-s})]/(p^3 - 1)(1 - p^{17-s}).$$

We let d_m be the coefficient of p^{-ms} in the power series expansion of d.

3.3.4. To complete the calculation of A'_m it remains only to deal with Case III, where we must compute $\beta_m(0)$, which is the number of

$$(a, x, y) \in p(\mathbf{Z} \times \mathfrak{o} \times \mathfrak{o})/p^m(\mathbf{Z} \times \mathfrak{o} \times \mathfrak{o})$$

that satisfy $N_c(x) \equiv N_c(y) \equiv x\bar{y} \equiv 0 \pmod{p^m}$. Clearly, if $m \geq 2$, then $\beta_m(0) = p^{m+15}D_{m-2}$, where D_k is the number of $(x, y) \in \mathfrak{o} \times \mathfrak{o}/p^k(\mathfrak{o} \times \mathfrak{o})$ with $N_c(x) = N_c(y) \equiv x\bar{y} \equiv 0 \pmod{p^k}$. From [8, Lemma 2.4],

(1) $$D_k = p^{4k}\left\{\sum_{j=0}^{k} p^{3j}\sigma_k(j) - \sum_{j=1}^{k} p^{3j-4}\sigma_k(j)\right\},$$

where $\sigma_k(j)$ is the number of $x \in \mathfrak{o}/p^k\mathfrak{o}$ with $x \equiv 0 \pmod{p^j}$ and $N_c(x) \equiv 0 \pmod{p^{k+j}}$. Since $\sigma_k(j) = \sigma_0(k - j)$, and, by [8, p. 182],

$$\sigma_0(k - j) = p^{4k-4j}\sum_{i=0}^{k-j} p^{3i} - p^{4k-4j-1}\sum_{i=0}^{k-(j+1)} p^{3i},$$

we find after a straightforward calculation that

(2) $$D_k = p^{8k}\left\{\sum_{i=0}^{k} p^{3i} - p^{-5}\sum_{i=0}^{k-1} p^{3i}\right\}.$$

Therefore, if $m \geq 2$, then

$$\beta_m(0) = p^{9m-1}\left\{\sum_{i=0}^{m-2} p^{3i} - \sum_{i=0}^{m-3} p^{3i-5}\right\}. \tag{3}$$

Therefore, we have,

$$\beta_m(0) = p^{4m}\left\{p^{5m}\sum_{i=0}^{m} p^{3i} - p^{5(m-1)}\sum_{i=0}^{m-1} p^{3i}\right\} + c_m \tag{4}$$

where, for $m \geq 2$,

$$c_m = [(p^5-1)(p-1)p^{9m-6} - (p^7-1)(p^8-1)p^{12m-12}]/(p^3-1). \tag{5}$$

Let $c = \sum_{m=2}^{\infty} c_m p^{-ms}$. Then

(6)

$$c = [(p-1)(p^5-1)/(1-p^{9-s}) - (p^7-1)(p^8-1)/(1-p^{12-s})]p^{12-2s}/(p^3-1).$$

Moreover, with d as in 3.3.3 (3),

$$c + d = (1 + p^4 - p^9 - p^{12})p^{17-2s}/(1-p^{17-s}). \tag{7}$$

3.3.6. We now calculate $A' = \sum_{m=0}^{\infty} p^{-ms}A'_m$ and $\langle \lambda_3, \phi_2\rangle = A - A'$. Let

$$\beta'_m(Z) = p^{4m+k}\sum_{i=0}^{k} p^{3i},$$

where

$$k = k_m(Z) = \min(m, \operatorname{ord}_p(b), \operatorname{ord}_p(bc) - m).$$

Note that $p^{4m+k-5} \sum_{i=0}^{k-1} p^{3i} = p^5\beta'_{m-1}(p^{-1}A) + p^{4m+4k}$ provided $Z \equiv 0 \pmod p$, so

$$\beta_m(Z) = (p^5-1)p^{4m}\beta'_{m-1}(p^{-1}Z) + p^{4m+4k} + c_m(Z) + d_m(Z),$$

where $d_m(Z) = 0$ unless $Z \equiv 0 \pmod{p^m}$ and $c_m(Z) = 0$ unless $c \equiv 0 \pmod{p^m}$ and $b \not\equiv 0 \pmod{p^m}$.

We let $a_m(Z) = (p^5-1)p^{8m}\beta'_{m-1}(p^{-1}Z)$, and $b_m(Z) = p^{8m+4k}$. We require the values of

$$a = \sum_{m=2}^{\infty}\left\{\sum_Z a_m(Z)\right\}p^{-ms} \quad \text{and} \quad b = \sum_{m=2}^{\infty}\left\{\sum_Z b_m(Z)\right\}p^{-ms},$$

where Z runs mod p^m with $Z \equiv 0 \pmod p$. We note that

$$a = p^{4-s}(p^5-1)\sum_{m=0}^{\infty} p^{(4-s)m}\sum_Z \beta'_m(Z),$$

where Z runs mod p^m without restriction. Comparing this with [8, 10.10], we see that

(1) $$a = (p^5 - 1)p^{4-s}\langle\lambda_3, \phi\rangle/(1 - p^{-s})(1 - p^{4-s}) - (p^5 - 1)p^{4-s}.$$

On the other hand, we can calculate b by counting Z (mod p^m) such that $Z \equiv 0$ (mod p) and $k_m(Z) = k$. For example, if $k \geq 1$, this is $\alpha_{m-k}(0) - \alpha_{m-k-1}(0)$, where $\alpha_n(0)$ is the number of Z (mod p^n) with $Q(Z) \equiv 0$ (mod p^n). If $k = 0$, we have $p^{10}\alpha_{m-2}(0) - \alpha_{m-1}(0)$ such values for Z. Using the fact [8, Section 6] that

(2) $$\alpha_n(0) = p^{5n} \sum_{i=0}^{n} p^{4i} - p^{5n-1} \sum_{i=0}^{n-1} p^{4i},$$

a lengthy but routine calculation yields

(3) $$b = [p^{12-s} + (p^8 - p^7 - p^3)p^{18-2s}]/(1 - p^{13-s})(1 - p^{17-s}) - p^{12-s}.$$

Recall that $\langle\lambda_3, \phi_2\rangle = A - A'$, $A' = a + b + c + d + p^{-s} + 1$. Assembling the results (1) and (3) above along with the values of A, c and d, we find that

(4) $$\langle\lambda_3, \phi_2\rangle = (1 - p^{8-s})(1 + p^{9-s} + p^{13-s})/(1 - p^{17-s}) - p^{4-s} - p^{-s} - 1.$$

3.4. To calculate $C_2(\chi)$ we need the values of $\langle\lambda_2, \phi\rangle$ and $\langle\Lambda_\psi, f_2\rangle$, where $f_2 = \phi_2 - \langle\lambda_3, \phi_2\rangle\, \phi_3$.

3.4.1. We have

$$\langle\lambda_2, \phi\rangle = \int_{N \cap w^{-1}Pw\backslash N} \phi(wn)d_w n,$$

where $w = w_2$. The subgroup $N \cap w^{-1}Pw$ is the product of root groups corresponding to the roots $\varepsilon_i + \varepsilon_3$, so we can identify $N \cap w^{-1}Pw\backslash N$ with the product of root groups $\varepsilon_i + \varepsilon_j$ with $1 \leq i, j \leq 2$. This makes it possible to identify the integral with the following sum, which is evaluated in [8, Section 6]:

(1) $$(1 - p^{-s}) \sum_{m=0}^{\infty} \alpha_m(0)p^{-ms} = (1 - p^{-s})(1 - p^{4-s}) \sum_{k=0}^{\infty} p^{4k} \sum_{m=k}^{\infty} p^{m(5-s)}$$

Thus

(2) $$\langle\lambda_2, \phi\rangle = (1 - p^{-s})(1 - p^{4-s})/(1 - p^{5-s})(1 - p^{9-s}).$$

3.4.2. We have $\langle\Lambda_\psi, f_2\rangle = \langle\Lambda_\psi, \phi_2\rangle - \langle\lambda_3, \phi_2\rangle$ because $\langle\Lambda_\psi, \phi_3\rangle = 1$; cf. 3.2 (1). Thus, we must calculate $\langle\Lambda_\psi, \phi_2\rangle$. Recall that $\phi_2(w_3 u_X) \neq 0$ if and only if X has exactly one p-adic order invariant < 0. Suppose that $\psi(X) = \tau_p((T, X))$ with T primitive in $J(\mathbf{Z}_p)$. Let $\omega_m^a = \tau_p(a/p^m)$, and let

$$A_m = \sum_X \omega_m^{(T,X)}$$

where the sum ranges over $X \in J(\mathbf{Z}_p)/p^m J(\mathbf{Z}_p)$ satisfying $X^* \equiv 0 \pmod{p^m}$ and $\det(X) \equiv 0 \pmod{p^{2m}}$. Let

$$A'_m = \sum_X \omega_m^{(T,X)}$$

where the sum runs over $X \in J(\mathbf{Z}_p)/p^m J(\mathbf{Z}_p)$ satisfying $X^* \equiv 0 \pmod{p^m}$, $\det(X) \equiv 0 \pmod{p^{2m}}$ and $X \equiv 0 \pmod{p}$. Then

$$\text{(1)} \qquad \langle \Lambda_\psi, \phi_2 \rangle = \sum_{m=1}^{\infty} p^{-ms}(A_m - A'_m) = A - A',$$

where $A = \sum_{m=1}^{\infty} p^{-ms} A_m$ and $A' = \sum_{m=1}^{\infty} p^{-ms} A'_m$.

3.4.3. From [8, Section 11] we have

$$\text{(2)} \qquad A_m = \sum_Z \omega_m^{(T,X)} \beta_m(T; Z),$$

summed over $Z \in J^{(2)}(\mathbf{Z}_p)/p^m J^{(2)}(\mathbf{Z}_p)$ with

$$\beta_m(T: Z) = p^{4m}\{\beta'_m(T; Z) - \beta'_{m-1}(T; p^{-1}Z)\},$$

and $\beta'_m(T; Z)$ is the characteristic function of the set of Z with $Q(Z) \equiv 0 \pmod{p^m}$. Then

$$\text{(3)} \qquad A = (1 - p^{4-s}) \sum_{m=0}^{\infty} \alpha'_m p^{m(4-s)} - 1,$$

with

$$\text{(4)} \qquad \alpha'_m = \sum_Z \omega_m^{(T,Z)},$$

where Z is summed modulo p^m with the restriction $Q(Z) \equiv 0 \pmod{p^m}$. As in [8, Section 6] one finds that

$$\text{(5)} \qquad A = (1 - p^{4-s})(1 - p^{8-s}) \sum_{k=0}^{\tau'} p^{4k} \sum_{m=k}^{d-k} p^{m(9-s)} - 1,$$

where $\tau' = \operatorname{ord}_p(t')$ and $d = \operatorname{ord}_p(t't'')$ if T is reduced with diagonal (t, t', t''). Let

$$\text{(6)} \qquad F_T(X) = \sum_{k=0}^{\tau'} p^{4k} \sum_{m=k}^{d-k} X^m.$$

3.4.4. The calculation of A'_m proceeds by the method of [8, Sections 8–9]. Let T' be the lower right 2 by 2 submatrix of T and let $\alpha_n(T')$ be as in [8, Section 6]. In particular, if T' is not integral, then $\alpha_n(T') = 0$. The result of the calculation for $m \geq 2$ is

$$\text{(1)} \qquad A'_m = p^{4m+10}\alpha_{m-2}(p^{-1}T') - p^{4m}\alpha_{m-1}(T').$$

From [8, Section 6], we have

$$\text{(2)} \qquad A' = p^{-s} + p^{4-s} + (1 - p^{8-s})[p^{18-2s}F_{T/p}(p^{9-s}) - p^{4-s}F_T(p^{9-s})].$$

Recall that $\langle \Lambda_\psi, \phi_2 \rangle = A - A'$. Since from 3.4.3 (5) we have

$$A = (1 - p^{4-s})(1 - p^{8-s})F_T(p^{9-s}) - 1$$

it follows that

(3) $\langle \Lambda_\psi, \phi_2 \rangle = -1 - p^{-s} - p^{4-s} - (1 - p^{8-s})[p^{18-2s}F_{T/p}(p^{9-s}) - F_T(p^{9-s})]$.

On the other hand, $\langle \Lambda_\psi, f_2 \rangle = \langle \Lambda_\psi, \phi_2 \rangle - \langle \lambda_3, \phi_2 \rangle$, so combining (3) with 3.3.6 (4) gives

$$(4) \qquad \langle \Lambda_\psi, f_2 \rangle = (1 - p^{8-s})[-(1 + p^{9-s} + p^{13-s})/(1 - p^{17-s}) + F_T(p^{9-s}) - p^{18-2s}F_{T/p}(p^{9-s})].$$

From 2.7.3 (5),

$$C_2(\chi) = \langle \lambda_2, \phi \rangle \langle \Lambda_\psi, f_2 \rangle / \langle \Lambda_\psi, \phi \rangle.$$

From 3.4.1 (2) and 3.1 (4) it follows that

$$(5) \qquad C_2(\chi) = [-(1 + (p^4 + 1)X)/(1 - p^8X) + F_T(X) - X^2F_{T/p}(X)]/[(1 - X)(1 - X/p^4)F_T(X)],$$

where $X = p^{9-s}$, $\chi = \chi_s$.

4. The value of the Whittaker function

Having evaluated $C_2(\chi)$ and $C_3(\chi)$ for $\chi = \chi_s$ and for primitive ψ we now compute $\langle \Lambda_\psi, \phi \rangle$ for all characters ψ on $N/\mathcal{N}(\mathbf{Z}_p)$ and all regular unramified characters χ on $A^\#$. Observe that

$$(1) \qquad \langle \Lambda_\psi, \pi(a)\phi \rangle = \chi^{-1}(a)\delta(a)\langle \Lambda_{\psi \circ \mathrm{Int}(a)}, \phi \rangle,$$

and recall that, for $a \in A^-$, by 2.7.3. (3),

$$(2) \qquad \langle \Lambda_\psi, \pi(a)\phi \rangle = \langle \Lambda_\psi, \phi \rangle \sum_{j=0}^{3} C_j(\chi)w_j^{-1}\chi(a)\delta_j(a).$$

Since each character of $\mathcal{N}(\mathbf{Q}_p)/\mathcal{N}(\mathbf{Z}_p)$ is expressible as $\psi \circ \mathrm{Int}\,(a)$ for some $a \in A^-$ and some primitive character ψ, it suffices to work with primitive characters ψ.

4.1. Suppose that Int (a) acts on N^- as multiplication by p^n. With $X = p^{9-s}$, then $X^{3n/2} = \delta(a)^{-1/2}\chi(a)$, and we have

$$(3) \qquad \delta(a)^{-1/2}\zeta_\psi(\chi, a) = C_0(\chi)X^{3n/2} + C_1(\chi)X^{n/2}p^{8n} + C_2(\chi)X^{-n/2}p^{8n} + C_3(\chi)X^{-3n/2}.$$

Therefore, by (1) and the functional equation

$$(4) \quad \langle \Lambda_{\psi \circ \mathrm{Int}\,(a)}, \phi \rangle = (C_3(\delta\chi^{-1})X^{3n} + C_2(\delta\chi^{-1})X^{2n}p^{8n} + C_2(\chi)X^n p^{8n} + C_3(\chi))\langle \Lambda_\psi, \phi \rangle.$$

We let $\mathscr{C}_0(X) = C_3(\chi)$, $\mathscr{C}_1(X) = C_2(\chi)$. Then, for regular χ, $\mathscr{C}_0(X^{-1}) = C_3(\delta\chi^{-1})$ and $\mathscr{C}_1(X^{-1}) = C_2(\delta\chi^{-1})$. From 7.7 and 9 (5) in [1] one sees that if $\psi(X) = \tau_p(\mathrm{tr}_J\,(T \circ X))$ for all X in $J(\mathbf{Q}_p)$, then $\langle \Lambda_\psi, \phi \rangle$ is the *pth* Euler factor $a_s(T)_p (= \mathbf{S}_p(T)$ in the notation of [1]) of the Fourier coefficient $a_s(T)$ of the Eisenstein series of weight s.

4.2. Let $q = p^4$. Then for primitive $T \in J(\mathbf{Z}_p)$ and for $m \in \mathbf{Z}$, $m \geq 0$, we have

(1) $a_s(p^m T)_p / a_s(T)_p = \mathscr{C}_0(X^{-1})X^{3m} + \mathscr{C}_1(X^{-1})q^{2m}X^{2m} + \mathscr{C}_1(X)q^{2m}X^m + \mathscr{C}_0(X)$.

Here, the rational functions $\mathscr{C}_0(X)$ and $\mathscr{C}_1(X)$ are given by the formulas

$$\mathscr{C}_0(X) = 1/(1 - X)(1 - qX)(1 - q^2X)F_T(X) \tag{2}$$

(see 3.2 (7) and 3.4.4 (5));

$$\begin{aligned} \mathscr{C}_1(X) = {} & [-(1 + (q + 1)X)/(1 - q^2X) + F_T(X) - X^2F_{T/p}(X)]/ \\ & (1 - X)(1 - X/q)F_T(X). \end{aligned} \tag{3}$$

4.3. The formulas for $a_s(T)_p$ have been checked against values computed directly from [8, Chapter III] for $T = p^m E$ with $m = 0, 1, 2$ and with E the identity element of J and for $T = p^m T_0$ with $m = 0, 1, 2$ and $T_0 = \mathbf{D}(1, 1, p)$. Further computations suggested the conjecture that, for T with p-adic order invariants $\tau(1) = 0$, $\tau(2)$ and $\tau(3)$,

$$a_s(p^m T)_p / a_s(E)_p = \sum_{i=0}^{m} q^i \left(\sum_{j=1}^{S(i)} q^j \left(\sum_{k=j}^{m''-j} X^k \right) \right), \tag{†}$$

where $m' = m + \tau(2)$, $m'' = m' + \tau(3)$ and

$$S(i) = \min\,(m' - i, [(m'' - i)/2],$$

where $[x]$ is the largest integer less than or equal to x.

A high-speed computer evaluation of formulas (1) and 4.2 (1) with $p = 2$ and $X = 1.4$, and with $p = 3$ and $X = 1.2$, produced additional evidence for the conjecture. The values of T and m considered were $0 \leq m \leq 5$, $0 \leq \tau(3) - \tau(2) \leq 5$ and $\tau(2) = 3, 4$.

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