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THE AUTOMORPHISMS OF $PU_4^+(K, f)$

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The automorphisms of the classical groups have been discussed in many places [1], [2], [3], [6], [7], [9], [10]. The paper presents a solution to the problem in a minimal case heretofore not discussed. Namely, let E be a vector space over a field K admitting an automorphism J such that $J^2 = 1$. Let there be a hermitian sesquilinear form $f\colon E \times E \to K$ defined relative to J. Then, designate by $U(E, f)$ the group of linear transformations leaving invariant f. When $\dim_K E = n$, we also use the notation $U_n(K, f)$. Let $Z(U_n(K, f))$ designate the center of $U_n(K, f)$. Then $U_n(K, f)/Z(U_n(K, f))$ acts on the projective geometry $P(E)$ obtained from E. Let $U_n^+(K, f)$ be the subgroup of $U_n(K, f)$ consisting of transformations of determinant 1. Let $PU_n^+(K, f)$ denote the image of $U_n^+(K, f)$ in $PU_n(K, f)$.

A sesquilinear form f is said to be *anisotropic* if its Witt index is zero. Then $f(x, x) = 0$ only if $x = 0$. It is known that f is never anisotropic if K is finite. Also, the group $U_n(K, f)$ contains no unipotent transformations when f is anisotropic. This means that the action of every element of $U_n(K, f)$ on E is completely reducible.

The group $U_n(K, f)$ acting on E is the group of semilinear transformations u acting on E relative to an automorphism $\sigma = \sigma(u)$ of K such that for all x, $y \in E$, $f(ux, uy) = ef(x, y)$ where e is an element of K such that $e^J = e$. It is known that $\Gamma U_n(K, f)$ is the normalizer of $U_n^+(K, f)$ in the group $\Gamma L_n(K)$ of semilinear transformations. Let $P\Gamma U_n(K, f)$ denote its image in the group $P\Gamma L_n(K)$ of collineations of $P(E)$. Then $P\Gamma U_n(K, f) \subseteq \operatorname{Aut} PU_n^+(K, f)$. When $n \geq 3$, it is known that $P\Gamma U_n(K, f) = \operatorname{Aut} PU_n(K, f)$ except when $n = 4$ and f is anisotropic, the case we treat in this paper.

Indeed, the most conclusive results in this direction are due to Wonenberger [10] who covered the cases when $n \neq 4$ and K has characteristic not 2, and Borel and Tits [1] who in a very general argument worked out the automorphisms of almost simple algebraic groups defined over K when the groups contain unipotent elements. This covers the case $PU_4^+(K, f)$ except when f is anisotropic. The result of this paper is the following.

THEOREM. *Let f be an anisotropic hermitian sesquilinear form defined over an infinite field of characteristic not 2, relative to an automorphism J of K of order 2. Then* $\operatorname{Aut} PU_4^+(K, f) = P\Gamma U_4(K, f)$.

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There have been essentially two approaches to the characterization of the automorphism of the classical groups. One approach is to study the unipotent elements or unipotent subgroups and then to study the action of G on either the internal BN-structure of the algebraic group or on the projective geometry. A second approach is to use the semisimple elements of G. Involutions are the most convenient elements of this type. Here one studies commuting involutions to obtain a characterization of involutions with 2-dimensional eigenspaces, and also sets of noncommuting involutions in order to obtain a characterization of the action of the automorphism on the underlying projective geometry. Because the dimension of E is small, there are not enough involutions for the methods which handle the general case to apply directly. To overcome this problem, other semisimple elements are used here in obtaining the characterization of exceptional pairs. Also, in the characterization of the underlying geometry in group-theoretic terms, dihedral subgroups are used.

The techniques we introduce can be extended to the case where f has nonzero Witt index. But this leads to a more complicated analysis due to the existence of isotropic vectors, and not all characterizations we establish here appear to go through directly. It seems that the approach using semisimple elements is particularly effective in the anisotropic case.

When K is the field of real numbers or, in general, locally compact, there exists results of Van der Waerden [8], Freudenthal [4] and Borel and Tits [2] covering this case.

1. Definitions and notations

We use the terminology of Dieudonné [3]. Let E denote a vector space of dimension n over an *infinite* field K of characteristic not 2. Assume that K admits an involutory automorphism $J: \alpha \mapsto \alpha^J$.

Let $f: E \times E \to K$ denote a nondegenerate hermitian sesquilinear form relative to J. Because f is anisotropic, all subspaces and vectors are nonisotropic. Set $V^\perp$ to be the orthogonal subspace to a subspace V. Then $E = V \oplus V^\perp$. Let 1_E be the identity transformation on E. Set

$$Z = \{\lambda \in K \mid \lambda\lambda^J = 1,\ \lambda^4 = 1\}, \tag{1.1}$$

$$Z_4 = \{\lambda 1 \mid \lambda \in Z\} \tag{1.2}$$

For convenience of notation, we set $G = U_4^+(K, f)$ and $G^* = PU_4^+(K, f)$. Denote images in G^* under the natural homomorphism by stars. Thus an element g^* in G^* may be regarded as a coset gZ_4 where $g \in g^*$. We also define for a subset S of G,

$$\begin{aligned} C_G^*(S) &= \{g \in G \mid [g, S] \subseteq Z_4\}, \\ C_G^\pm(S) &= \{g \in G \mid [g, S] \subseteq \{1, -1\}\}, \\ C_{G^*}^\pm(S^*) &= C_G^\pm(S)^*, \\ C_{G^*}^+(S^*) &= C_G(S)^*. \end{aligned} \tag{1.3}$$

All these sets are subgroups. Clearly $C_G^*(S)^* = C_{G^*}(S^*)$. Two elements g^* and h^* are said to *strongly commute* if $g^* \in C_{G^*}^+(S^*)$. Then for $g \in g^*$ and $h \in h^*$, $[g, h] = 1$ when g^* and h^* strongly commute.

Let n be a positive integer. For any group H, set

$$H^n = \langle h^n \mid h \in H \rangle.$$

PROPOSITION 1.1. *Let S^* be a subset of G^*. Then*

$$G_{G^*}(S^*)^4 \subseteq C_{G^*}^+(S^*). \tag{1.4}$$

When S^* consists of involutions,

$$C_{G^*}(S^*)^2 \subseteq C_{G^*}^+(S^*) \subseteq C_{G^*}^{\pm}(S^*) = C_{G^*}(S^*) \tag{1.5}$$

Proof. Let $s^* \in S^*$ and take $s \in s^*$. Let $g^* \in C_{G^*}(S^*)$ and take $g \in g^*$. Then $[s, g] \in Z_4$. But Z_4 has exponent 4. So $[s, g^4] = [s, g]^4 = 1$. This implies (1.4). When $s^2 = 1$, $1 = [s^2, h] = [s, h]^2$. So $[s, h] = \pm 1$, and $[s, h^2] = 1$. This implies (1.5).

An involution u^* of G^* is said to *regularly commute* with a second involution v^* provided

$$[u^*, \hat{v}^*] = 1 \tag{1.6}$$

for some $\hat{v}^* \in G^*$ with $\hat{v}^{*2} = v^*$. Then $v^* \in C_{G^*}(u^*)^2$. So u^* and v^* strongly commute by virtue of (1.5).

We omit consideration of the case that K is finite since the results are well known in that case. Then we can use the following lemma to obtain the nontriviality of $C_G(u^*)^2$ when u^* is an involution.

LEMMA 1.2. *Let K be an infinite field of characteristic not 2 admitting an involutionary automorphism J. Then there are an infinite number of elements η such that $\eta^J \eta = 1$.*

Proof. There exists $\delta \in K$ such that $\delta^J = -\delta$ and $K = F[\delta]$ where F is the fixed field of J. Let $\mu = \delta^2$. Then $\mu \in F$. Let $\gamma \in F$ and set $\eta_\gamma = \alpha_\gamma + \beta_\gamma \delta$ where

$$\alpha_\gamma = (\gamma^2 + \mu)/(\gamma^2 - \mu), \quad \beta_\gamma = 2\gamma/(\gamma^2 - \mu).$$

Then $\eta_\gamma \eta_\gamma^J = \alpha_\gamma^2 - \beta_\gamma^2 \mu = 1$. The lemma now follows since $\gamma \mapsto \eta_\gamma$ is an injection.

PROPOSITION 1.3. *Let V be a 2-dimensional nonisotropic subspace of E. Let G_V be the subgroup of G leaving invariant V. Then G_V^4 acts irreducibly on V and G_V^4 is nonabelian.*

Proof. Let e_1 be any nonisotropic vector in V and let $e_2 \in Ke_1^\perp$ so that $V = Ke_1 + Ke_2$. By Lemma 1.2, there exists $\eta \in K$ such that $\eta\eta^J = 1$ and $\eta^8 \neq 1$. Let $\zeta = \eta^4$. Let $\tilde{t}_\eta$ be an element of $GL(Ke_1 + Ke_2)$ defined by $\tilde{t}_\eta(e_1) =$

ηe_1 and $\tilde{t}_\eta(e_2) = \eta^{-1}e_2$. Then $\tilde{t}_\eta$ has determinant 1. Hence there exists an extension t_η of $\tilde{t}_\eta$ to an element of G since $f(\tilde{t}_\eta e_1, \tilde{t}_\eta e_1) = \eta^J \eta f(e_1, e_1) = f(e_1, e_1)$ and $f(\tilde{t}_\eta e_2, \tilde{t}_\eta e_2) = \eta^{-J}\eta^{-1}f(e_2, e_2) = f(e_2, e_2)$. Let $\zeta = \eta^4$, and let $t_\zeta = t_\eta^4$. Then as $\zeta \neq \pm 1$, t_ζ is a nontrivial element of G_V^4 with two distinct eigenspaces Ke_1 and Ke_2. Then only Ke_1 and Ke_2 are left invariant by t_ζ. Then no subspace of V is left invariant by G_V^4 since V always contains a vector not in Ke_1 or Ke_2 from which a transformation not leaving invariant Ke_1 or Ke_2 can be defined. It is now clear that G_V^4 is nonabelian since $G_V^4 \neq \langle t_\xi \rangle$.

COROLLARY 1.4. *Let $J \neq 1$. Let V be a nonisotropic 2-dimensional subspace. Then $C_{G_V}(G_V^4)$ contains only one involution.*

Proof. By Schur's lemma, $C_{G_V}(G_V^4)$ can be embedded in the multiplicative group of a division ring, which necessarily contains only one involution.

2. Classification of involutions in G^*

Let u^* be an involution in G^*. Then $u^2 = \gamma 1$ where $\gamma \in Z$ when $u \in u^*$. Thus $\gamma\gamma^J = 1$ and γ is a fourth root of 1. We distinguish three types of involutions:

(i) u^* is *ordinary* if $\gamma = \lambda^2$ where $\lambda\lambda^J = 1$ for some $\lambda \in K$
(ii) u^* is *isotropic* if $\gamma = \lambda^2$ where $\lambda\lambda^J = -1$ for some $\lambda \in K$
(iii) u^* is *nonordinary* if γ is a nonsquare in K.

Clearly the type of involution is a well defined concept. Accordingly any element u such that $u^* = uZ_4$ is said to be a *projective involution* in G. It is said to be an *ordinary*, *isotropic*, or *nonordinary* projective involution according as u^* is ordinary, isotropic, or nonordinary. When $\gamma\gamma^J = 1$ and $\gamma = \lambda^2$, $\lambda\lambda^J = \pm 1$. So all involutions in G^* are ordinary, isotropic, or nonordinary.

Ordinary Involutions. Let u^* be an ordinary involution in G^*. Let $u' \in u^*$. Then $u'^2 = \gamma 1$ where $\gamma = \lambda^2$, $\lambda^J\lambda = 1$. Set $u = \lambda^{-1}u'$. Then u is an involution in G. So $E = U^+ \oplus U^-$ where U^+ and U^- are the eigenspaces of u associated with the eigenvalues 1 and -1. Then U^+, U^- are the eigenspaces for all the projective involutions in $u^* = uZ_4$, and they will be called the spaces of u^*. As $-u \in uZ_4$, we may always choose u so that $\dim U^- \leq \dim U^+$. Set $p = \dim U^-$. As $\dim E = 4$, $p = 1$ or 2. We call u as well as u^* a *p-involution.*

PROPOSITION 2.1. *The group G^* always contains 2-involutions. It contains 1-involution if and only if Z contains a primitive fourth root of* 1.

Proof. Consider the determinants of these involutions.

Let u^*, v^* be a pair of strongly commuting ordinary involutions of G^* with eigenspaces U^+, U^- and V^+, V^-, respectively. Let $v \in v^*$. Then v leaves invariant both U^+ and U^-. Set

$$P_1 = U^+ \cap V^+, \quad P_2 = U^+ \cap V^-, \quad P_3 = U^- \cap V^+ \text{ and } P_4 = U^- \cap V^-.$$

Then $E = P_1 \oplus P_2 \oplus P_3 \oplus P_4$. The subspaces P_i, $i = 1, 2, 3, 4$, are mutually orthogonal.

Suppose v^* anticommutes with u^*. Let $v \in v^*$ and let U^+, U^- be the eigenspaces of u^*. Then it is clear that $v(U^+) = U^-$. In particular U^+ and U^- are isometric.

Isotropic involutions. Let u^* be an isotropic involution in G^* and choose a representative u in u^*. Then $u^2 = \gamma 1$ where $\gamma = \lambda^2$, λ is in K and $\lambda^J\lambda = -1$. Then $u' = \lambda^{-1}u$ is an involution of $\Gamma U_4(K, f)$ with multiplier -1, and $E = U^+ \oplus U^-$ where U^+ and U^- are the eigenspaces of u' and of u for any u in u^*; the subspaces U^+ and U^- are totally isotropic of dimension 2 [3, p. 27]. *Because f is anisotropic, G^* contains no isotropic involutions.*

Nonordinary Involutions. Let u^* be a nonordinary involution in G^* and choose a representative u in u^*. Then $u^2 = \gamma$ where γ is a nonsquare in K. Let L be a quadratic extension of K obtained by adjoining a root ρ of γ. Now E can be considered a vector space F of dimension 2 over L by setting $x\rho = u(x)$ for x in E by [3, p. 25]. The involutory automorphism J may be extended to L by setting $\rho^J = \rho^{-1}$. Corresponding to F, we define a new nondegenerate hermitian sesquilinear form g by setting

$$g(x, y) = f(x, y) + \rho^{-1} f(x, u(y)) \tag{2.1}$$

Let t be an element of $U_4(K, f)$ which is relative to an automorphism τ of K and which commutes with u. Extend τ to L by setting $\rho^\tau = \rho$. Then t belongs to $\Gamma U_2(L, g)$. Conversely, if t is an element of $\Gamma U_2(L, g)$ which is relative to an automorphism τ of L such that $K^\tau = K$, then t is also an element of $\Gamma U_4(K, f)$ which is relative to the restriction of τ to K and which commutes with u.

3. Regular commuting pairs of involutions

The characterization of 2-involutions is based on a study of exceptional strongly commuting pairs of involutions. To obtain such pairs, we take an involution u^* in G^* and consider the involutions which regularly commute with u^*.

PROPOSITION 3.1. *Let u^* be an nonordinary involution in G^*.*

(i) *There exists a 2-involution with which u^* regularly commutes.*
(ii) *u^* commutes only with nonordinary and 2-involutions.*

Proof. By Section 2, $C_G(u)$ is a subgroup of $U(F, g)$ where F is a 2-dimensional space over $L = K(\rho)$. Let v be a 1-involution in $U(F, g)$ then v is also a 2-involution in $U(E, f)$ and so also in G. If Z contains a primitive fourth root i of 1, then let $\hat{v}$ be defined by $\hat{v}|_{V^-} = i1_{V^-}$ and $\hat{v}|_{V^+} = 1_{V^+}$ where 1_{V^-} and 1_{V^+} are the identity transformations on V^+ and V^- respectively. Then $v \in C_G(u)$ and $\hat{v}^2 = v$. This implies that u regularly commutes with v^*.

On the other hand, suppose that $Z = \{\pm 1\}$. Then $u^2 = -1$. Because $[u. v] = 1$, μ leaves invariant both V^+ and V^-. Let now $\hat{v}|_{V^-} = u|_{V^-}$ and $\hat{v}|_{V^+} = 1|_{V^+}$. Then clearly $\hat{v}^2 = v$ and $[\hat{v}, u] = 1$. So again u^* regularly commutes with v^*. This proves (i).

To prove (ii), it is required to show that u^* does not commute with any 1-involution. So let v^* be a 1-involution with eigenspaces V^+, V^-. As $\dim_K V^+ \neq \dim_K V^-$, $[u, v] = 1$. Then $v \in U_2(L, g)$ acting on $F = E \otimes_K L$. So $\dim_L V^+ = \dim_L V^- = 1$. Then $\dim_L V^+ = \dim_K V^- = 2$, which gives the desired contradiction.

PROPOSITION 3.2. *Let u^* be a 1-involution in G^*.*

(i) *Then u^* regularly commutes with a 2-involution.*
(ii) *The field K contains a primitive eight root λ of 1 with $\lambda^J \lambda = 1$.*

Proof. We first argue that (ii) holds. Indeed, if u^* is a 1-involution, then $\det u = 1$ for $u \in u^*$. But there exists a scalar transformation $\lambda 1$ in E such that u is a 1-involution in G. Then $\lambda^4 = \det(\lambda 1) = \det(\lambda u) = -1$. So λ is the desired primitive eight root 1. It is clear that $\lambda^J \lambda = 1$.

To prove (i), let U^+, U^- be the eigenspaces of u^*. Then $\dim_K U^+ = 3$. Let e_1, e_2, e_3, e_4 be an orthogonal basis such that

$$U^- = Ke_4 \quad \text{and} \quad U^+ = Ke_1 \oplus Ke_2 \oplus Ke_3.$$

Let $V^- = Ke_1 \oplus Ke_2$ and $V^+ = Ke_3 \oplus Ke_4$. Define $\hat{v}$ on E by setting $\hat{v}(e_1) = ie_1$, $\hat{v}(e_2) = ie_2$, $\hat{v}(e_3) = e_3$, $\hat{v}(e_4) = e_4$. Then $\hat{v}$ has eigenspaces V^+ and V^- and $\hat{v}^2 \neq 1$. Clearly $[u, v] = 1$. So $\hat{v}^{*2}$ is a 2-involution and u^* commutes with v^*.

PROPOSITION 3.3. *Let u^* be a 2-involution in G^*. Then if u^* regularly commutes with some involution v^* distinct from u^*, it regularly commutes with some 2-involution.*

Proof. Let v^* be an involution in G^* which regularly commutes with u^*. Then there exists $\hat{v}^* \in C_{G^*}(u^*)$ such that $v^{*2} = v^*$. Take $\hat{v} \in \hat{v}^*$ and set $v = \hat{v}^2$. Then $v \in C_G(u)$ where u is a 2-involution in u^*.

First, take the case that v^* is a nonordinary involution. Set $L = K[v]$ and $F = E \otimes_K L$. Let g be the sesquilinear extension of f to $F \times F$ such that $C_G(v) \subseteq U_2(F, g)$. Then $v, u \in U_2(F, g)$. Hence $F = U^+ \oplus U^-$ where U^+, U^- are the eigenspaces of u. As $L = K[v]$, v is a nonsquare in L. As $\hat{v}^2 = v$, $\hat{v}$ is a nonordinary projective involution in $U_2(F, g)$. Then $\hat{v}$ acts irreducibly on F and hence also on E. Consequently $K[\hat{v}]$ is a maximal subfield in the simple algebra $\operatorname{Hom}_K(E, E)$. So $C_G(\hat{v}) \subseteq K[\hat{v}]$. But $K[\hat{v}]$ has but one involution, namely, the scalar -1. Hence $u \notin C_G(\hat{v})$; so by (1.5), $[u, \hat{v}] = -1$.

Then $\hat{v}(U^+) = U^-$. Let e_1, e_2 be orthogonal elements of U^+, and set $e_3 = v(e_1)$ and $e_4 = v(e_2)$. Take $W^+ = Ke_1 \oplus Ke_3$ and $W^- = Ke_2 \oplus Ke_4$. Set $w(e_1) = e_3$, $w(e_3) = -e_1$, $w(e_2) = e_2$, and $w(e_4) = e_4$. Then w^2 is a 2-involution w with eigenspaces W^+, W^-. Clearly u^* regularly commutes with w^{*2}.

Consider next that v^* is a 1-involution. Then by Proposition 2.1, Z contains a primitive fourth root of unity which we designate by λ. Again let e_1, e_2 be orthogonal vectors in U^+ and e_3, e_4 be orthogonal vectors in U^-. Now set $W^+ = Ke_1 \oplus Ke_3$ and $W^- = Ke_2 \oplus Ke_4$. Let w be the 2-involution with eigenspaces W^+, W^-. Then $[u, w] = 1$; so u^* and w^* strongly commute. Let $\hat{w}$ be the transformation defined by $\hat{w}(e_1) = e_1$, $\hat{w}(e_3) = e_3$, $\hat{w}(e_2) = \lambda e_2$ and $\hat{w}(e_4) = \lambda e_4$. Then $\hat{w} \in G$ as $\lambda\lambda^J = 1$. Clearly $\hat{w}^2 = w$ and $w \in C_G(u)$. So u^* regularly commutes with w^*. Thus in all cases u^* regularly commutes with a 2-involution.

4. Characterization of 2-involutions

Now, when 2-involutions fail to regularly commute with any other involution, they are distinguished by group-theoretic properties by virtue of Propositions 3.1, 3.2, and 3.3. This section treats the remaining case. By virtue of Proposition 3.3, we may begin with an involution u^* which regularly commutes with a second involution v^*. Then $\{u^*, v^*\}$ form a strongly commuting pair.

Let $\{u^*, v^*\}$ be a pair of strongly commuting involutions of G^*. We say that $\{u^*, v^*\}$ is an *exceptional pair of involutions* provided there exist a nontrivial element $t^* \in C_{G^*}(u^*, v^*)^2$ such that $C_{G^*}(t^*)^4$ is not contained in any of $C_{G^*}(u^*)$, $C_{G^*}(v^*)$, $C_{G^*}(u^*v^*)$.

The following is a direct consequence of Proposition 1.1.

LEMMA 4.1. *Let $\{u^*, v^*\}$ be a pair of strongly commuting involutions in G^* and take $u \in u^*$ and $v \in v^*$. Then $\{u^*, v^*\}$ is nonexceptional if $C_G(t)$ is contained in one of $C_G(u)$, $C_G(v)$ and $C_G(uv)$ whenever $t \in C_G(u, v)$ and $t \neq 1$.*

Proof. From (1.5) and the hypothesis, it follows that

$$C_{G^*}(t^*)^4 \subseteq C^+_{G^*}(t^*) = C_G(t)^* \subseteq C_G(w)^* \tag{4.1}$$

for $w = u$, v, or uv.

We now consider a variety of cases.

PROPOSITION 4.2. *Let $\{u^*, v^*\}$ be a strongly commuting pair of involutions of G^* where u^* is nonordinary and v^* is ordinary. Then $\{u^*, v^*\}$ is an nonexceptional pair.*

Proof. Take $u \in u^*$ and $v \in v^*$. Then $u^2 = \gamma 1$ where $\gamma \in Z$, $\gamma \notin K^{*2}$ and $\gamma^J\gamma = 1$. Let $L = K[\rho]$ be a quadratic extension of K obtained by adjoining ρ where $\rho^2 = \gamma$, and the action of ρ on E is given by $\rho x = ux$. Let $F = E \otimes_K L$, and let g be an L-sesquilinear defined by (2.1). Then $C_G(u) \subseteq U(F, g)$ and v is a

1-involution of $U(F, g)$. Thus the eigenspaces V^+ and V^- of v are 1-dimensional subspaces of F.

For the purpose of showing that $\{u^*, v^*\}$ is nonexceptional, take

$$t^* \in C_{G^*}(u^*, v^*)^2 \quad \text{with} \quad t^{*2} \neq 1.$$

Let $t \in t^*$; then $t \in C_G(u, v)$ by Lemma 4.1. So both V^+ and V^- are t-invariant. Hence $t(x) = \tau^\varepsilon x$ where $\tau^\varepsilon \in L$ for $x \in V^\varepsilon$, $\varepsilon = +, -$, since $[t, u] = 1$. So V^+ and V^- are t-eigenspaces when $\tau^+ \neq \tau^-$.

Let $s^* \in C_{G^*}(t^*)^4$, and take $s \in s^*$. Then $[s, t] = 1$ by Proposition 1.1. Consider first that $\tau^+ \neq \tau^-$. Then $s(V^\varepsilon)$ is also a t-eigenspace. So either $s(V^+) = V^+$ and $s(V^-) = V^-$ or $s(V^+) = V^-$ and $s(V^-) = V^+$. Then $[s, v] = \pm 1$. Hence $C_{G^*}(t^*)^4 \subseteq C_{G^*}(v^*)$ in this case.

So it remains to consider that $\tau^+ = \tau^-$. Then E itself is an L-eigenspace for t. Let $\tau^+ = \alpha + \beta\rho$ where $\alpha, \beta \in K$. By virtue of the action of u on E, $\tau = \alpha + \beta u$. Since $[t, s] = 1$, $[u, s] = 1$. Hence $C_{G^*}(t^*)^4 \subseteq C_{G^*}(u^*)$ in this case. This shows that $\{u^*, v^*\}$ is nonexceptional.

PROPOSITION 4.3. *Let $\{u^*, v^*\}$ be a strongly commuting pair of nonordinary involutions. Then $\{u^*, v^*\}$ is nonexceptional.*

Proof. Take $u \in u^*$ and $v \in v^*$. Then $u^2 = \gamma 1$ and $v^2 = \beta 1$ where β and γ are nonsquares in K. By hypothesis u^* and v^* are a strongly commuting pair. So $[u, v] = 1$. Let $L = K(\rho)$ be a quadratic extension of K obtained by adjoining a root ρ of γ and designate by F the L-space obtained from E by setting $x\rho = u(x)$ for all $x \in E$. Let g be a sesquilinear extension of f to F defined in (2.1). Then $C_G(u) \subseteq U_2(F, g)$.

If v is ordinary in $U(F, g)$, then ρ is a square in L. Since L is the splitting field of $x^2 - \gamma$, $\beta = (b\rho)^2$ for some $b \in K$. Thus $v^2 = (bu)^2$. Then u^*v^* is ordinary and the result from the previous proposition applies to show that $C_G(t^*)^4 \subseteq C_{G^*}(u^*)$ or $C_{G^*}(u^*v^*)$. Hence we may suppose that both v and uv are nonordinary projective involutions in $U(F, g)$. Then we may extend L to $M = L(\rho')$ where $\rho'^2 = \beta$ and $y\rho' = v(y)$ for $y \in F$. Hence M is a quadratic extension of L. The Galois group of the extension M over K is a four group, and M has intermediate fields $K(\rho)$, $K(\rho')$ and $K(\rho\rho')$.

In order to show that $\{u^*, v^*\}$ in nonexceptional, take $t^* \in C(u^*, v^*)^2$ so that $t^{*2} = 1$, and take $t \in t^*$. Then $t \in M$. If $t \in K(\rho)$, $K(\rho')$ or $K(\rho\rho')$, then $t = \beta_1 u$, $\beta_2 v$, $\beta_3 uv$ with $\beta_i \in K$. So $C_G(t) = C_G(u)$, $C_G(v)$ or $C_G(uv)$, respectively.

In the remaining case, $K[t] = M$. So

$$C_G(t) = C_G(K[t]) = C_G(M) = C_G(u, v).$$

Therefore, $\{u^*, v^*\}$ is nonexceptional by Lemma 4.1.

PROPOSITION 4.4. *Let $\{u^*, v^*\}$ be a pair of strongly commuting ordinary involutions of G^*. Then $\{u^*, v^*\}$ is an exceptional pair.*

Proof. Let $u \in u^*$ and $v \in v^*$, and let U^+, U^- and V^+, U^- be the eigenspaces of u and v, respectively. Set

$$P_1 = U^+ \cap V^+, \quad P_2 = U^+ \cap V^-, \quad P_3 = U^- \cap V^+ \quad \text{and} \quad P_4 = U^- \cap V^-.$$

As $[u, v] = 1$, $E = P_1 \oplus P_2 \oplus P_3 \oplus P_4$. Let $P_i = Ke_i$, $i = 1, 2, 3, 4$. There are three cases according as both, one, or none of u, v are 2-involutions.

(i) Suppose u and v are 2-involutions. Then the subspaces P_i, $i = 1, 2, 3, 4$, are 1-dimensional and mutually orthogonal. By Lemma 1.2, there exists $\eta \in K$ such that $\eta\eta^J = 1$ and $\eta^4 \neq 1$. Define $t \in G$ by taking $te_1 = \eta^3 e_1$ and $te_i = \eta^{-1}e_i$, $i = 2, 3, 4$. Then $T_1 = Ke_1$ and $T_2 = Ke_2 + Ke_3 + Ke_4$ are the eigenspaces of t. Clearly $t \in C_G(u, v)$. As $\dim T_1 \neq \dim T_3$, $C_G(t)^* = C^+_{G^*}(t^*) = C_{G^*}(t^*)$ by (1.5). So when $w \in C_G(t)^4$, $w^* \in C_{G^*}(t^*)^4$.

Again by Lemma 1.3, there exists $\zeta \in K$ with $\zeta^{16} \neq 1$ and $\zeta\zeta^J = 1$. Also it easily follows that there exists $e' \in T_2$ such that

$$e' = \alpha_2 e_2 + \alpha_3 e_3 + \alpha_4 e_4 \quad \text{with } \alpha_2\alpha_3\alpha_4 \neq 0.$$

Set $W_1 = Ke'$ and $W_2 = W_1^\perp$. Let w be the transformation with eigenspaces W_1 and W_2 and respective eigenvalues ζ^3 and ζ^{-1}. Now $\zeta^{16} \neq 1$ implies $\zeta^{12} \neq \zeta^{-4}$. Thus W_1 and W_2 are eigenspaces for w^4. Clearly $w^4 \in C_G(t)^4$. As $W_1 \nsubseteq P_i + P_j$ for $1 \leq i < j \leq 4$, $w^4 \notin C_G(u)$, $C(v)$, or $C_G(uv)$. As $\dim W_1 \neq \dim W_2$, $[w^4, r] \neq -1$ for $r = u$, v, or uv. Hence $w^4 \notin C_G^{\pm}(r)$, $r = u, v, uv$. Hence $w^{*4} \notin C_{G^*}(r^*)$, $r^* = u^*, v^*$, or u^*v^*. Thus $\{u^*, v^*\}$ is exceptional.

(ii) Let now u be a 1-involution and v be a 2-involution. We may now suppose $P_1 = 0$, $\dim P_2 = 2$, $\dim P_3 = \dim P_4 = 1$. Let Q_1be a 1-dimensional subspace of P_1 and let $Q_2 = Q_1^\perp \cap P_2$. Then $E = Q_1 \oplus Q_2 \oplus P_3 \oplus P_4$. Set $Q_1 = Kd_1$, $Q_2 = Kd_2$. Define a transformation t on E by setting $td_1 = \eta^3 d_1$ and $td_2 = \eta^{-1}d_2$, $te_i = \eta^{-1}e_i$, $i = 3, 4$, where $\eta^J\eta = 1$ and $\eta^4 \neq 1$ for some $\eta \in K$ as before. Then $T_1 = Q_1$ and $T_2 = Q_2 \oplus P_3 \oplus P_4$ are the eigenspaces for t. As before, there exists a vector $\mathbf{e}'$ which is not contained in Q_2 or $P_3 \oplus P_4$. Defining w as in the preceding part, we again conclude that $\{u^*, v^*\}$ is an exceptional pair.

(iii) Consider finally that u and v are 1-involutions. Then uv is a 2-involution, and this case is then the same as case (ii).

The following result is the desired group theoretic characterization of 2-involutions.

THEOREM 4.5. *An involution u^* of G^* is a 2-involution if and only if one of the following conditions hold.*

(i) *u^* does not regularly commute with any other involution of G^*.*

(ii) *u^* regularly commutes with an involution $v^* \neq u^*$ such that $\{u^*, v^*\}$ is an exceptional pair of commuting involutions and, when G^* contains 1-involutions, $C_{G^*}(u^*, v^*)$ contains an involution not in $\langle u^*, v^*\rangle$.*

Proof. Let u^* be a 2-involution. By virtue of Proposition 3.3 we may assume that u^* regularly commutes with another 2-involution v^*. Then, by Proposition 4.4, $\{u^*, v^*\}$ form an exceptional pair. When G^* contains 1-involutions, it follows that K contains a primitive eighth root λ of 1 such that $\lambda^J\lambda = 1$ by virtue of Proposition 3.2. Then because u^* and v^* strongly commute, $E = P_1 \oplus P_2 \oplus P_3 \oplus P_4$ where each P_i is the intersection of an eigenspace of u and an eigenspace of v. Then $C_G(u, v)$ contains a transformation t with eigenspaces P_1 and $P_2 \oplus P_3 \oplus P_4$ corresponding to eigenvalues λ^5 and λ. Then t^* is an involution in $C_{G^*}(u^*, v^*)$, and t is a 1-involution. Thus $t^* \in \langle u^*, v^*\rangle$ since u^*, v^* and u^*v^* are all 2-involutions. Thus 2-involutions satisfy either conditions (i) or (ii).

Conversely let u^* be an involution in G^* satisfying (i) or (ii). We wish to show that u^* is a 2-involution. By virtue of Propositions 3.1 and 3.2, we may assume that u^* regularly commutes with an involution v^*. By assumption, $\{u^*, v^*\}$ is an exceptional pair. Then by Propositions 4.2, 4.3 and 4.4, u^* and v^* are ordinary involutions. Then they are both 2-involutions when G^* contains no 1-involution.

Otherwise $C_{G^*}(u^*, v^*)$ contains a 1-involution. Assume that u^* is 1-involution in order to obtain a contradiction to (ii). Then $E = P_1 \oplus P_3 \oplus P_4$ where $P_1 = U^+ \cap V^+$, $P_3 = U^- \cap V^+$ and $P_4 = U^- \cap V^-$, U^+, U^- and V^+, V^- being the eigenspaces of u and v, respectively. Hence $\dim P_1 = 2$ and $\dim P_3 = \dim P_4 = 1$. In this case if $t \in C_G^{\pm}(u, v)$, $t(P_i) = P_j$ for some j. Then $t(P_1) = P_1$, which implies $t(U^+) = U^+$ and $t(V^+) = V^+$. So $C_G^{\pm}(u, v) = C_G(u, v)$. Clearly $C_G(u, v)$ has seven involutions since E is the direct sum of exactly three subspases P_1, P_3, P_4. Then $C_{G^*}(u^*, v^*) = C_G^{\pm}(u, v)^*$ contains exactly three involutions, namely u^*, v^*, and u^*v^*. This contradiction to (ii) implies that u^* is a 2-involution.

THEOREM 4.6. *The following conditions on a pair $\{u^*, v^*\}$ of commuting 2-involutions are necèssary and sufficient for $\{u^*, v^*\}$ to be strongly commuting.*

(i) $C_{G^*}(u^*, v^*)^2$ *is abelian when* $-1 \in K^2$

(ii) u^*v^* *is a 2-involution when* $-1 \notin K^2$.

Proof. Let $u \in u^*$ and $v \in v^*$ be involutions. Then $[u, v] = \varepsilon 1_E$ where $\varepsilon = \pm 1$. Let $w = uv$. Then $w^2 = \varepsilon 1_E$.

(i) Take first the case $-1 \in K^2$. Let $\varepsilon = -1$. Then $uv = -vu$. Let U^+, U^- be the eigenspaces for u. Then $v(U^+) = U^-$. Let $C = C_G(u)$. Then $C = C^+C^-\langle t\rangle$ where C^+ is the normal subgroup of C leaving fixed U^- and C^- is the normal subgroup of C leaving fixed U^+, and t is a 2-involution in $C_G(u, v)$ with $U^{\pm} = U^{\pm} \cap T^+ + U^{\pm} \cap T^-$ where T^+, T^- are the eigenspaces for t. Then C^+ and C^- are the unitary groups $U_2^+(U^+, f|_{U^+})$ and $U_2^+(U^-, f|_{U^-})$. By Proposition 1.3, $U_2^+(U^\varepsilon, f|_{U^\varepsilon})^2$ is nonabelian for $\varepsilon = +, -$. But as $v(U^+) = U^-$, $v(C^+) = C^-$. As $v^2 = 1$, $C_{C^+C^-}(v) = \{cc^v \mid c \in C^+\}$. Then $C_{C^+C^-}(v)$ is isomorphic to C^+. So $C_G(u, v)^2$ has a nonabelian subgroup of index 2. Since

$C_G^*(u, v) = C_G(u, v)\langle u, v\rangle$, $C_G^*(u, v)^2$ is nonabelian. Then $C_{G*}(u^*, v^*)^2$ is nonabelian as required.

Next, let $\varepsilon = 1$. Let V^+, V^- be the eigenspaces for v. Then

$$E = P_1 \oplus P_2 \oplus P_3 \oplus P_4$$

where $P_1 = U^+ \cap V^+$, $P_2 = U^+ \cap V^-$, $P_3 = U^- \cap V^+$ and $P_4 = U^- \cap V^-$ are all 1-dimensional. Consequently $C_G(u, v)$ is abelian as $C_G(u, v)|_{P_i}$ is clearly abelian, $i = 1, 2, 3, 4$. Then, as $C_G^+(u, v) \subseteq C_G^*(u, v)^2$, $C_{G*}^+(u, v)$ is a abelian subgroup of $C_G^*(u^*, v^*)$ containing $C_{G*}(u^*, v^*)$ as required.

(ii) Suppose $-1 \notin K^2$. If $\varepsilon = -1$, $w^2 = -1$. So w is a nonordinary involution. If $\varepsilon = 1$, then $w = uv$ is a 2-involution with eigenspaces

$$U^+ \cap V^+ \oplus U^- \cap V^- \quad \text{and} \quad U^+ \cap V^- \oplus U^- \cap V^+$$

where U^+, U^- and V^+, V^- are the eigenspaces of u and v respectively.

COROLLARY 4.7. *Let $\mathscr{S}^*$ be a set of 2-involutions in G^*. Let σ be an automorphism of G^*, and let $C_{G*}^{++}(\mathscr{S}^*)$ denote the group generated by the 2-involutions of $C_{G*}(\mathscr{S}^*)$ which strongly commute with the involutions of $\mathscr{S}^*$. Then*

$$\sigma(C_{G*}^{++}(\mathscr{S}^*)) = C_{G*}^{++}(\sigma(\mathscr{S}^*)).$$

Proof. This follows from the previous theorem together with the group-theoretical characterization of the set of 2-involutions given by Theorem 4.5.

5. Noncommuting pairs of involutions

On the basis of Theorem 4.5, we have distinguished group-theoretically the set of 2-involutions from the set of nonordinary involutions. We will consider pairs of noncommuting involutions on the basis of the following lemma.

LEMMA 5.1. *Let u^*, v^* be noncommuting 2-involutions of G^* with eigenspaces U^+, U^- and V^+, V^-, respectively. Then* $\dim(U^+ \cap V^+) = 1$ *implies that* $\dim(U^- \cap V^-) = 1$.

Proof. Let $W_1 = U^+ \cap V^+$. Then $W_1^\perp = (U^+)^\perp + (V^+)^\perp = U^- + V^-$. So $\dim(U^- + V^-) = 3$. Hence $\dim(U^- \cap V^-) = 1$.

We define a pair of noncommuting 2-involutions $\{u^*, v^*\}$ to be *intersecting* if for some eigenspace U of u^* and V of v^*, $U \cap V \neq 0$. Otherwise we say that $\{u^*, v^*\}$ is *nonintersecting*. When $\{u^*, v^*\}$ is intersecting, we may always choose $u \in u^*$ and $v \in v^*$ to be 2-involutions with eigenspaces U^+, U^-, and V^+, V^-, respectively, so that $\dim(U^+ \cap V^+) = \dim(U^- \cap V^-) = 1$. Then set $w = uv$ and $W = U^+ \cap V^+ + U^- \cap V^-$. Clearly W is the fixed subspace of E for w and the dihedral group $\langle u, v\rangle$ acts faithfully on $W^\perp$. As $[u^*, v^*] \neq 1$, $w^2 \notin Z_4$. So $\langle u^*, v^*\rangle$ is nonabelian.

It is easy to see that

$$C_G^*(u, v) = C_G^*(u) \cap C_G^*(w) \subseteq C_G^*(w) \tag{5.1}$$

Since $C_G^*(u, v)/C_G(u, v)$ has exponent 2 and $C_G^*(w)/C_G(w)$ is cyclic,

$$|C_G^*(u, v)/(C_G(w) \cap C_G^*(u, v))| \leq 2.$$

Thus

$$|C_G^*(u, v)C_G(w)/C_G(w)| \leq 2.$$

Because $C_G^*(g)$ maps onto $C_{G^*}(g^*)$ for $g \in G$, we can study $C_{G^*}(u^*, v^*)$ by studying $C_G(w^*)$.

Let $\bar{E} = E \otimes_K \bar{K}$ where $\bar{K}$ is an algebraic closure of K. Then

$$G \subseteq \bar{G} = U_4^+(\bar{E}, \bar{f})$$

where $\bar{f}$ is an extension of f to $\bar{f}$: $\bar{E} \times \bar{E} \to K$. Then $\bar{f}$ is a sesquilinear hermitian form relative to an extension $\bar{J}$ of J to an element of Aut $\bar{K}$.

LEMMA 5.2. *Let $\{u^*, v^*\}$ be a noncommuting pair of 2-involutions. Take $u \in u^*$ and $v \in v^*$ so that u, v are 2-involutions. Set $w = uv$. Then $uwu = w^{-1}$. If $w\bar{e} = \eta\bar{e}$ for some $\bar{e} \in \bar{E}$ and $\eta \in \bar{K}$, $w(u\bar{e}) = \eta^{-1}(u\bar{e})$.*

If $\eta = \pm 1$, then $\bar{e} = e$ where $e \in E$ and, replacing u by $-u$ and v by $-v$, if necessary, we have that $e \in W$ where

$$W = U^+ \cap V^+ + U^- \cap V^- \tag{5.2}$$

With this choice of u and v, $\eta = 1$ and W is the fixed subspace of w on E. In this case, $\{u^*, v^*\}$ is an intersecting pair.

Proof. Because $\langle u, v\rangle$ is dihedral, $uwu = w^{-1}$. Then direct calculation shows that $w(u\bar{e}) = \eta^{-1}(u\bar{e})$ where $w\bar{e} = \eta\bar{e}$. Let $\eta = \pm 1$. Then $\eta = \eta^{-1}$, so w has an eigenspace $\bar{W}$ with respect to η. Also, $w^2|_{\bar{W}} = 1$. So $\bar{W} \neq \bar{E}$. As $\langle u, v\rangle$ is a dihedral group acting faithfully on $\bar{E}$, $\langle u, v\rangle$ acts faithfully on $\bar{W}$, which then has dimension at least 2. Then dim $\bar{W} = 2$. This decomposition of $\bar{E}$ into $\langle u, v\rangle$-irreducible modules can be obtained over K. Then $\bar{W} = W \otimes_K \bar{K}$ where W is a u-variant subspace of E. Then $W = W \cap U^+ + W \cap U^-$. As $v = uw$, $W \cap U^+$ and $W \cap U^-$ are also v-invariant. Hence, either $W \cap U^+ \cap V^+ \neq 0$ or $W \cap U^+ \cap V^- \neq 0$. We choose u and v so that the former case occurs. Then $U^+ \cap V^+$ is a 1-dimensional subspace of W. Similarly, $U^- \cap V^- \subseteq W$. Then, as $w = uv$, $w|_W = 1_W$; that is, $\eta = 1$. Clearly $\{u^*, v^*\}$ is an intersecting pair. This proves the lemma.

LEMMA 5.3. *Assume the notation of Lemma 5.2.*

(i) *When $C_G^*(w) \neq C_G(w)$, neither 1 nor -1 is an eigenvalue for w. In particular, $\{u^*, v^*\}$ is a nonintersecting pair of noncommuting 2-involutions.*

(ii) *When* $\{u^*, v^*\}$ *is an intersecting pair,*

$$C_G^*(u, v) = C_G^*(u) \cap C_G(w) \tag{5.3}$$

Proof. (i) Suppose that $C_G^*(w) = C_G(w)\langle g\rangle$ and that $\eta = \pm 1$ is an eigenvalue for w. Let $e \in E$ so that $we = \eta e$. Now, $[g, w] = \lambda 1$ with $\lambda \in Z$ and $\lambda \neq 1$. So either ge or g^2e is an eigenvector e' for w with $we' = -\eta e'$ according as $\lambda = -1$ or $\lambda^2 = -1$. By Lemma 5.2, both e and e' belong to $W = U^+ \cap V^+ + U^- \cap V^-$. By the choice of u and v given in Lemma 5.2, $w|_W = 1_W$. Then a contradiction results. So $\eta \neq \pm 1$. Then $\{u^*, v^*\}$ is nonintersecting.

(ii) Let $\{u^*, v^*\}$ be an intersecting pair. Then, from (i), $C_G^*(w) = C_G(w)$. Thus, $C_G^*(u, v) = C_G^*(u, w) = C_G^*(u) \cap C_G(w)$.

The following theorem uses the notation introduced in Corollary 4.7.

THEOREM 5.4. *A pair* $\{u^*, v^*\}$ *of noncommuting involutions is intersecting, if and only if the following holds.*

(i) *Each abelian normal subgroup of* $C_{G*}(u^*v^*)$ *has index greater than* 4.
(ii) $Z(C_{G*}(u^*v^*)) \cap C_{G*}^{++}(u^*, v^*)$ *contains a* 2-*involution.*

Furthermore, one of the spaces of the 2-*involution is also the fixed subspace of the element* $w \in u^*v^*$ *described in Lemma* 5.2.

Proof. Assume (i) and (ii). Let $w^* = u^*v^*$, and let t^* be a 2-involution in

$$Z(C_{G*}(w^*)) \cap C_{G*}^{++}(u^*, v^*).$$

Choose $u \in u^*$ and $v \in v^*$ as described in Lemma 5.2, and take t to be a 2-involution in t^*. Set $w = uv$. As $t^* \in C_{G*}^{++}(u^*, v^*)$, $t \in C_G(u, v)$. Then w leaves invariant the eigenspaces T^* and T^- of t.

Assume that neither T^+ nor T^- is an eigenspace for w. Then T^ε, $\varepsilon = +, -$, is either a direct sum of 1-dimensional eigenspaces of w or a cyclic $K[w]$-module. Consequently $C_G(w)$ is abelian. But $C_G(w)$ is a normal subgroup of $C_G^*(w)$ of index at most 4. This contradicts (i).

From this contradiction, it follows that T^ε is a w-eigenspace for $\varepsilon = +, -$. We may suppose that T^+ is a w-eigenspace with eigenvalue η_1. By Lemma 5.2, $w(ue_1) = \eta_1^{-1}(ue_1)$ for $e_1 \in T^+$. As T^+ is u-invariant, $ue_1 \in T^+$. So $\eta_1 = \eta_1^{-1}$. Then by the choice of u and v in accordance with Lemma 5.2, $\eta_1 = 1$ and $T^+ = W$. By Lemma 5.2, $\{u^*, v^*\}$ is an intersecting pair.

Conversely assume that $\{u^*, v^*\}$ is an intersecting pair of noncommuting 2-involutions and use the same notation as in the first part of this theorem. Then

$$W = U^+ \cap V^+ \oplus U^- \cap V^-,$$

and W is the fixed subspace of $w = uv$ as described in Lemma 5.2. Let t be the 2-involution with eigenspaces W and $W^\perp$. Since $C_G(w)$ leaves W and $W^\perp$

invariant, $t \in Z(C_G(w))$. By Lemma 5.3, $C_{G*}(w) = C_G(w)$. So $t^* \in Z(C_{G*}(w^*))$. Clearly $t \in C_{G*}^{++}(u^*, v^*)$. So (ii) holds. Since $C_G(w)|_W \supseteq U_2^+(W, f|_W)$, it follows from Proposition 1.3 that $(C_G(w)|_W)^4$ is nonabelian. So (i) follows inasmuch as $C_G(w)^4$ is contained in all normal subgroups of $C_G(w)$ of index 4 and $C_G(w) = C_G^*(w)$. This proves the theorem.

Utilizing the group-theoretical characterization of intersecting noncommuting pairs of 2-involution given in Theorem 5.4 on the basis of Corollary 4.7 and Theorem 4.5, we have the following corollary.

COROLLARY 5.5. *Every automorphism of G^* leaves invariant the set of intersecting noncommuting pairs of 2-involutions.*

6. Mappings of 1-dimensional spaces

Let σ denote an automorphism of G^*. Then σ leaves invariant the set of 2-involutions and also the set of intersecting noncommuting pairs of 2-involutions. Our aim is to associate with σ a collineation ψ of the projective space $P(E)$ determined by E and show that σ is obtained by conjugation by ψ.

Choose a vector $e \in E$, $e \neq 0$. Denote by $\mathscr{S}(Ke)$ the set of all 2-involutions in G^*, one of whose spaces contains Ke. Let $\{u_1^*, u_2^*\}$ be an intersecting noncommuting pair of 2-involutions with spaces U_1^+, U_1^- and U_2^+, U_2^-, respectively, and with the property that $U_1^+ \cap U_2^+ = Ke_1$. Let u_3^* be a 2-involution in $\mathscr{S}(Ke)$ distinct from u_1^* and u_2^* such that the subspace of u_3^* which contains Ke is itself contained in $U_1^+ + U_2^+$. The triple $(u_1^*, u_2^*; u_3^*)$ with u_1^*, u_2^*, and u_3^* in $\mathscr{S}(Ke)$ will be called a *tight triple in* $\mathscr{S}(Ke)$. Because $u_i^* \in \mathscr{S}(Ke)$, $i = 1, 2, 3$, $\{u_1^*, u_2^*\}$ is an intersecting noncommuting pair and the pairs $\{u_1^*, u_3^*\}$ and $\{u_2^*, u_3^*\}$ are either strongly commuting pairs or intersecting noncommuting pairs of 2-involutions.

LEMMA 6.1. *Take $e \in E$, $e \neq 0$. Let $(u_1^*, u_2^*; u_3^*)$ be a tight triple of 2-involutions in $\mathscr{S}(Ke)$. Let U_i^+, U_i^-, $i = 1, 2, 3$, be the respective subspaces of u_i^* chosen so that*

$$U_1^+ \cap U_2^+ \cap U_3^+ = Ke_1.$$

Then

$$U_1^- \cap U_2^- \cap U_3^- \neq 0.$$

Conversely, suppose for distinct 2-involutions u_i^*, $i = 1, 2, 3$, with subspaces U_i^+, U_i^-, both

$$U_1^+ \cap U_2^+ \cap U_3^+ \neq 0 \quad \text{and} \quad U_1^- \cap U_2^- \cap U_3^- \neq 0.$$

Then $(u_i^*, u_j^*; u_k^*)$ is a tight triple for some $\{i, j, k\} = \{1, 2, 3\}$.

Proof. Assume $(u_1^*, u_2^*; u_3^*)$ is a tight triple. Then $U_3^+ \subseteq U_1^+ + U_2^+$. So

$$U_1^+ + U_2^+ + U_3^+ = U_1^+ + U_2^+,$$

and

$$U_1^- \cap U_2^- = (U_1^+ + U_2^+)^\perp = (U_1^+ + U_2^+ + U_3^+)^\perp = U_1^- \cap U_2^- \cap U_3^-.$$

Conversely, let $E_1 = U_1^+ \cap U_2^+ \cap U_3^+$ and $E_2 = U_1^- \cap U_2^- \cap U_3^-$, and assume $E_1 \neq 0$ and $E_2 \neq 0$. Then $E_2^\perp \supseteq U_1^+ + U_2^+ + U_3^+$. So dim $(U_1^+ + U_2^+ + U_3^+) = 3$. Hence

$$U_i^+ \subseteq U_j^+ + U_k^+ \quad \text{for } \{i, j, k\} = \{1, 2, 3\}.$$

Now $U_i^+ \cap U_j^+ = E_1$; so $U_i^+ \cap U_j^+ \cap U_k^- = 0$. Similarly, $U_i^- \cap U_j^+ \cap U_k^- = U_j^+ \cap E_2 = 0$. But u_j and u_k commute if and only if $U_j^+ \cap U_k^- \neq 0$ and $U_j^- \cap U_k^+ \neq 0$. Should u_i commute with both u_j and u_k, we would obtain $U_j^+ \cap U_k^- \subseteq U_i^+$ or $U_j^+ \cap U_k^- \subseteq U_i^-$, which is not possible. Hence $[u_i, u_j] \neq 1$ for some pair i, j, and $(u_i^*, u_j^*; u_k^*)$ is a tight triple.

LEMMA 6.2. *Let $(u_1^*, u_2^*; u_3^*)$ be a tight triple of 2-involutions. Then there exists a 2-involution $t^* \in C_{G^*}^{++}(u_1, u_2^*; u_3)$.*

Proof. As $(u_1^*, u_2^*; u_3^*)$ is a tight triple, $\{u_1^*, u_2^*\}$ is a noncommuting intersecting pair. Then there exists $e_1, e_2 \in E$ such that $U_1^+ \cap U_2^+ = Ke_1$ and $U_1^- \cap U_2^- = Ke_2$. Let $T^+ = Ke_1 + Ke_2$ and $T^- = (T^+)^\perp$. Let t^* be the 2-involution in G^* with subspaces T^+, T^-. As $u_3^* \in \mathscr{S}(Ke_1)$, one space of u_3^* contains e_1. We may assume that $U_1^+ \cap U_2^+ \cap U_3^+ = Ke_1$; then $U_1^- \cap U_2^- \cap U_3^- = Ke_2$. So $Ke_1 = U_3^+ \cap T^+$ and $Ke_2 = U_3^- \cap T^-$. Let

$$Kd_i^+ = (U_i^-)^\perp \cap T^+ \quad \text{and} \quad Kd_i^- = (U_i)^\perp \cap T^-, \quad i = 1, 2, 3.$$

Then $E = Ke_1 \oplus Ke_3 \oplus Kd_i^+ \oplus Kd_i^-$, $i = 1, 2, 3$. These three decompositions show that $t^* \in C_{G^*}^{++}(u_1^*, u_2^*; u_3^*)$.

LEMMA 6.3. *Let $(u_1^*, u_2^*; u_3)$ be a tight triple of 2-involutions. Let $v_i^* = \sigma(u_i^*)$, $i = 1, 2, 3$, and let V_i^+, V_i^- be the spaces of v_i^* chosen so that $V_1^+ \cap V_2^+ = Kd_1$ and $V_1^- \cap V_2^- = Kd_2$ for $d_1, d_2 \in E$. Then d_1 and d_2 belong to distinct subspaces V_3^+, V_3^- and the notation can be chosen so that*

$$Kd_1 = V_1^+ \cap V_2^+ \cap V_3^+ \quad \textit{and} \quad Kd_2 = V_1^- \cap V_2^- \cap V_3^-.$$

In particular, $(v_1^*, v_2^*; v_3^*)$ is a tight triple of 2-involutions in $\mathscr{S}(Kd_1)$ and in $\mathscr{S}(Kd_2)$.

Proof. By Theorem 5.4, $\{v_1^*, v_2^*\}$ is an intersecting pair of noncommuting 2-involutions. Then after an appropriate choice of sign, $V_1^+ \cap V_2^+ = Kd_1$ and $V_1^- \cap V_2^- = Kd_2$. Let $S^+ = Kd_1 + Kd_2$. By Lemma 6.2, $C_{G^*}^{++}(u_1^*, u_2^*, u_3^*)$ contains a 2-involution t^*. Let $s^* = \sigma(t^*)$. By virtue of Corollary 4.7,

$$s^* \in C^{++}_{G^{**}}(v_1^*, v_2^*, v_3^*).$$

Let s be a 2-involution in s^*. Then $s \in C_G(v_1, v_2, v_3)$.

As (v_1^*, v_2^*) is an intersecting pair of noncommuting 2-involutions, it follows from Theorem 5.2 that S^+ is an eigenspace of s. When $[v_2^*, v_3^*] \neq 1$, we may choose the sign $+$, $-$ so that $V_3^+ \cap V_2^+ \neq 0$ since $\{v_2^* \, v_3^*\}$ is an intersecting pair. When $[v_2^*, v_3^*] = 1$, v_2^* and v_3^* strongly commute by virtue of Corollary 4.7. Then $V_3^+ \cap V_2^+ \neq 0$ in this case as well. As $[\langle v_2, v_3\rangle, s] = 1$, s leaves invariant $V_3^+ \cap V_2^+$. So $V_3^+ \cap V_2^+ \subseteq S^+$ or $V_3^+ \cap V_2^+ \subseteq S^-$ where $S^- = (S^+)^\perp$.

First, take the case that $V_3^+ \cap V_2^+ \subseteq S^+$. Then

$$V_3^+ \cap V_2^+ = V_2^+ \cap S^+ = V_2^+ \cap V_1^+ = Kd_1.$$

Hence $V_3^- \cap V_2^- \neq 0$. So $V_3^- \cap V_2^- \subseteq S^\varepsilon$ where $\varepsilon = +$ or $-$. If $V_3^- \cap V_2^- \subseteq S^+$, then

$$V_3^- \cap V_2^- = V_2^- \cap S^+ = V_2^- \cap V_1^- = Kd_2.$$

So suppose that $\varepsilon = -$. Then

$$V_3^- \cap S^- = V_2^- \cap S^- \neq V_1^- \cap S^-.$$

By symmetry, we also conclude that

$$V_3^- \cap S^- = V_1^- \cap S^- \neq V_2^- \cap S^-,$$

which is a contradiction. The same contradiction occurs if $V_3^- \cap V_2^- \subseteq S^+$.

In the remaining case, $S^- = V_3^+ \cap V_2^+ + V_3^- \cap V_2^-$. Again by symmetry. $S^- = V_3^+ \cap V_1^+ + V_3^- \cap V_1^-$. As the components in these two decompositions are all distinct and v_3-invariant, S^- is an v_3-eigenspace. But clearly $S^- \neq V_3^+$ and $S^- \neq V_3^-$. This contradiction proves the lemma.

Take $e \in E$, $e \neq 0$. Let $\{u_1^*, u_2^*\}$ be an intersecting noncommuting pair of 2-involutions with subspaces U_1^+, U_1^- and U_2^+, U_2^-, respectively, such that $U_1^+ \cap U_2^+ = Ke$. Let $u_3^* \in \mathscr{S}(Ke)$, with subspaces U_3^+, U_3^- such that $U_3^+ \supseteq Ke$. The triple $(u_1^*, u_2^*; u_3^*)$ is said to be a *loose triple* in $\mathscr{S}(Ke)$ provided

$$U_3^+ \nsubseteq U_1^+ + U_2^+ \quad \text{and} \quad U_1^- \cap U_2^- \nsubseteq U_3^+.$$

LEMMA 6.4. *Take $e \in E$, $e \neq 0$. Let $\{u_1^*, u_2^*\}$ be an intersecting noncommuting pair in $\mathscr{S}(Ke)$. Let U_i^+, U_i^- be subspaces of u_i^*, $i = 1, 2, 3$, chosen so that $e \in U_i^+$, $i = 1, 2, 3$. Let $u_3^* \in \mathscr{S}(Ke)$ be chosen with subspaces U_3^+ with $e \in U_3^+$. Then $(u_1^*, u_2^*; u_3^*)$ forms a loose triple in $\mathscr{S}(Ke)$ if and only if*

$$U_1^- \cap U_2^- \nsubseteq U_3^+ \quad \textit{and} \quad U_1^- \cap U_2^- \nsubseteq U_3^-. \tag{6.1}$$

or if and only if for $\varepsilon_i = +, -, i = 1, 2, 3,$

$$U_1^{\varepsilon_1} \cap U_2^{\varepsilon_2} \cap U_3^{\varepsilon_3} \neq 0 \tag{6.2}$$

only when $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = +$ and $U_1^+ \cap U_2^+ \cap U_3^+ = Ke.$

Proof. The condition $U_1^- \cap U_2^- \nsubseteq U_3^-$ is equivalent to

$$U_3^+ \nsubseteq (U_1^- \cap U_2^-)^\perp = U_1^+ + U_2^+.$$

So the conditions (6.1) are equivalent to $(u_1^*, u_2^*; u_3^*)$ being loose. When $(u_1^*, u_2^*; u_3^*)$ is a loose triple in $\mathscr{S}(Ke)$, it follows that $U_1^+ \cap U_2^- = U_1^- \cap U_2^+ = 0$ since $\{u_1^*, u_2^*\}$ is a noncommuting intersecting pair, and therefore $U_1^{\varepsilon_1} \cap U_2^{\varepsilon_2} \cap U_3^{\varepsilon_3} \neq 0$ only if $\varepsilon_1 = \varepsilon_2$. From (6.1), it also follows that $\varepsilon_1 = \varepsilon_2 = +$. Now $U_1^+ \cap U_2^+ \cap U_3^+ = Ke_1$; so $U_1^+ \cap U_2^+ \cap U_3^- = 0$ as $e \notin U_1^-$. Conversely, assume (6.2) holds. Then $\{u_1^*, u_2^*\}$ clearly forms intersecting noncommuting pair for which (6.1) holds.

Because of the symmetry in the conditions (6.2), it is clear that $(u_1^*, u_3^*; u_2^*)$ and $(u_2^*, u_3^*; u_1^*)$ form loose triples in $\mathscr{S}(Ke)$ whenever $(u_1^*, u_2^*; u_3^*)$ is loose in $\mathscr{S}(Ke)$. Consequently we write (u_1^*, u_2^*, u_3^*) for any of these triples. When $(u_1^*, u_2^*; u_3^*)$ forms a tight triple, so does $(u_i^*, u_j^*; u_k^*)$ for any rearrangement (i, j, k) of $(1, 2, 3)$ provided $[u_i^*, u_j^*] \neq 1$.

LEMMA 6.5. *Let* u_1^*, u_2^*, u_3^* *belong to* $\mathscr{S}(Ke)$ *for some* $e \in E$, $e \neq 0$. *Then, either* $u_3^* = u_1^* u_2^*$ *or a pair of involutions, say,* u_1^*, u_2^*, *do not commute, and one and only one of the following holds:*

(i) $(u_1^*, u_2^*; u_3^*)$ *is a tight triple in* $\mathscr{S}(Ke)$.
(ii) $(u_1^*, u_2^*; u_3^*)$ *is a loose triple in* $\mathscr{S}(Ke)$.
(iii) $u_3^* \in C_{G^*}^{+,+}(u_1^*, u_2^*)$.

Let U_i^+, U_i^- *be the spaces of* $u_i^*, i = 1, 2, 3,$ *chosen so that* $U_1^+ \cap U_2^+ \neq 0$ *and* $U_1^- \cap U_2^- \neq 0$. *Then for* $\varepsilon = +$ or $-$,

$$U_3^\varepsilon = U_1^+ \cap U_2^+ + U_1^- \cap U_2^-. \tag{6.3}$$

if and only if (iii) *holds.*

Proof. We may suppose that $(u_1^*, u_2^*; u_3^*)$ is neither a tight nor a loose triple, and that $U_1^+ \cap U_2^+ \cap U_3^+ = Ke$. Then $U_1^- \cap U_2^- \nsubseteq U_3^-$ by Lemma 6.1 but $U_1^- \cap U_2^- \subseteq U_3^+$ by (6.1). This gives (6.3), which implies that U_3^+ is both u_1-invariant and u_2-invariant. Then $u_3^* \in C_{G^*}^{+,+}(u_1^*, u_2^*)$ and (iii) holds. It is clear that (iii) implies (6.3).

LEMMA 6.6. *Take* $e \in E$, $e \neq 0$. *Let* (u_1^*, u_2^*, u_3^*) *be a loose triple in* $\mathscr{S}(Ke)$. *Set* $v_i^* = \sigma(u_i^*)$, $i = 1, 2, 3$. *Then* (v_1^*, v_2^*, v_3^*) *is a loose triple in* $\mathscr{S}(Kd)$ *for some* $d \in E$, $d \neq 0$.

Proof. Let V_i^+, V_i^- be the subspaces of v_i^*, $i = 1, 2, 3$. By Theorem 5.4, $\{v_1^*, v_2^*\}$ is an intersecting noncommuting pair. We may suppose that $V_1^+ \cap V_2^+ = Kd_1$, $V_1^- \cap V_2^- = Kd_2$ for $d_1, d_2 \in E$, $d_1 \neq 0$, $d_2 \neq 0$, and $V_1^+ \cap V_2^- = V_1^- \cap V_2^+ = 0$. Assume to the contrary that (v_1^*, v_2^*, v_3^*) is not loose. Suppose first that $V_1^{\varepsilon_1} \cap V_2^{\varepsilon_1} \cap V_3^{\varepsilon_3} \neq 0$ for some choice of signs $\varepsilon_1, \varepsilon_3 = +, -$. Then by Lemma 6.5, either $(v_1^*, v_2^*; v_3^*)$ forms a tight triple or $v_3^* \in C_{G^+}^{+,+}(v_1^*, v_2^*)$. But now a contradiction to the looseness of (u_1^*, u_2^*, u_3^*) follows using the automorphism σ^{-1} and Lemma 6.3 or Corollary 4.7.

It remains to consider the case

$$V_1^{\varepsilon_1} \cap V_2^{\varepsilon_2} \cap V_3^{\varepsilon_3} = 0 \tag{6.4}$$

where $\varepsilon_i = +, -$, $i = 1, 2, 3$. We will contradict (6.4). By Lemma 6.4, $\{u_i^*, u_3^*\}$ is an intersecting noncommuting pair of 2-involutions for $i = 1, 2$. By Theorem 5.4, the same is true for $\{v_i^*, v_3^*\}$. Then V_3^+ nontrivially intersects exactly one of V_1^+, V_1^- and exactly one of V_2^+, V_2^-.

Suppose that $V_3^+ \cap V_1^{\varepsilon_1} \neq 0$ and $V_3^+ \cap V_2^{\varepsilon_2} \neq 0$ for $\varepsilon_i = +, -$, $i = 1, 2$. Then by (6.4)

$$V_3^+ = V_3^+ \cap V_1^{\varepsilon_1} + V_3^+ \cap V_2^{\varepsilon_2} \subseteq V_1^{\varepsilon_1} + V_2^{\varepsilon_2} = (V_1^{-\varepsilon_1} \cap V_2^{-\varepsilon_2}) \tag{6.5}$$

where $-\varepsilon_i = -, +$ according as $\varepsilon_i = +, -$. Then from (6.5) $V_1^{-\varepsilon_1} \cap V_2^{-\varepsilon_2} \subseteq (V_3^+)^\perp = V_3^-$. But this contradicts (6.4). Thus (v_1^*, v_2^*, v_3^*) is loose.

PROPOSITION 6.7. *Let $e \in E$, $e \neq 0$. Then there exists $d \in E$, $d \neq 0$ such that $\sigma(\mathscr{S}(Ke)) = \mathscr{S}(Kd)$.*

Proof. Choose (u_1^*, u_2^*, u_3^*) to be a loose triple in $\mathscr{S}(Ke)$. Set $v_i^* = \sigma(u_i^*)$, $i = 1, 2, 3$. By Lemma 6.6, (v_1^*, v_2^*, v_3^*) is a loose triple in $\mathscr{S}(Kd)$ for some $d \in E$, $d \neq 0$. Let V_i^+, V_i^- be the subspaces of v_i^*, $i = 1, 2, 3$. We may assume that

$$V_1^+ \cap V_2^+ \cap V_3^+ = Kd. \tag{6.6}$$

By (6.2), if $\varepsilon_i = +, -$, $i = 1, 2, 3$,

$$V_1^{\varepsilon_1} \cap V_2^{\varepsilon_2} \cap V_3^{\varepsilon_3} = 0 \tag{6.7}$$

except when $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = +$.

Now, let $u^* \in \mathscr{S}(Ke)$. Let $v^* = \sigma(u^*)$, and let V^+, V^- be the subspaces of v^*. It is required to show that $V_i^+ \cap V_j^+ \cap V^\varepsilon \neq 0$ either for $\varepsilon = +$ or for $\varepsilon = -$ where $i, j \in \{1, 2, 3\}$. When $(v_1^*, v_2^*; v^*)$ forms a tight triple, it follows by the definition of tightness that $V_1^+ \cap V_2^+ \cap V^+ \neq 0$. When $u^* \in C_{G*}^{+,+}(u_1^*, u_2^*)$, $v^* \in C_{G*}^{+,+}(v_1^*, v_2^*)$ by Corollary 4.7. Then by (6.3),

$$V_3^+ = V_1^+ \cap V_2^+ + V_1^- \cap V_2^-.$$

In particular, $V_1^+ \cap V_2^+ \cap V_3^+ \neq 0$.

By virtue of Lemma 6.5, it remains to consider that (u_1^*, u_2^*, u^*) is a loose triple in $\mathscr{S}(Ke)$. By Lemma 6.6 (v_1^*, v_2^*, v^*) is also loose. Then by Lemma 6.4, either

$$V_1^+ \cap V_2^+ \cap V^{\varepsilon_1} \neq 0 \quad \text{and} \quad V_1^- \cap V_2^- \cap V^{\varepsilon_1} = 0$$

or

$$V_1^+ \cap V_2^+ \cap V^{\varepsilon_1} = 0 \quad \text{and} \quad V_1^- \cap V_2^- \cap V^{\varepsilon_1} \neq 0$$

for some choice $\varepsilon_1 = +, -$. When the first alternative occurs, $v^* \in \mathscr{S}(Kd)$ since $V_1^+ \cap V_2^+ = Kd$. Thus, consider that the second alternative occurs. As $[u_1^*, u_3^*] \neq 1$ and $[u_2^*, u_3^*] \neq 1$, we obtain, by symmetry, the relations

$$V_1^- \cap V_2^- \cap V^{\varepsilon_1} = Kd_{12} \neq 0,$$
$$V_2^- \cap V_3^- \cap V^{\varepsilon_2} = Kd_{23} \neq 0,$$
$$V_3^- \cap V_1^- \cap V^{\varepsilon_3} = Kd_{31} \neq 0,$$

where $\varepsilon_i = +, -$, $i = 1, 2, 3$. By (6.2), $V_1^- \cap V_2^-$, $V_2^- \cap V_3^-$, and $V_3^- \cap V_1^-$ are distinct 1-dimensional subspaces. Then the spaces Kd_{ij} are distinct for $(i, j) = (1, 2), (2, 3)$ or $(3, 1)$. For some pair (i, j), $\varepsilon_i = \varepsilon_j$. Then

$$V^{\varepsilon_j} = Kd_{ij} + Kd_{jk}$$

where $\{i, j, k\} = \{1, 2, 3\}$. But d_{ik} and d_{jk} belong to V_j. Hence $V^{\varepsilon_j} \subseteq V_j$. Let $v_j \in v_j^*$. Then $v_j(V^{\varepsilon_j}) = V^{\varepsilon_j}$. Thus $v_j(V^+) = V^+$ and $v_j(V^-) = V^-$ since $V^+ = (V^-)^\perp$. So $[v_j^*, v^*] = 1$, in contradiction to (v_1^*, v_2^*, v^*) being loose. This proves the proposition.

Now we are in a position to relate σ to the projective geometry $P(E)$ consisting of the lattice of subspaces of E. Let $P_m(E)$ be the set of subspaces of E of dimension at most $m + 1$. The elements of $P_0(E)$ are called *points in* $P(E)$ and the elements of $P_1(E)$ are called *lines in* $P(E)$. Each point in $P(E)$ has the form Ke, $e \in E$, $e \neq 0$. Clearly $\mathscr{S}(Ke_1) = \mathscr{S}(Ke_2)$ if and only if $Ke_1 = Ke_2$. So we define

$$\psi_0 \colon P_0(E) \to P_0(E)$$

by setting

$$\mathscr{S}(\psi_0(Ke)) = \sigma\mathscr{S}(Ke) \quad \text{for } Ke \in P_0(E). \tag{6.8}$$

LEMMA 6.8. *The mapping ψ_0 is a bijection which maps orthogonal pairs in $P_0(E)$ to orthogonal pairs.*

Proof. Define $\psi'_0 \colon P_0(E) \to P_0(E)$ by setting

$$\mathscr{S}(\psi'_0(Ke)) = \sigma^{-1}\mathscr{S}(Ke) \quad \text{for } Ke \in P_0(E).$$

Then $\psi_0\psi'_0 = \psi'_0\psi_0 = 1$. So ψ_0 is a bijection. Let e_1 and e_2 be nonzero orthogonal vectors in E. Then $\mathscr{S}(Ke_1) \cap \mathscr{S}(Ke_2)$ contains no loose triples by virtue of Lemma 6.4, and it clearly contains all tight triples of 2-involutions, one of whose spaces contains Ke_1 and the other contains Ke_2. By virtue of Lemma 6.3 and (6.8), $\mathscr{S}(\psi_0(Ke_1)) \cap \mathscr{S}(\psi_0(Ke_2))$ contains all tight triples of 2-involutions, one of whose spaces contains $\psi_0(Ke_1)$ and the other contains $\psi_0(Ke_2)$. As the eigenspaces of an involution are orthogonal, $\psi_0(Ke_1)$ is orthogonal to $\psi_0(Ke_2)$.

If remains to extend ψ_0 to a mapping on $P_1(E)$.

LEMMA 6.9. *Let $Ke_i \in P_0(E)$, $i = 1, 2, 3, 4$, and suppose*

$$Ke_1 + Ke_2 = Ke_3 + Ke_4. \tag{6.9}$$

Then

$$\psi_0(Ke_1) + \psi_0(Ke_2) = \psi_0(Ke_3) + \psi_0(Ke_4). \tag{6.10}$$

Furthermore, if $Ke_1 + Ke_2$ is a subspace for a 2-involution u^*, then $\psi_0(Ke_1) + \psi_0(Ke_2)$ is a subspace for σw^*.

Proof. Let u^* be the 2-involution in G^* with a space $U^+ = Ke_1 + Ke_2$. Then $u^* \in \bigcap_{i=1}^4 \mathscr{S}(Ke_i)$ as $e_i \in U^+$ for $i = 1, 2, 3, 4$. By (6.8), $\sigma(u^*) \in \bigcap_{i=1}^4 \mathscr{S}(\psi_0(Ke_i))$. Set $v^* = \sigma(u^*)$, and let V^+, V be the spaces of v^*. Then each $\psi_0(Ke_i)$ belongs either to V^+ or to V^-. But each Ke_i can be orthogonal to at most one of Ke_1, Ke_2, Ke_3, Ke_4. So each Ke_i belongs to a triple obtained from $\{Ke_1, Ke_2, Ke_3, Ke_4\}$, all of whose elements are contained in the same space U^ε, $\varepsilon = +$ or $-$. Then $\psi_0(Ke_i) \subseteq V^\varepsilon$, $i = 1, 2, 3, 4$. So

$$V^\varepsilon = \psi_0(Ke_1) + \psi_0(Ke_2) = \psi_0(Ke_3) + \psi_0(Ke_4).$$

This proves the lemma.

On the basis of Lemma 6.9, we extend ψ_0 to a mapping ψ_1 on $P_1(E)$ by setting

$$\psi_1(Ke_1 + Ke_2) = \psi_0(Ke_1) + \psi_0(Ke_2) \quad \text{for } Ke_1, Ke_2 \in P_0(E). \tag{6.11}$$

THEOREM 6.10. *To each automorphisim σ of G^*, there exists a collineation ψ of $P(E)$ which commutes with the polarity determined by the sesquilinear form f such that*

$$\sigma\colon g^* \to \psi g^* \psi^{-1}. \tag{6.12}$$

Proof. The mapping ψ_1, given by (6.11) maps collinear points onto collinear points. Then it is well known that ψ_1 extends to a collineation ψ of $P(E)$. Furthermore, for every 2-involution u^* with spaces U^+, U^-, $\sigma(u^*)$ has spaces $\psi(U^+)$ and $\psi(U^-)$ by virtue of (6.10) and (6.11). Therefore (6.12) holds when $g^* = u^*$.

Let G_0^* be the subgroup of G^* generated by its 2-involutions. Then (6.12) holds when $g \in G_0^*$. Let σ_ψ be the inner automorphism of G^* induced by the element ψ given in (6.12). Set $\tau = \sigma_\psi^{-1}\sigma = \sigma_{\psi^{-1}}\sigma$. Then $\tau \in C_{G^*}(G_0^*)$. Now G_0^* is a normal subgroup of G^*. So by Phillip Hall's three subgroups lemma,

$$[G^*, \tau, G_0^*] \subseteq [G_0^*, G^*, \tau][\tau, G_0^*, G^*] \subseteq [G_0^*, \tau] = 1.$$

As G_0^* acts irreducibily on E, $C_{G^*}(G_0^*) \subseteq Z(G^*) = 1$. Thus $[G^*, \tau] = 1$. So $\tau = 1$ and $\sigma = \sigma_\psi$.

In the usual way, using the fundamental theorem of projective geometry, it can be shown that ψ is induced by a semilinear transformation of E which preserves the sesquilinear form f up to a multiplier. Such a semilinear transformation is called a semimulitude. Thus Aut G^* is isomorphic to the group $P\Gamma U_4(K, f)$ of project semisimilitudes.

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