ILLINOIS JOURNAL OF MATHEMATICS
Volume 28, Number 3, Fall 1984

CHARACTERIZATIONS OF CLOSED DECOMPOSABLE OPERATORS

BY

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In extending the spectral theory beyond the class of spectral operators, one can no longer produce a spectral measure to represent the operator or to reveal its spectral structure. In this paper, we extend the use of some substitutes for the unavailable spectral measure, such as spectral capacity [1], [2], [6] and spectral resolvent [3] for the spectral-theoretic study of closed operators in an abstract Banach space.

The program of the paper is as follows. After a preliminary section, we supplement the given closed operator with a weaker constituent than that of the spectral resolvent and obtain a new criterion for its spectral decomposition (Corollary 2.3). In Section 3, we introduce a concept weaker than that of spectral capacity for a closed operator and again, we obtain a simpler description of its spectral decomposition (Theorem 3.3). Moreover, we extend a property of a bounded decomposable operator to the unbounded case (Corollary 3.5) and find conditions for a specific linear manifold occuring in the theory of spectral capacities, to be dense in the underlying space (Theorem 3.6).

If not mentioned otherwise, throughout this paper T is an unbounded closed operator with domain D_T and range in a Banach space X over the complex field $\mathbf{C}$. For a set S, $\bar{S}$ is the closure, S^c is the complement, ∂S is the boundary, $d(\lambda, S)$ is the distance from a point $\lambda \in \mathbf{C}$ to $S \subset \mathbf{C}$, and we write cov S for the collection of all finite open covers of S. If S is a subset of $\mathbf{C}$, then the above mentioned topological constructs are referred to the topology of $\mathbf{C}$. Without loss of generality, we assume that for $S \subset \mathbf{C}$, each $\{G_i\}_{i=0}^n \in$ cov S has, at most, one unbounded set G_0. A set $G \subset \mathbf{C}$ is said to be a neighborhood of ∞, in symbols $G \in V_\infty$, if for $r > 0$ sufficiently large,

$$\{\lambda \in \mathbf{C}: |\lambda| > r\} \subset G.$$

We write $S^\perp$ for the annihilator of $S \subset X$ in the dual space X^*. We write $\mathfrak{G}$ and $\mathfrak{F}$ for the families of all open and closed subsets of $\mathbf{C}$, respectively. Further, we denote by $\mathfrak{G}^K$ and $\mathfrak{F}^K$ the collection of all relatively compact open and that of all compact subsets of $\mathbf{C}$, respectively. $\mathbf{N}$ is the set of all positive integers.

We use the notations $\sigma(T)$, $\rho(T)$ and $R(\cdot; T)$ for the spectrum resolvent set and resolvent operator, respectively of T. If T has the single valued extension

Received April 26, 1982.

property (SVEP) then, for $x \in X$, $\sigma_T(x)$ is the local spectrum, $\rho_T(x)$ is the local resolvent set, $x(\cdot)$ is the resolvent function and, for $S \subset \mathbf{C}$,

$$X(T, S) = \{x \in X: \sigma_T(x) \subset S\}$$

is the spectral manifold of T. We write $\mathfrak{S}(X)$ for the lattice of all subspaces (closed linear manifolds) of X and Inv T for the sublattice of $\mathfrak{S}(X)$, whose members are invariant under T. For $Y \in \operatorname{Inv} T$, $T \mid Y$ is the restriction of T to Y. T^* denotes the conjugate (if defined) of T.

1. Introduction

We review the definitions and some properties which will serve as a basis for our study.

1.1. Definition [5]. Given $n \in \mathbf{N}$, T is said to have the *n-spectral decomposition property* ($n-$ SDP) if, for any $\{G_i\}_{i=0}^n \in \operatorname{cov} \sigma(T)$ with $G_0 \in V_\infty$, there exists a system $\{X_i\}_{i=0} \subset \operatorname{Inv} T$ satisfying the following conditions:

(i) $X_i \subset D_T$ if $G_i \in \mathfrak{G}^K$ $(1 \leq i \leq n)$;

(ii) $X = \sum_{i=0}^{n} X_i$ and $\sigma(T \mid X_i) \subset G_i$, $0 \leq i \leq n$.

If, for every $n \in \mathbf{N}$, T has the n-SDP then T is said to have the *spectral decomposition property* (SDP) [4].

If the system $\{X_i\}_{i=0}^n$, satisfying conditions (i) and (ii), consists of spectral maximal spaces of T then T is a decomposable operator.

Some properties of closed operators with the SDP are summarized in the following result.

1.2. Theorem [5], [8]. (a) *If T has the* SDP *then, for every $F \in \mathfrak{F}$, $X(T, F)$ is a spectral maximal space of T* and

$$\sigma[T \mid X(T, F)] \subset F \cap \sigma(T).$$

(b) *Given T, let $Y \in \operatorname{Inv} T$ be such that $\sigma(T \mid Y) \in \mathfrak{F}^K$. Then there exist Υ, $W \in \operatorname{Inv} T$ with the following properties:*

$$Y = \Upsilon \oplus W,\ \Upsilon \subset D_T,\ \sigma(T \mid \Upsilon) = \sigma(T \mid Y),\ \sigma(T \mid W) = \emptyset.$$

Moreover, if Δ is a bounded Cauchy domain that contains $\sigma(T \mid Y)$ then $\Upsilon = PY$, where P is the spectral projection

$$P = \frac{1}{2\pi i} \int_{\partial\Delta} R(\lambda; T \mid Y)\, d\lambda. \tag{1.1}$$

In particular, if T has the SDP *then, for every $F \in \mathfrak{F}^K$,*

$$X(T, F) = \Xi(T, F) \oplus X(T, \emptyset)$$

with $\Xi(T, F) \in \operatorname{Inv} T$, $\Xi(T, F) \subset D_T$, *and* $\sigma[T \mid \Xi(T, F)] = \sigma[T \mid X(T, F)]$. *Moreover,* $\Xi(T, F) = PX(T, F)$ *where, for* $Y = X(T, F)$, P *is the spectral projection* (1.1).

(c) *T is decomposable iff T has the SDP and $X(T, \emptyset) = \{0\}$.*

(d) *Let T be densely defined with $\rho(T) \neq \emptyset$. Then T has the SDP iff T^* has the SDP. Extend the definition of $\Xi(T, \cdot)$ to all open sets in $\mathbf{C}$ by*

$$\Xi(T, G) = \{x \colon x \in \Xi(T, F), F (\in \mathfrak{F}^K) \subset G\}.$$

Then

$$X(T, G) = \Xi(T, G) \oplus X(T, \emptyset), \ G \text{ open in } \mathbf{C}; \tag{1.2}$$

$$X^*(T^*, F) = [\Xi(T, \mathbf{C} - F)]^\perp, \ F \text{ closed in } \mathbf{C}.$$

(e) *If T has the 1-SDP then T has the SDP.*

1.3. Definition. A map $E\colon \mathfrak{G} \to \operatorname{Inv} T$ is called a *spectral resolvent* of T if the following conditions are satisfied:

(I) $E(G) \subset D_T$, if $G \in \mathfrak{G}^K$;
(II) $\sigma[T \mid E(G)] \subset \bar{G}$, for all $G \in \mathfrak{G}$;
(III) $X = \sum_{i=0}^n E(G_i)$, for every $\{G_i\}_{i=0}^n \in \operatorname{cov} \sigma(T)$ with $G_0 \in V_\infty$.

It follows from (I) and (II) that $E(\emptyset) = \{0\}$, and (III) implies that, for every $G \in V_\infty$ with $G \supset \sigma(T)$, $E(G) = X$.

Evidently, T endowed with a spectral resolvent has the SDP.

Given $F \subset \mathbf{C}$ with $\mathbf{C} \not\subset F$, for $\lambda \in F^c$, let

$$G(\lambda) = \{\mu \in \mathbf{C} \colon |\mu - \lambda| > \tfrac{1}{2} d(\lambda, F)\},$$

provided that $F \neq \emptyset$. Define

$$G_F = \begin{cases} \{G(\lambda) \colon \lambda \in F^c\} & \text{if } F \neq \emptyset; \\ \mathfrak{G} \cap V_\infty, & \text{if } F = \emptyset. \end{cases} \tag{1.3}$$

1.4. Lemma. *Let T have a spectral resolvent E. Then, for every $F \in \mathfrak{F}$,*

$$X(T, F) = \cap \{E(G) \colon G \in G_F\}$$

Proof. Formally the same as in [7, Theorem 4.1]. ∎

2. Prespectral resolvents and the SDP

If we drop condition (I) from Definition 1.3, the resulting weaker concept of prespectral resolvent will give a characterization of the SDP.

2.1. Definition. We call a map $\tilde{E}\colon \mathfrak{G} \to \operatorname{Inv} T$ a *prespectral resolvent of* T, if it satisfies the following conditions:

(i) $\sigma[T \mid \tilde{E}(G)] \subset \bar{G}$ for every $G \in \mathfrak{G}$;
(ii) $X = \sum_{i=0}^{n} \tilde{E}(G_i)$ for every $\{G_i\}_{i=0}^{n} \in \operatorname{cov} \sigma(T)$ with $G_0 \in V_\infty$.

In contrast to E, we may have $\tilde{E}(\emptyset) \neq \{0\}$, but for every open $G \in V_\infty$ with $G \supset \sigma(T)$, (ii) implies that $\tilde{E}(G) = X$. If T is bounded then the concepts of prespectral and spectral resolvents coincide.

Given T, let Z be the set of all $x \in X$ satisfying the following property: for every $\lambda_0 \in \mathbf{C}$ there is a neighborhood δ of λ_0 and a function $f\colon \delta \to D_T$, analytic on δ, such that

$$(\lambda - T) f(\lambda) = x \quad \text{on } \delta.$$

2.2. Theorem. *Let $\tilde{E}$ be a prespectral resolvent of T. The following assertions are equivalent:*

(i) $Z \subset \tilde{E}(G)$, *for every* $G \in V_\infty$.
(ii) *There exists a spectral resolvent E of T such that*

$$E(G) \subset \tilde{E}(G) \quad \text{if} \quad G \in \mathfrak{G}^K, \quad E(G) = \tilde{E}(G) \quad \text{if} \quad G \text{ is unbounded.} \tag{2.1}$$

Proof. (i) $\Rightarrow$ (ii) Let $H \in \mathfrak{G}^K$. Then $\sigma[T \mid \tilde{E}(H)]$ is compact and by Theorem 1.2 (b), we have

$$\tilde{E}(H) = \tilde{E}'(H) \oplus W, \tag{2.2}$$

where $\tilde{E}'(H)$, $W \in \operatorname{Inv} T$, $\tilde{E}'(H) \subset D_T$, $\sigma[T \mid \tilde{E}'(H)] = \sigma[T \mid \tilde{E}(H)] \subset \bar{H}$ and $\sigma(T \mid W) = \emptyset$. Then $T \mid \tilde{E}'(H)$ is bounded and $W \subset Z$. If $G \in V_\infty$ then, for every $H \in \mathfrak{G}^K$, the direct summand W in (2.2) is contained in $\tilde{E}(G)$ since, by hypothesis, $W \subset Z \subset \tilde{E}(G)$. Define $E\colon G \to \operatorname{Inv} T$, by

$$E(G) = \begin{cases} \tilde{E}'(G), & \text{if } G \in \mathfrak{G}^K; \\ \tilde{E}(G), & \text{if } G \text{ is unbounded.} \end{cases} \tag{2.3}$$

Then, for every $\{G_i\}_{i=0}^{n} \in \operatorname{cov} \sigma(T)$ with $G_0 \in V_\infty$ and $G_i \in \mathfrak{G}^K$ $(1 \leq i \leq n)$, we have

$$X = \sum_{i=0}^{n} \tilde{E}(G_i) = \tilde{E}(G_0) + \sum_{i=1}^{n} \tilde{E}'(G_i) = \sum_{i=0}^{n} E(G_i);$$
$$\sigma[T \mid E(G_0)] = \sigma[T \mid \tilde{E}(G_0)] \subset \bar{G}_0;$$
$$\sigma[T \mid E(G_i)] = \sigma[T \mid \tilde{E}'(G_i)] = \sigma[T \mid \tilde{E}(G_i)] \subset \bar{G}_i, \quad 1 \leq i \leq n.$$

Consequently, E is a spectral resolvent of T that satisfies condition (2.1).

(ii) ⇒ (i) Let E be any spectral resolvent of T that satisfies condition (2.1). By Lemma 1.4, we have $X(T, \emptyset) = \cap \{E(G)\colon G \in \mathfrak{G} \cap V_\infty\}$. Since, for every $G \in \mathfrak{G} \cap V_\infty$, $E(G) = \tilde{E}(G)$ as defined by (2.3), we have

$$Z = X(T, \emptyset) \subset \tilde{E}(G),\ G \in \mathfrak{G} \cap V_\infty. \qquad \blacksquare$$

The following properties are immediate consequences of Theorem 2.2.

2.3. COROLLARY. *If T has a prespectral resolvent E that satisfies condition* (i) *of Theorem* 2.2, *then T has the SDP.*

2.4. COROLLARY. *If $Z = \{0\}$, then every prespectral resolvent of T is a spectral resolvent of T.*

3. Spectral capacities and decomposable operators

Spectral capacities play a role in the theory of decomposable operators analogous to the role of spectral resolvents in the skeletal structure of operators with the SDP. The spectral capacity concept introduced axiomatically in [1], subsequently evolved to an auxiliary of bounded decomposable operators in [6]. The concepts of spectral resolvent and spectral capacity come in junction in an extreme case: the maximal spectral resolvent is a spectral capacity for bounded operators. In this section, we shall both weaken and generalize the concept of spectral capacity for further characterizations of unbounded closed decomposable operators.

3.1. DEFINITION. Given $n \in \mathbf{N}$, a map $\mathfrak{E}\colon \mathfrak{F} \to S(X)$ is called a *n-prespectral capacity* if it satisfies the following conditions:

(i) $\{F_k\}_{k=1}^\infty \subset \mathfrak{F}$ implies $\mathfrak{E}(\bigcap_{k=1}^\infty F_k) = \bigcap_{k=1}^\infty \mathfrak{E}(F_k)$;
(ii) $X = \sum_{i=0}^n \mathfrak{E}(\bar{G}_i)$, for any $\{G_i\}_{i=0}^n \in \operatorname{cov} \mathbf{C}$ with $G_0 \in V_\infty$.

Note that (ii) implies that $\mathfrak{E}(\mathbf{C}) = X$.

Given T, $\mathfrak{E}$ is said to be a n-prespectral capacity of T if $\mathfrak{E}$ satisfies conditions (i), (ii) above and the following:

(iii) $\mathfrak{E}(F) \in \operatorname{Inv} T$, $\sigma[T \mid \mathfrak{E}(F)] \subset F$ for all $F \in \mathfrak{F}$.

$\mathfrak{E}$ is called a *n-spectral capacity* of T if it satisfies conditions (i)–(iii) above and the following

(iv) $\mathfrak{E}(F) \subset D_T$, if $F \in \mathfrak{F}^K$.

If, for all $n \in \mathbf{N}$, $\mathfrak{E}$ is a n-prespectral (n-spectral) capacity of T then $\mathfrak{E}$ is said to be a prespectral (spectral) capacity of T.

3.2. *Remarks.* (a) Condition (i) of Definition 3.1 can be replaced by the following:

(i′) For every collection $\mathfrak{F}' \subset \mathfrak{F}$, $\mathfrak{E}(\bigcap_{F \in \mathfrak{F}'} F) = \bigcap_{F \in \mathfrak{F}'} \mathfrak{E}(F)$.

(b) If $\mathfrak{E}$ is a n-prespectral capacity of T then $\mathfrak{E}(\emptyset) \subset Z$. Moreover, if $\mathfrak{E}$ is a n-spectral capacity of T then $\mathfrak{E}(\emptyset) = \{0\}$.

3.3. THEOREM. *Given T, the following assertions are equivalent:*

(I) *T has the n-SDP.*

(II) *T has a n-prespectral capacity $\mathfrak{E}$ such that $\mathfrak{E}(\emptyset)$ is a spectral maximal space of T. In this case, $\mathfrak{E}$ is unique.*

Proof. (II) ⇒ (I). Let $F \in \mathfrak{F}^K$. In view of Theorem 1.2 (b), we have

$$\mathfrak{E}(F) = \mathfrak{E}'(F) \oplus W \tag{3.1}$$

where $\mathfrak{E}'(F) \subset D_T$, $\sigma[T \mid \mathfrak{E}'(F)] = \sigma[T \mid \mathfrak{E}(F)] \subset F$ and $\sigma(T \mid W) = \emptyset$. Since $\mathfrak{E}(\emptyset)$ is a spectral maximal space of T, we have $W \subset \mathfrak{E}(\emptyset)$. For $G \in \mathfrak{G}$, let

$$E(G) = \begin{cases} \mathfrak{E}'(\bar{G}), & \text{if } G \in \mathfrak{G}^K; \\ \mathfrak{E}(\bar{G}), & \text{if } G \text{ is unbounded.} \end{cases} \tag{3.2}$$

It follows from Definition 3.1 (i) that, for every $G \in \mathfrak{G}$, we have $\mathfrak{E}(\emptyset) \subset \mathfrak{E}(\bar{G})$ and hence (3.2) implies that $\mathfrak{E}(\emptyset) \subset E(G)$ whenever $G \in \mathfrak{G}$ is unbounded. Thus, by Definition 3.1 (ii), for any $\{G_i\}_{i=0}^n \in \operatorname{cov} \sigma(T)$ with $G_0 \in V_\infty$ and $G_i \in \mathfrak{G}^K$ $(1 \le i \le n)$, we have

$$X = \sum_{i=0}^n \mathfrak{E}(\bar{G}_i) = \mathfrak{E}(\bar{G}_0) + \sum_{i=1}^n \mathfrak{E}'(\bar{G}_i) = \sum_{i=0}^n E(G_i).$$

Consequently, T has the n-SDP. In case that $\mathfrak{E}$ is a prespectral capacity of T, E as defined by (3.2), is a spectral resolvent of T. To see that $\mathfrak{E}$ is unique, note that for every $F \in \mathfrak{F}$, the members of G_F (1.3) are unbounded. Apply Remark 3.2 (a), (3.2) and Lemma 1.4 to infer that

$$\begin{aligned} X(T, F) &= \cap \{E(G) \colon G \in G_F\} = \cap \{\mathfrak{E}(\bar{G}) \colon G \in G_F\} \\ &= \mathfrak{E}(\cap \{\bar{G} \colon G \in G_F\}) = \mathfrak{E}(F). \end{aligned} \tag{3.3}$$

(I) ⇒ (II) In view of Theorem 1.2 (e), (a), we can define $\mathfrak{E}(F) = X(T, F)$, $F \in \mathfrak{F}$. By the properties of $X(T, \cdot)$, $\mathfrak{E}$ is a n-prespectral capacity of T. Since $\mathfrak{E}(\emptyset) = X(T, \emptyset)$ is a spectral maximal space of T, (II) holds. ▌

3.4 COROLLARY. *If E is a 1-prespectral capacity of T such that $E(\emptyset)$ is a spectral maximal space of T then E is a prespectral capacity of T.*

Proof. Let $\mathfrak{E}$ have the properties stated by the corollary. By Theorem 3.3, T has the 1-SDP. In view of Theorem 1.2 (e), for every $n \in \mathbf{N}$, T has the n-SDP. Thus, again by Theorem 3.3, $\mathfrak{E}$ is a n-prespectral capacity of T for all $n \in \mathbf{N}$. ▌

Next, we extend a property of a bounded decomposable operator [6] to the unbounded case.

3.5. COROLLARY. *T is a decomposable operator iff T has a spectral capacity $\mathfrak{E}$. Moreover, in such a case, $\mathfrak{E}$ is unique.*

Proof. By Theorem 1.2 (c), T is decomposable iff T has the SDP and $X(T, \emptyset) = \{0\}$. Then, assuming that T has the SDP and $X(T, \emptyset) = \{0\}$, Theorem 3.3 implies that T has a unique prespectral capacity $\mathfrak{E}$. Since, for $F \in \mathfrak{F}^K$, (3.1) implies

$$\mathfrak{E}(F) = \mathfrak{E}'(F) \oplus W \subset \mathfrak{E}'(F) + \mathfrak{E}(\emptyset) = \mathfrak{E}'(F) + \{0\} = \mathfrak{E}'(F) \subset D_T,$$

$\mathfrak{E}$ is a spectral capacity of T.

Conversely, if T has a spectral capacity $\mathfrak{E}$, then $\mathfrak{E}(\emptyset) = \{0\}$, by Remark 3.2 (b). Since, for $G \in \mathfrak{G}^K$, $\mathfrak{E}(\bar{G}) = \mathfrak{E}'(\bar{G})$, E defined by (3.2) is a spectral resolvent of T. For $F = \emptyset$ and $G_F = \mathfrak{G} \cap V_\infty$, it follows from (3.3) that $X(T, \emptyset) = \mathfrak{E}(\emptyset) = \{0\}$. Then, T is decomposable, by Theorem 1.2 (c). ∎

In [2] an extension of the decomposable operator concept to the unbounded case was obtained by means of a strong version of the spectral capacity. An extra feature of the strong spectral capacity ε is that, for any closed F in $\mathbf{C}$, the linear manifold

$$M(F) = \{x \in \varepsilon(K) : K \in \mathfrak{F}^K, K \subset F\}$$

is dense in $\varepsilon(F)$. In this vein we have the following:

3.6. THEOREM. *Let T have a spectral capacity $\mathfrak{E}$.*

(i) $\overline{M(\mathbf{C})} = X$ *iff T is densely defined and T^* is decomposable.*

(ii) *For every closed $F \in V_\infty$, $\overline{M(F)} = \mathfrak{E}(F)$ if T is densely defined and T^* is decomposable.*

Proof. By Corollary 3.5, T is decomposable and $\mathfrak{E}(F) = X(T, F)$, $F \in \mathfrak{F}$. Assume that T is densely defined.

(i) In particular, T has the SDP and hence T^* has the SDP, by Theorem 1.2 (d). Moreover, for every $F \in \mathfrak{F}$, (1.2) holds. Let $\{G_n\}_{n=0}^{\infty} \subset \mathfrak{G}^K$ be such that $G_0 = \emptyset$, $\bar{G}_n \subset G_{n+1}$ $(n \in \mathbf{N})$, and $\bigcup_{n=1}^{\infty} G_n = \mathbf{C}$. Putting $F_n = \mathbf{C} - G_n$, $n \in \mathbf{N}$, we have $\bigcap_{n=1}^{\infty} F_n = \emptyset$. The monotonicity of the spectral manifold implies $X(T, \bar{G}_n) \subset X(T, G_{n+1})$, $n \in \mathbf{N}$, and since T is decomposable we have

$$\Xi(T, G_n) = X(T, G_n), \quad n = 0, 1, \ldots$$

Assume that T is densely defined, apply (1.2) and obtain successively

$$X^*(T^*, \emptyset) = \bigcap_{n=1}^{\infty} X^*(T^*, F_n) = \left[\bigcup_{n=1}^{\infty} \Xi(T, G_n)\right]^{\perp} = \left[\bigcup_{n=1}^{\infty} X(T, G_n)\right]^{\perp}$$

$$= \left[\bigcup_{n=0}^{\infty} X(T, \bar{G}_n)\right]^{\perp} = \left[\bigcup_{n=1}^{\infty} \mathfrak{E}(\bar{G}_n)\right]^{\perp}.$$

Property (i) now follows from the sequel of the equivalent statements:

(a) T^* is decomposable;

(b) $X^*(T^*, \emptyset) = \{0\}$, by Theorem 1.2 (c);

(c) $\overline{[\bigcup_{n=1}^{\infty} \mathfrak{E}(\bar{G}_n)]^{\perp}} = \{0\}$;
(d) $\overline{\bigcup_{n=1}^{\infty} \mathfrak{E}(\bar{G}_n)} = X$.

(ii) Assume that T is densely defined and T^* is decomposable. Let $F \in V_\infty$ be closed. let H_1 be the interior of F and choose $H_2 \in \mathfrak{G}^K$ such that $H_1 \cup H_2 = \mathbf{C}_\infty$. Since T is decomposable, we have

$$X = X(T, \bar{H}_1) + X(T, \bar{H}_2). \tag{3.4}$$

Let $x \in X(T, F)$. Since, by (i), $\overline{M(\mathbf{C})} = X$, there is a sequence $\{x_n\} \subset D_T$ such that $\sigma_T(x_n)$ is compact for each n and $x_n \to x$. In view of (3.4), there is a representation

$$x - x_n = x_{n1} + x_{n2},\ x_{ni} \in X(T, \bar{H}_i),\ i = 1, 2,$$

and there is $M > 0$ (independent of x) such that $\|x_{n1}\| + \|x_{n2}\| \leq M\|x - x_n\|$. Consequently, $\|x - x_n\| \to 0$ and this implies that $\|x_{ni}\| \to 0$, $i = 1, 2$. For the vectors

$$y_n = x - x_{n1} = x_n + x_{n2},\ n \in \mathbf{N}$$

we have

$$\sigma_T(y_n) \subset F \cap [\sigma_T(x_n) \cup \bar{H}_2].$$

Since $\sigma_T(x_n) \cup \bar{H}_2$ is compact, $y_n \in M(F)$. Thus, $\|x - y_n\| = \|x_{n1}\| \to 0$ implies $\overline{M(F)} = \mathfrak{E}(F)$. ∎

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