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NONRECTIFIABLE LEVEL SETS FOR UNIVERSAL COVERING MAPS

BY

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Let Δ denote the open unit disk in the complex plane, and let K be a relatively closed subset of Δ such that $0 \notin K$ and $\Delta \backslash K$ is connected. Let ϕ denote the universal covering map of Δ onto $\Delta \backslash K$ with $\phi(0) = 0$, and let

$$\left\{ T_n(z) = e^{i\theta_n}(a_n - z)/(1 - \bar{a}_n z) \right\}_{n=1}^{\infty}$$

be the group of automorphisms of Δ under which ϕ is invariant. Finally, let γ denote an arbitrary compact rectifiable Jordan arc in $\Delta \backslash \{0\}$, and let $l(\cdot)$ denote linear Lebesgue measure.

Belna and Piranian [1] showed that the equivalence

$$l(\phi^{-1}(\gamma)) = \infty \quad \text{if and only if } \gamma \text{ meets } K$$

is valid when K is a singleton set; subsequently, Belna, Cohn, Piranian, and Stephenson [2] proved that it remains valid when K is of capacity 0. However, the characterization may fail when K has positive capacity; for example, if $K = [0, 1/2]$, then each "level set" $\phi^{-1}(\gamma)$ is rectifiable.

Here we shall present for the general case a condition that implies the nonrectifiability of $\phi^{-1}(\gamma)$.

THEOREM. *If γ contains an irregular boundary point of $\Delta \backslash K$, then $l(\phi^{-1}(\gamma)) = \infty$.*

We note that the converse is not necessarily true. Let

$$K = (-1, 0] \cup \{1/2, 1/3, \ldots\}.$$

According to our theorem, $\phi^{-1}([1/(n+1), 1/n])$ has infinite length for each integer $n \geq 2$. Choose numbers a_n and b_n that satisfy $1/(n+1) < a_n < b_n < 1/n$ and for which $\phi^{-1}([a_n, b_n])$ has length greater than 1. For each index n connect the segment $[a_n, b_n]$ to the segment $[a_{n+1}, b_{n+1}]$ by an arc in $\Delta \backslash K$ in such a way that the resulting arc τ is rectifiable. If $\gamma = \tau \cup \{0\}$, then $\gamma \cap K = \{0\}$ and 0 is a regular boundary point of $\Delta \backslash K$.

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Proof of the theorem. Because $0 \notin \gamma$, the non-euclidean version of Schwarz's lemma implies that for some $\lambda \in (0, 1)$ the set

$$\Lambda = \{z: |z - a_n|/|1 - \bar{a}_n z| \leq \lambda \quad \text{for some } n = 1, 2, \ldots\}$$

satisfies $\gamma \cap \phi(\Lambda) = \emptyset$. If G is the Green function for $\Delta \backslash K$ with singularity at 0, then

$$(G \circ \phi)(z) = -\sum_{n=1}^{\infty} \log (|z - a_n|/|1 - \bar{a}_n z|)$$

(see [4; p. 210]). Since there exists a positive number A such that

$$-\log x < A(1 - x^2) \quad \text{for } \lambda < x < 1$$

and since

$$1 - (|z - a_n|/|1 - \bar{a}_n z|)^2 = (1 - |z|^2)(1 - |a_n|^2)/|1 - \bar{a}_n z|^2,$$

we have

$$(1) \qquad (G \circ \phi)(z) < A(1 - |z|^2) \sum_{n=1}^{\infty} (1 - |a_n|^2)/|1 - \bar{a}_n z|^2 \quad (z \in \Delta \backslash \Lambda).$$

For each $p \in \Delta \backslash \Lambda$, the function $G \circ \phi$ is positive and harmonic in the disk $|z - p|/|1 - \bar{p}z| < \lambda$; thus it readily follows from Harnack's inequality [3; p. 29] that there exists a universal constant $\lambda_0 \in (0, \lambda)$ such that

$$(2) \quad (G \circ \phi)(z) \leq 2(G \circ \phi)(p) \quad \text{for} \quad p \in \Delta \backslash \Lambda \quad \text{and} \quad |z - p|/|1 - \bar{p}z| < \lambda_0.$$

Now suppose $w \in \gamma$ and w is an irregular boundary point of $\Delta \backslash K$. Then G has a fine limit at w that is greater than 2ε for some $\varepsilon > 0$ [3; combine Theorems 10.11, 10.15 and 10.16]. Set $Q = \{z: (G \circ \phi)(z) \leq \varepsilon\}$. Let $\gamma_1, \gamma_2, \ldots$ be the components of $\gamma \backslash K$, and for each index n let α_n be a Jordan arc in Δ that is mapped homeomorphically onto γ_n by ϕ. (Each α_n reaches $\partial\Delta$.) Let $\beta_1, \beta_2, \ldots$ be the components of the set $(\bigcup_n \alpha_n) \backslash Q$. Then

$$(3) \qquad l(\phi^{-1}(\gamma)) \geq \sum_j \sum_{n=1}^{\infty} l(T_n(\beta_j)).$$

Because of the identities

$$l(T_n(\beta_j)) = \int_{\beta_j} |T_n'(z)|\,|dz| = \int_{\beta_j} [(1 - |a_n|^2)/|1 - \bar{a}_n z|^2]\,|dz|,$$

it follows from (1) and (3) that

$$(4) \qquad l(\phi^{-1}(\gamma)) > (\varepsilon/A) \sum_j \int_{\beta_j} (1 - |z|^2)^{-1}\,|dz|.$$

Thus $l(\phi^{-1}(\gamma)) = \infty$ if some β_j reaches $\partial\Delta$.

It remains to consider the case when each β_j fails to reach $\partial\Delta$. In this case there must be infinitely many components β_j. If not, there would exist a nondegenerate subarc γ^* of γ with $w \in \gamma^*$ and $G \leq \varepsilon$ on $\gamma^*\backslash K$, and since $\gamma^*\backslash K$ is not thin at w this would contradict the fact that G has a fine limit greater than 2ε at w.

Each β_j must have at least one endpoint p_j in Q. If $\chi(Z, p_j)$ denotes the non-euclidean hyperbolic distance between p_j and a point $Z \in \beta_j$, then we have the identities

$$\chi(Z, p_j) = \tanh^{-1} (|Z - p_j|/|1 - \bar{p}_j Z|) = \inf_{\sigma} \int_{\sigma} (1 - |z|^2)^{-1} |dz|$$

where σ varies over all rectifiable Jordan arcs in Δ that join Z to p_j. Therefore

$$\int_{\beta_j} (1 - |z|^2)^{-1} |dz| \geq \chi(Z, p_j) \quad \text{for each} \quad Z \in \beta_j; \tag{5}$$

and because of (4) and (5), we can conclude the proof by showing that for infinitely many indices j there exists a point $Z_j \in \beta_j$ for which $\chi(Z_j, p_j) \geq \tanh^{-1} \lambda_0$.

To the contrary, suppose there exists a positive integer J such that

$$|Z - p_j|/|1 - \bar{p}_j Z| < \lambda_0 \quad \text{for each} \quad Z \in \beta_j \ (j > J).$$

By (2) we would have

$$(G \circ \phi)(Z) \leq 2(G \circ \phi)(p_j) = 2\varepsilon \quad \text{for each} \quad Z \in \beta_j \quad (j > J).$$

Consequently there would exist a nondegenerate subarc γ^* of γ with $w \in \gamma^*$ and $G \leq 2\varepsilon$ on $\gamma^*\backslash K$. But this would contradict the fact that G has a fine limit greater than 2ε at w, and the proof is complete.

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