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CURVATURE PROPERTIES OF HARMONIC FOLIATIONS[1]

BY

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Introduction

A foliation $\mathscr{F}$ on a manifold M is harmonic, if the canonical projection $\pi\colon TM \to Q$ of the tangent bundle to the normal bundle Q is a harmonic Q-valued 1-form [5], [6]. For this one needs a connection ∇ in Q, and a Riemannian metric g_M in M. In this paper we examine the interplay of the harmonicity property with the curvature of the Riemannian metric g_M and the curvature of the connection ∇. As a typical application we mention the result of Proposition 2.36: a harmonic foliation of codimension one on a compact flat Riemannian manifold is necessarily induced from a hyperplane foliation of Euclidean space. Other representative conclusions are Corollaries 2.15, 2.27 and Theorem 2.34.

A rich variety of harmonic foliations were discussed in [6]. See also [7]. If ∇ is defined by formulas (2.16) below in terms of the foliated bundle structure in Q, and the Riemannian connection ∇^M on M, then $\mathscr{F}$ is harmonic if and only if all leaves of $\mathscr{F}$ are minimal submanifolds of M [6, 3.3]. A foliation is said to be taut, if there exists a Riemannian metric g_M such that all leaves of $\mathscr{F}$ are minimal. The study of such foliations was begun by Gluck [2], Rummler [14] and Sullivan [16].

The main tools exploited in this paper are the Weitzenböck formulas (1.1) and (1.2). One novelty is that we prove (1.1) under slightly different hypotheses than usual: the formula is established for a 1-form ω on a Riemannian manifold M with values in a vector-bundle $E \to M$ carrying any connection ∇ (no metric on E is involved), but on the other hand we require ω to be d_∇-closed. After the applications to harmonic and to Riemannian foliations in Section 2, we prove (1.1) as a consequence of an equation of Codazzi type (3.8), which may be of independent interest. The last section is devoted to the formula (4.3) for the tension of the Gauss section of a foliation $\mathscr{F}$. It is a consequence of the Weitzenböck formula (1.1). Besides the derivative of the torsion $\tau(\mathscr{F})$ of $\mathscr{F}$ there is an additional term involving the normal curvature of $\mathscr{F}$ and the curvature of M. Restricted to a leaf $\mathscr{L}$ of a Riemannian foliation of Euclidean space, it reduces to the Ruh-Vilms type for the tension of the Gauss map of $\mathscr{L}$ [15].

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1. Weitzenböck formulas

We begin with the statement of the main technical tool of this paper. Let $E \to M$ be a smooth vectorbundle over a Riemannian manifold. A connection ∇ in E induces a connection on E-valued forms $\Omega^{\cdot}(M, E)$, and an exterior derivative

$$d_\nabla \colon \Omega^r(M, E) \to \Omega^{r+1}(M, E), \quad r \geq 0.$$

For a Riemannian metric g_M on M the star operator on forms $\Omega^{\cdot}(M)$ on M extends to E-valued forms.

$$* \colon \Omega^r(M, E) \to \Omega^{n-r}(M, E), \quad n = \dim M.$$

The codifferential $d_\nabla^* \colon \Omega^r(M, E) \to \Omega^{r-1}(M, E)$, $r > 0$ is given by

$$d_\nabla^* \omega = (-1)^{n(r+1)+1} * d_\nabla * \omega, \quad \omega \in \Omega^r(M, E),$$

and the Laplacian Δ by $d_\nabla d_\nabla^* + d_\nabla^* d_\nabla$. At a point $x \in M$ we fix an orthonormal basis $e_1, \dots, e_n$ of $T_x M$. Let $E_1, \dots, E_n$ be a local framing of TM in a neighborhood of x, coinciding with $e_1, \dots, e_n$ at x and satisfying $\nabla^M_{e_A} E_B = (\nabla^M_{E_A} E_B)_x = 0$. Here ∇^M denotes the Riemannian connection of (M, g_M). The Weitzenböck formula evaluates the Laplacian of $\omega \in \Omega^1(M, E)$ as follows:

$$(\Delta\omega)_x = -\sum_A \tilde{\nabla}_{e_A} \tilde{\nabla}_{E_A} \omega + S(\omega)_x, \tag{1.1}$$

where, for a vector field X,

$$S(\omega)_x(X) = \sum_A \{R_\nabla(e_A, X)\omega(e_A) - \omega(R_{\nabla^M}(e_A, X)e_A)\}.$$

Only the value $X_x \in T_x M$ enters into this formula. Here R_∇ denotes the curvature of the connection ∇ in E, and R_{∇^M} the curvature of the Riemannian connection ∇^M in TM.

This formula is usually proved (see [1], [10], [11]) under the additional assumption that $E \to M$ is a Riemannian vectorbundle, i.e., equipped with a metric g_E compatible with the connection ∇ in the sense that $\nabla g_E = 0$. In Section 3 we give a proof of (1.1) for the special case of a closed 1-form ω. but *without the assumption of any metric* on E. This is a consequence of the Codazzi type equation (3.8).

If g_E is a metric on E such that $\nabla g_E = 0$, we can form the scalar product $g_{\Omega^1}(\Delta\omega, \omega)$. Formula (1.1) yields then the following well-known "scalar" Weitzenböck formula

$$-\tfrac{1}{2}\Delta^M |\omega|^2 = |\tilde{\nabla}\omega|^2 - g_{\Omega^1}(\Delta\omega, \omega) + g_{\Omega^1}(S(\omega), \omega). \tag{1.2}$$

The Laplacian Δ^M on the left hand side is the ordinary Laplacian d^*d on M (note the sign convention adopted) applied to the function $|\omega|^2 = g_{\Omega^1}(\omega, \omega)$ given by

$$|\omega|_x^2 = \sum_{A=1}^{n} g_E(\omega(e_A), \omega(e_A)).$$

The first term on the right hand side is given by

$$|\nabla\omega|_x^2 = \sum_{A=1}^{n} g_{\Omega^1}(\tilde{\nabla}_{e_A}\omega, \tilde{\nabla}_{e_A}\omega).$$

For M closed and oriented, the global scalar product $\langle \omega, \omega' \rangle$ for forms ω, $\omega' \in \Omega^r(M, E)$ is defined by

$$\langle \omega, \omega' \rangle = \int_M g_E(\omega \wedge *\omega'), \quad \|\omega\|^2 = \langle \omega, \omega \rangle.$$

d_∇^* is then the formal adjoint of d_∇ with respect to $\langle \ , \ \rangle$ and, by Green's theorem, formula (1.2) yields

(1.3) $$\langle \Delta\omega, \omega \rangle = \|\tilde{\nabla}\omega\|^2 + \langle S(\omega), \omega \rangle, \quad \omega \in \Omega^1(M, E).$$

2. Harmonicity of foliations and curvature

The context for the applications in this section is as follows. Let $L \subset TM$ be an integrable subbundle defining a foliation $\mathscr{F}$, and $Q = TM/L$ the normal bundle. If $\mathscr{F}$ is Riemannian, i.e., if there exists a holonomy invariant metric g_Q on Q, there is a unique metric and torsion-free connection ∇ in Q [6, 1.11].

A Riemannian metric g_M on M defines a splitting σ of the exact sequence

(2.1) $$0 \to L \to TM \underset{\pi}{\overset{\sigma}{\leftrightarrows}} Q \to 0$$

with σQ the orthogonal complement of L. The induced connections $\tilde{\nabla}$ on Q-valued forms involve ∇ and the Riemannian connection ∇^M of g_M.

In the presence of metrics g_Q and g_M we refine the choice of local framings as follows. We begin with an orthonormal basis of T_xM with $e_i \in L_x$ for $i = 1, \ldots, p$ and $e_\alpha \in \sigma Q_x$ for $\alpha = p + 1, \ldots, n$. Then $E_1, \ldots, E_n$ is a local framing of TM in a neighborhood of x with $e_A = (E_A)_x$ and $\nabla_{e_A}^M E_B = 0$ (A, $B = 1, \ldots, n$). But we neither claim nor require that $(E_i)_y \in L_y$ for $1 \leq i \leq p$ or $(E_\alpha)_y \in \sigma Q_y$ for $p + 1 \leq \alpha \leq n$ at points $y \neq x$. We do have $(\pi E_i)_x = \pi e_i = 0$ for $1 \leq i \leq p$. In the case of a bundle-like metric g_M the vectors $(\pi E_\alpha)_x = \pi e_\alpha$ for $p + 1 \leq \alpha \leq n$ form, in addition, an orthonormal basis of Q_x.

Consider the canonical projection $\pi\colon TM \to Q$ as Q-valued 1-form. Then

(2.2) $$d_\nabla \pi = 0,$$

since $d_\nabla \pi$ equals the torsion T_∇ given by

$$T_\nabla(X, Y) = \nabla_X \pi(Y) - \nabla_Y \pi(X) - \pi[X, Y] \tag{2.3}$$

which is zero.

The Ricci operator $\rho_{\nabla^M}: TM \to TM$ of ∇^M is given for $X \in T_x M$ by

$$\rho_{\nabla^M}(X) = \sum_A R_{\nabla^M}(X, e_A)e_A. \tag{2.4}$$

With these notations one obtains the following statement from (1.2).

2.5 PROPOSITION. *Let $\mathscr{F}$ be a Riemannian foliation of codimension q on (M^n, g_M) with holonomy-invariant metric g_Q on Q (g_M is not assumed to be bundle-like). Then for $\pi \in \Omega^1(M, Q)$ one has the identity*

$$-\tfrac{1}{2}\Delta^M |\pi|^2 = |\tilde{\nabla}\pi|^2 - g_{\Omega^1}(\Delta\pi, \pi) + g_{\Omega^1}(S(\pi), \pi), \tag{2.6}$$

where

$$|\pi|_x^2 = \sum_A g_Q(\pi(e_A), \pi(e_A)) = \sum_\alpha g_Q(\pi(e_\alpha), \pi(e_\alpha)),$$

$$|\tilde{\nabla}\pi|_x^2 = \sum_{A,B} g_Q((\tilde{\nabla}\pi)(e_A, e_B), (\tilde{\nabla}\pi)(e_A, e_B))$$

and

$$g_{\Omega^1}(S(\pi), \pi)_x = \sum_{\alpha \neq \beta} g_Q(R_\nabla(e_\alpha, e_\beta)\pi(e_\alpha), \pi(e_\beta)) + \sum_\alpha g_Q(\pi(\rho_{\nabla^M}(e_\alpha)), \pi(e_\alpha)).$$

For compact oriented M one obtains the following identity for the global scalar products:

$$\langle \Delta\pi, \pi \rangle = \|\tilde{\nabla}\pi\|^2 + \langle S(\pi), \pi \rangle. \tag{2.7}$$

Note that $d_\nabla \pi = 0$ by (2.2), so that $\Delta\pi = d_\nabla d_\nabla^* \pi$, and therefore $\langle \Delta\pi, \pi \rangle = \|d_\nabla^* \pi\|^2$.

To analyze the sign of the term $g_{\Omega^1}(S(\pi), \pi)$, it is convenient to introduce the self-adjoint operator $B_\pi: TM \to TM$ by

$$g_M(B_\pi X, Y) = g_Q(\pi(X), \pi(Y)) \quad \text{for } X, Y \in \Gamma TM. \tag{2.8}$$

Clearly $\ker B_\pi = L$, $\operatorname{im} B_\pi = \sigma Q \cong L^\perp$. We further refine the choice of local framings by requiring that the orthogonal basis $e_1, \ldots, e_n$ of $T_x M$ also diagonalize B_π, i.e.

$$B_\pi(e_i) = 0 \ (i = 1, \ldots, p); \quad B_\pi(e_\alpha) = \lambda_\alpha e_\alpha \ (\alpha = p+1, \ldots, n) \tag{2.9}$$

where $\lambda_\alpha > 0$, since g_Q is positive definite. By (2.8) we then clearly have

$$g_Q(\pi(e_\alpha), \pi(e_\beta)) = \lambda_\alpha \delta_{\alpha\beta} \tag{2.10}$$

Now consider the normal sectional curvature $K_\nabla(e_\alpha, e_\beta)$ in direction of the normal 2-plane spanned by e_α, e_β defined by

$$K_\nabla(e_\alpha, e_\beta) = \frac{1}{\lambda_\alpha \lambda_\beta} g_Q(R_\nabla(\pi(e_\alpha), \pi(e_\beta))\pi(e_\beta), \pi(e_\alpha)). \tag{2.11}$$

Note that since ∇ is a basic connection, $i(X)R_\nabla = 0$ for $X \in \Gamma L$ [6, 1.13], hence $R_\nabla(\pi(e_\alpha), -) = R_\nabla(e_\alpha, -)$.

Further, by (2.8) and (2.9),

$$\begin{aligned} g_Q(\pi(\rho_{\nabla M}(e_\alpha)), \pi(e_\alpha)) &= g_M((B_\pi \circ \rho_{\nabla M})e_\alpha, e_\alpha) \\ &= g_M(\rho_{\nabla M}(e_\alpha), B_\pi e_\alpha) \\ &= \lambda_\alpha\, g_M(\rho_{\nabla M}(e_\alpha), e_\alpha). \end{aligned} \tag{2.12}$$

Using (2.11) and (2.12) we then obtain

$$g_{\Omega^1}(S(\pi), \pi)_x = -\sum_{\alpha \neq \beta} \lambda_\alpha \lambda_\beta K_\nabla(e_\alpha, e_B) + \sum_\alpha \lambda_\alpha\, g_M(\rho_{\nabla M}(e_\alpha), e_\alpha). \tag{2.13}$$

Thus non-negative Ricci curvature on M and non-positive normal sectional curvature K_∇ imply $g_{\Omega^1}(S(\pi), \pi) \geq 0$ and, a fortiori, $\langle S(\pi), \pi \rangle \geq 0$. We note that in the case of a bundle-like metric g_M these curvature assumptions run contrary to the spirit of the Gray–O'Neill formula (see (2.20) below), unless both curvatures vanish and $\sigma Q \subset TM$ is an involutive subbundle. We discuss the bundle-like case after finishing the present discussion. From (2.5) and (2.13) we obtain the following result.

2.14 PROPOSITION. *Let $\mathcal{F}$ be a Riemannian foliation of codimension q on a closed oriented manifold M. Let g_M be a metric on M with non-negative Ricci curvature, and assume the normal sectional curvature K_∇ of g_Q to be non-positive. Then*

$$d_\nabla^* \pi = 0 \Leftrightarrow \tilde{\nabla}\pi = 0 \quad \textit{and} \quad g_{\Omega^1}(S(\pi), \pi) = 0.$$

Now for $X, Y \in \Gamma TM$ we have

$$(\tilde{\nabla}\pi)(X, Y) \equiv (\tilde{\nabla}_X \pi)(Y) = \nabla_X \pi(Y) - \pi(\nabla_X^M Y).$$

It follows that $(\tilde{\nabla}\pi)(X, Y) = 0$ for $X, Y \in \Gamma L$ iff $\nabla_X^M Y \in \Gamma L$ for $X, Y \in \Gamma L$. This condition means that each leaf $\mathcal{L}$ is a totally geodesic submanifold of M [8, vol. II, p. 56, 57].

2.15 COROLLARY. *Let $\mathcal{F}$ be a foliation satisfying the conditions in* (2.14).

(i) *If π is a harmonic form, then each leaf $\mathcal{L}$ is a totally geodesic submanifold of M.*

(ii) *If $g_{\Omega^1}(S(\pi), \pi)_{x_0} \neq 0$ for at least some $x_0 \in M$, then π is not a harmonic form.*

One way to satisfy the condition $K_\nabla \leq 0$ is to assume $q = 1$, in which case $K_\nabla = 0$ (for lack of normal 2-planes). Corollary 2.15 holds then under the assumption that the Ricci curvature of g_M is non-negative. If the Ricci operator $(\rho_{\nabla^M})_{x_0}$ is positive for at least some $x_0 \in M$, we are in case (ii). But for $q = 1$ we obtain sharper results in Theorem 2.34, where $\mathscr{F}$ is not required to be a Riemannian foliation.

Next we apply the Weitzenböck formula to harmonic foliations [5], [6]. We have two types of applications: (a) results assuming g_M to be bundle-like; and (b) results in the codimension one case, where less assumptions on the metrics g_M and g_Q are needed.

We discuss first the bundle-like case, i.e., g_Q can be assumed to be induced by g_M. The projection $\pi\colon TM \to Q$ is then an orthogonal projection. The particular connection ∇ in Q, given by [6, 1.3],

$$(2.16)\qquad \nabla_X s = \begin{cases} \pi[X, Y_s] & \text{for } X \in \Gamma L \\ \pi(\nabla^M_X Y_s) & \text{for } X \in \Gamma\sigma Q, \end{cases} \qquad s \in \Gamma Q,\ Y_s = \sigma(s) \in \Gamma\sigma Q,$$

is then the unique metric and torsion-free connection in Q [6, 1.11]. The harmonicity of π, i.e., the condition $d^*_\nabla \pi = 0$ (since we already have $d_\nabla \pi = 0$), is then equivalent to the property that all leaves of $\mathscr{F}$ are minimal submanifolds of (M, g_M) [6, 2.28]. The Q-valued symmetric bilinear form $\alpha = -\tilde{\nabla}\pi$ restricted to any leaf $\mathscr{L} \subset M$ of $\mathscr{F}$ is then the second fundamental form of the Riemannian submanifold $\mathscr{L} \subset M$. By [6, 2.26], the tension $\tau = \operatorname{Tr} \alpha$ of $\mathscr{F}$ is evaluated at $x \in M$ by

$$(2.17)\qquad \tau_x = \operatorname{Tr} \alpha = \sum_A \alpha(e_A, e_A) = \sum_i \alpha(e_i, e_i) \in Q_x.$$

It is immediate that

$$(2.18)\qquad \tau = d^*_\nabla \pi,$$

and $\mathscr{F}$ is harmonic iff $\tau = 0$ [6, 2.28].

The operator $B_\pi\colon TM \to TM$ defined by (2.5) is the map $\sigma \circ \pi$ and the non-zero eigenvalues λ_α equal 1. Then formula (2.13) becomes

$$(2.19)\qquad g_{\Omega^1}(S(\pi), \pi)_x = -\sum_{\alpha \neq \beta} K_\nabla(e_\alpha, e_B) + \sum_{\alpha \neq A} K_{\nabla^M}(e_\alpha, e_A)$$

where $K_{\nabla^M}(e_\alpha, e_A)$ denotes the sectional curvature of the metric g_M in direction of the 2-plane spanned by e_α, e_A ($\alpha = p+1, \ldots, n$; $A = 1, \ldots, n$). The sectional curvatures $K_{\nabla^M}(e_\alpha, e_\beta)$ and $K_\nabla(e_\alpha, e_\beta)$ are related as the sectional curvatures in the total space and base space of a Riemannian submersion. Therefore by the formula of Gray [3] and O'Neill [12] we have

$$(2.20)\qquad K_\nabla(e_\alpha, e_\beta) - K_{\nabla^M}(e_\alpha, e_\beta) = \frac{3}{4}\left|\pi^\perp[E_\alpha, E_\beta]\right|^2_x$$

where $\pi^\perp\colon TM \to L$ denotes the orthogonal projection $\pi^\perp = \mathrm{id} - \pi$.

At this point it is convenient to refer to the self-adjoint map

$A(v)\colon TM \to TM$ uniquely associated to α by the formula

$$g_M(A(v)X, Y) = g_Q(\alpha(X, Y), v) \tag{2.21}$$

for $v \in \Gamma Q$ and $X, Y \in \Gamma TM$ [6, 2.8]. In terms of the matrix representation [6, 2.9]

$$A = \begin{pmatrix} A_1 & A_2 \\ A_2^* & 0 \end{pmatrix}, \tag{2.22}$$

formula (2.20) then reads

$$K_\nabla(e_\alpha, e_\beta) - K_{\nabla^M}(e_\alpha, e_\beta) = 3\,|A_2(e_\alpha)e_\beta|^2, \tag{2.23}$$

and (2.19) takes the form

$$g_{\Omega^1}(S(\pi), \pi)_x = -3 \sum_{\alpha \neq \beta} |A_2(e_\alpha)e_\beta|^2 + \sum_{\alpha, i} K_{\nabla^M}(e_\alpha, e_i) \tag{2.24}$$

where i ranges over $1, \dots, p$. Note that the first sum on the right hand side is just $|A_2|^2$ for $A_2\colon Q \to \operatorname{Hom}(\sigma Q, L)$. Clearly, from (2.22),

$$|\alpha|^2 = |A_1|^2 + 2\,|A_2|^2. \tag{2.25}$$

Since $|\pi|^2 = q$ is constant, $\Delta^M |\pi|^2 = 0$. Thus from (2.6) (2.24) (2.25) we obtain the following result.

2.26 THEOREM. *Let $\mathcal{F}$ be a Riemannian foliation of codimension q on M with bundle-like metric g_M. Then*

$$|A_1|^2 = |A_2|^2 + g_{\Omega^1}(\Delta\pi, \pi) - \sum_{\alpha, i} K_{\nabla^M}(e_\alpha, e_i).$$

$A_1 = 0$ iff all leaves of $\mathcal{F}$ are totally geodesic [6, 2.6]; and $A_2 = 0$ iff $\sigma Q \subset TM$ is involutive [6, 2.15]. Then $\mathcal{F}$ has a totally geodesic complementary foliation by [6, 2.15]. We obtain the following conclusion.

2.27 COROLLARY. *Let $\mathcal{F}$ be a Riemannian foliation of codimension q on M with bundle-like metric g_M. Assume $\sigma Q \subset TM$ to be involutive. If $K_{\nabla^M} \geq 0$, $\mathcal{F}$ harmonic implies $\mathcal{F}$ totally geodesic.*

Note that no compactness is required for the proof. Therefore we obtain the following application.

2.28 COROLLARY. *Let (M, g_M) be a Riemannian manifold of the form $M \cong \Gamma\backslash\mathbf{R}^n$, Γ a discrete torsionfree subgroup of the Euclidean group. Let $\mathcal{F}$ be a codimension one Riemannian foliation such that g_M is bundle-like. If $\mathcal{F}$ is harmonic, then $\mathcal{F}$ is induced from a Γ-invariant hyperplane foliation on $\mathbf{R}^n$.*

Proof. $\mathcal{F}$ is induced from a Γ-invariant foliation $\tilde{\mathcal{F}}$ on $\mathbf{R}^n$. $\tilde{\mathcal{F}}$ is harmonic, hence totally geodesic, and hence a hyperplane foliation of $\mathbf{R}^n$. ∎

This result should be compared with the closely related Proposition 2.36.

Finally we discuss foliations of codimension one in more detail. We assume that $\mathscr{F}$ is transversely oriented by a unit normal section $Z \in \Gamma\sigma Q$. Then $\mathscr{F}$ is defined by the 1-form $\omega(X) = g_M(X, Z)$ for $X \in \Gamma TM$ and $\pi(X) = \omega(X) \cdot Z$. Then $\mathscr{F}$ is harmonic iff $d^*\omega = 0$ [6, 3.9] and Riemannian iff $d\omega = 0$ [6, 3.14]. For the real-valued 1-form ω the identity (1.2) holds without any restriction on g_M. The term $g_{\Omega^1}(S(\omega), \omega)$ involves only the Ricci operator and equals $\mathrm{Ric}^M(Z, Z)$. Since ω is of unit length, we have

$$g_{\Omega^1}(\Delta\omega, \omega) = |\nabla\omega|^2 + \mathrm{Ric}^M(Z, Z). \tag{2.29}$$

For compact oriented M, by integration with respect to the Riemannian volume η_M, we obtain

$$\|d\omega\|^2 + \|d^*\omega\|^2 = \|\nabla\omega\|^2 + \int_M \mathrm{Ric}^M(Z, Z)\eta_M. \tag{2.30}$$

We compare this with the integral formula of Yano (see [9, p. 154])

$$\|d^*\omega\|^2 = \int_M \mathrm{Tr}((\nabla^M Z)^2) \cdot \eta_M + \int_M \mathrm{Ric}^M(Z, Z) \cdot \eta_M, \tag{2.31}$$

where $\nabla^M Z \colon TM \to TM$ is given by $(\nabla^M Z)(X) = \nabla_X^M Z$ for $X \in \Gamma TM$. Thus

$$\|\nabla\omega\|^2 = \|d\omega\|^2 + \int_M \mathrm{Tr}((\nabla^M Z)^2) \cdot \eta_M. \tag{2.32}$$

Observe that in fact $\nabla^M Z \colon TM \to L$, since $g_M(\nabla_X^M Z, Z) = \frac{1}{2} X g_M(Z, Z) = 0$. Thus the restriction of $-\nabla^M Z$ to L is the Weingarten map $W(Z) \colon L \to L$ [6, 2.10], which is self-adjoint. It follows that $\mathrm{Tr}(W(Z)^2) \geq 0$. But also

$$\mathrm{Tr}((\nabla^M Z)^2) = \mathrm{Tr}(W(Z)^2) \geq 0. \tag{2.33}$$

The equality of the traces follows from $-\nabla^M Z/L = W(Z)$ and

$$g_M((\nabla^M Z)^2 Z, Z) = g_M(\nabla^M_{\nabla_Z^M Z} Z, Z) = \tfrac{1}{2}\nabla_Z^M Z g_M(Z, Z) = 0.$$

The following statement sharpens the results of (2.15) and (2.27) in the case $q = 1$. Note that $\mathscr{F}$ is not assumed to be Riemannian. *Note added in proof*: The following result is also contained in the paper by G. Oshikiri, *A remark on minimal foliations*, Tôhoku Math. J., vol. 33 (1981), pp. 133–137.

2.34 THEOREM. *Let $\mathscr{F}$ be a transversally orientable foliation of codimension one on a compact oriented Riemannian manifold M with non-negative Ricci curvature.*

(i) *If the Ricci operator is positive for at least one $x_0 \in M$, the foliation is not harmonic.*
(ii) *If $\mathscr{F}$ is harmonic, then $\mathscr{F}$ is totally geodesic.*

Proof. (i) Under the Ricci curvature assumption made,

$$\int_M \mathrm{Ric}^M(Z, Z)\eta_M > 0.$$

Since $\mathrm{Tr}((\nabla^M Z)^2) \geq 0$, it follows from (2.31) that $d^*\omega \neq 0$. Thus $\mathscr{F}$ is not harmonic [6, 3.9].

(ii) If $d^*\omega = 0$, (2.31) implies $\mathrm{Tr}((\nabla^M Z)^2) = 0$. By (2.33) it follows that $\mathrm{Tr}(W(Z)^2) = 0$. Since $W(Z)$ is self-adjoint, all eigenvalues of $W(Z)$ are zero, hence $W(Z) = 0$. The vanishing of the Weingarten map of the foliation implies that $\mathscr{F}$ is totally geodesic. ▮

In this context it is worthwhile to note the following fact.

2.35 Proposition. *On a compact oriented Riemannian manifold M with non-negative Ricci curvature and positive Ricci operator at some $x_0 \in M$, or with strictly negative sectional curvature, there is no transversally orientable Riemannian foliation of codimension one.*

Proof. Under the stated assumptions on the Ricci curvature, the harmonic 1-forms vanish by Bochner's method (for example, see Wu [18]). Thus $H^1_{DR}(M) = 0$, so that every closed 1-form ω is of the form df for some function $f\colon M \to \mathbf{R}$. But a transversally orientable Riemannian foliation of codimension one is given by a nowhere zero closed 1-form [6, 3.14], which does not exist on M. Similarly if the sectional curvature is strictly negative, by a result of Tsagas [17] every closed 1-form has zeroes. ▮

We return finally to the situation discussed in Corollary 2.28. As a consequence of Theorem 2.34, we obtain the following related result.

2.36 Proposition. *Let $\mathscr{F}$ be a transversally oriented harmonic foliation of codimension one on a compact oriented flat Riemannian manifold M. Then $\mathscr{F}$ is induced from a hyperplane foliation on the universal covering $M^n \cong \mathbf{R}^n$.*

Proof. By (2.34), $\mathscr{F}$ is totally geodesic. Its lift $\tilde{\mathscr{F}}$ to $\mathbf{R}^n$ is therefore totally geodesic, hence a hyperplane foliation. ▮

3. Codazzi equation

We return to the context of the first section. For $\omega \in \Omega^1(M, E)$ the form $\tilde{\nabla}\omega \in \Omega^1(M, T^*M \otimes E)$ is defined by

$$(\tilde{\nabla}_X \omega)(Y) = \nabla_X \omega(Y) - \omega(\nabla^M_X Y) \quad \text{for } X, Y \in \Gamma TM. \tag{3.1}$$

3.2 LEMMA. $(d_\nabla \omega)(X, Y) = (\tilde{\nabla}_X \omega)(Y) - (\tilde{\nabla}_Y \omega)(X)$.

Proof.

$$\begin{aligned}(\tilde{\nabla}_X \omega)(Y) - (\tilde{\nabla}_Y \omega)(X) &= (\nabla_X \omega(Y) - \omega(\nabla^M_X Y)) - (\nabla_Y \omega(X) - \omega(\nabla^M_Y X)) \\ &= \nabla_X \omega(Y) - \nabla_Y \omega(X) - \omega([X, Y]) - \omega(T_{\nabla M}(X, Y)) \\ &= (d_\nabla \omega)(X, Y) - \omega(T_{\nabla M}(X, Y)).\end{aligned}$$

Since the torsion $T_{\nabla M} = 0$, the result follows. ∎

Let $\alpha_\omega = -\tilde{\nabla}\omega$. By (3.2) it follows that $d_\nabla \omega = 0$ implies

$$\alpha_\omega(X, Y) = \alpha_\omega(Y, X) \quad \text{for } X, Y \in \Gamma TM. \tag{3.3}$$

Next consider $\tilde{\nabla}\alpha_\omega = -\tilde{\nabla}^2\omega \in \Omega^1(M, T^*M \otimes T^*M \otimes E)$ given by

$$(\tilde{\nabla}_X \alpha_\omega)(Y, Z) = \nabla_X \alpha_\omega(Y, Z) - \alpha_\omega(Y, \nabla^M_X Y, Z) - \alpha_\omega(Y, \nabla^M_X Z). \tag{3.4}$$

3.5 LEMMA. $(\tilde{\nabla}_X \alpha_\omega)(Y, Z) = (\tilde{\nabla}_X \alpha_\omega)(Z, Y)$.

Proof. This follows from the symmetry (3.3). ∎

Note that (3.2) applied to $\alpha_\omega \in \Omega^1(M, T^*M \otimes E)$ yields

$$(d_\nabla \alpha_\omega)(X, Y; Z) = (\tilde{\nabla}_X \alpha_\omega)(Y, Z) - (\tilde{\nabla}_Y \alpha_\omega)(X, Z). \tag{3.6}$$

The expression need not vanish. In fact we have the following result.

3.7 THEOREM. *Let $E \to M$ be a smooth vectorbundle with connection ∇ over a Riemannian manifold M. For $\omega \in \Omega^1(M, E)$ satisfying $d_\nabla \omega = 0$, and $\alpha_\omega = -\tilde{\nabla}\omega$ we have the Codazzi type equation*

$$(\tilde{\nabla}_X \alpha_\omega)(Y, Z) - (\tilde{\nabla}_Y \alpha_\omega)(X, Z) = -R_\nabla(X, Y)\omega(Z) + \omega(R_{\nabla M}(X, Y)Z) \tag{3.8}$$

for $X, Y, Z \in \Gamma TM$.

Proof. By (3.4),

$$\begin{aligned}(\tilde{\nabla}_X \alpha_\omega)(Y, Z) = &-\nabla_X(\nabla_Y \omega(Z) - \omega(\nabla^M_Y Z)) \\ &+ \nabla_{\nabla^M_X Y}\, \omega(Z) - \omega(\nabla^M_{\nabla^M_X Y}\, \omega Z) + \nabla_Y \omega(\nabla^M_X Z) - \omega(\nabla^M_Y \nabla^M_X Z).\end{aligned}$$

The terms $\nabla_X \omega(\nabla^M_Y Z) + \nabla_Y \omega(\nabla^M_X Z)$ will also appear in the corresponding expression for $(\tilde{\nabla}_Y \alpha_\omega)(X, Z)$. Therefore

$$\begin{aligned}(\tilde{\nabla}_X \alpha_\omega)(Y, Z) - (\tilde{\nabla}_Y \alpha_\omega)(X, Z) = &-\{R_\nabla(X, Y)\omega(Z) + \nabla_{[X,Y]}\omega(Z)\} \\ &+ \{\nabla_{T_{\nabla M}(X,Y)}\omega(Z) + \nabla_{[X,Y]}\omega(Z)\} \\ &- \omega\{\nabla_{T_{\nabla M}(X,Y)} Z + \nabla_{[X,Y]} Z\} \\ &- \omega\{R_{\nabla M}(Y, X)Z + \nabla_{[Y,X]} Z\}.\end{aligned}$$

The vanishing of the torsion $T_{\nabla M}$ implies the desired formula. ∎

3.9 *Remark.* To explain in which sense equation (3.8) is of Codazzi type, let Q be the normal bundle of a foliation $\mathscr{F}$ on M, and $\pi\colon TM \to Q$ be the canonical projection. A Riemannian metric g_M gives rise to a connection ∇ in Q by (2.16), and the second fundamental form $\alpha = -\tilde{\nabla}\pi$. The restriction of α to any leaf $\mathscr{L}$ is the second fundamental form of the Riemannian submanifold $\mathscr{L} \subset M$. If X, Y and Z are tangent vectorfields to $\mathscr{L}$, then (3.8) is of Lodazzi type for $\mathscr{L} \subset M$, expressing the normal component of $R_{\nabla^M}(X, Y)Z$ by

$$(\tilde{\nabla}_X \alpha)(Y, Z) - (\tilde{\nabla}_Y \alpha)(X, Z)$$

Now we turn to the proof of the Weitzenböck formula (1.1). First we consider the trace $\tau(\omega) = \operatorname{Tr} \alpha_\omega \in \Gamma Q$ of the symmetric bilinear Q-valued form α_ω. In terms of an orthonormal basis e_A $(A = 1, \ldots, n)$ at $x \in M$ it is evaluated by

$$\tau(\omega)_x = \sum_A \alpha_\omega(e_A, e_A) \in Q_x. \tag{3.10}$$

Let $E_1, \ldots, E_n$ be a local framing of TM in a neighborhood of x, coinciding with $e_1, \ldots, e_n$ at x, and satisfying $\nabla^M_{e_A} E_B = (\nabla^M_{E_A} E_B)_x = 0$.

3.11 LEMMA. $(\nabla_X \tau(\omega))_x = \sum_A (\tilde{\nabla}_X \alpha_\omega)(e_A, e_A)$ *for* $X \in \Gamma TM$.

Proof. From (3.3) and (3.4) we get

$$(\tilde{\nabla}_X \alpha_\omega)(Y, Y) = \nabla_X \alpha_\omega(Y, Y) - 2\alpha_\omega(\nabla^M_X Y, Y).$$

Thus

$$\sum_A (\tilde{\nabla}_X \alpha_\omega)(e_A, e_A) = \nabla_X \sum_A \alpha_\omega(e_A, e_A) - 2 \sum_A \alpha_\omega((\nabla^M_X E_A)_x, e_A).$$

But $(\nabla^M_X E_A)_x = 0$, which yields the result. ∎

This formula can equivalently be expressed by

$$\nabla_X \tau(\omega) \equiv \nabla_X \operatorname{Tr} \alpha_\omega = \operatorname{Tr} (\tilde{\nabla}_X \alpha_\omega) \tag{3.12}$$

and is a consequence of the symmetry of α_ω.

Next we observe that

$$\tau(\omega) = d^*_\nabla \omega. \tag{3.13}$$

This is immediate from the evaluation formula

$$(d^*_\nabla \omega)_x = -\sum_A (\tilde{\nabla}_{e_A} \omega)(e_A) = \operatorname{Tr} \alpha_\omega.$$

Since $d_\nabla \omega = 0$ by assumption, we obtain for the Laplacian $\Delta\omega$ the expression

$$i(X)\, \Delta\omega = i(X) d_\nabla d^*_\nabla \omega = i(X) d_\nabla \tau(\omega) = \nabla_X \tau(\omega). \tag{3.14}$$

The Codazzi type equation (3.8) now yields

$$\nabla_X \tau(\omega) - \sum_A (\tilde{\nabla}_{e_A} \alpha_\omega)(X, e_A) = \sum_A \{R_\nabla(e_A, X)\omega(e_A) - \omega(R_{\nabla^M}(e_A, X)e_A\}. \tag{3.15}$$

Further

$$(3.16)\qquad \begin{aligned}(\tilde{\nabla}_{e_A}\alpha_\omega)(X, e_A) &= \nabla_{e_A}\alpha_\omega(X, E_A) - \alpha_\omega(\nabla^M_{e_A}X, e_A) - \alpha_\omega(X, \nabla^M_{e_A}E_A)\\ &= -\nabla_{e_A}((\tilde{\nabla}_{E_A}\omega)(X)) + (\tilde{\nabla}_{e_A}\omega)(\nabla^M_{e_A}X)\\ &= -(\tilde{\nabla}_{e_A}\tilde{\nabla}_{E_A}\omega)(X).\end{aligned}$$

Substituting (3.14) and (3.16) in (3.15) yields (1.1).

To derive the scalar Weitzenböck formula (1.2), we need a metric g_E on $E \to M$ compatible with the connection ∇, i.e., satisfying

$$(3.17)\qquad (\nabla_X g_E)(s, t) = Xg_E(s, t) - g_E(\nabla_X s, t) - g_E(s, \nabla_X t) = 0$$

for $X \in \Gamma TM$ and $s, t \in \Gamma Q$. Then

$$(3.18)\qquad \begin{aligned}-(\Delta^M |\omega|^2)_x = -(d^*d|\omega|^2)_x &= \sum_A (\nabla^M_{e_A} d|\omega|^2)(e_A)\\ &= \sum_A (\nabla^M_{e_A}(d|\omega|^2)(E_A) - (d|\omega|^2)(\nabla^M_{e_A}E_A))\\ &= \sum_A (E_A E_A g_{\Omega^1}(\omega, \omega))_x.\end{aligned}$$

Using (3.17) we have

$$E_A g_{\Omega^1}(\omega, \omega) = 2g_{\Omega^1}(\tilde{\nabla}_{E_A}\omega, \omega),$$

$$E_A E_A g_{\Omega^1}(\omega, \omega) = 2(g_{\Omega^1}(\tilde{\nabla}_{E_A}\tilde{\nabla}_{E_A}\omega, \omega) + g_{\Omega^1}(\tilde{\nabla}_{E_A}\omega, \tilde{\nabla}_{E_A}\omega)).$$

This identity and (1.1) substituted in (3.18) yield then the well-known formula (1.2).

4. Ruh-Vilms formula

Let $\mathscr{F}$ be a foliation of codimension q on a Riemannian manifold M. Let

$$G^q(M) \xrightarrow{p} M$$

be the Grassmannian bundle of codimension q subspaces in TM. The subbundle $L \subset TM$ is characterized by the Gauss section $\gamma\colon M \to G^q(M)$. Its derivative γ_* is a 1-form on M with values in the tangent bundle of $G^q(M)$. Since γ is a section, the horizontal component $\gamma^H_* \in \Omega^1(M, p^*TM)$ is given at $x \in M$ by

$$\gamma^H_{*x} = \mathrm{id}\colon T_x M \to (p^*TM)_{\gamma(x)} \cong T_x M.$$

To describe the vertical component γ^V_*, we first observe that the tangent bundle along the fibres of $G^q(M) \to M$ is given at $\gamma(x) = L_x$ by $\mathrm{Hom}\,(L_x, Q_x)$. Then

$$(4.1)\qquad \gamma^V_* = \alpha,$$

i.e.,

$$((\gamma_*^V)(X))(Y) = \alpha_x(X, Y) \quad \text{for } X \in T_x M,\ Y \in L_x,$$

where $\alpha = -\nabla\pi$ is the second fundamental form of $\mathscr{F}$ (see Section 2).

According to [1], the tension $\tau(\gamma)$ is given by

$$\tau(\gamma) = -d_\nabla^* \gamma_*^V \in \Gamma \operatorname{Hom}(L, Q)$$

We prove the following result.

4.2 THEOREM. *Let $\mathscr{F}$ be a foliation of codimension q on a Riemannian manifold M, and ∇ a torsion-free connection in the normal bundle. For the tension $\tau(\gamma) \in \operatorname{Hom}(L, Q)$ of the Gauss section $\gamma\colon M \to G^q(M)$ of $\mathscr{F}$ we have the formula*

$$\tau(\gamma)_x(X) = \nabla_x \tau(\mathscr{F}) - S(\pi)_x(X) \tag{4.3}$$

for $x \in M$ and $X \in L_x$.

To describe the Q-valued 1-form $S(\pi)$ on the right hand side, we choose a local framing as in Section 2, by starting with an orthonormal basis of $T_x M$ with $e_i \in L_x$ $(i = 1, \ldots, p)$ and $e_\alpha \in \sigma Q_x$ $(\alpha = p+1, \ldots, n)$. Then for $X \in L_x$,

$$S(\pi)_x(X) = \sum_\alpha R_\nabla(e_\alpha, X)\pi(e_\alpha) + \pi(\rho_{\nabla M}(X)) \tag{4.4}$$

Proof. If ∇ is a torsion-free connection in Q, then $d_\nabla \pi = 0$ (see (2.2) and (2.3)). Therefore $\Delta\pi = d_\nabla d_\nabla^* \pi$. But $d_\nabla^* \pi = \tau(\mathscr{F})$ (2.18), thus $\Delta\pi = d_\nabla \tau(\mathscr{F}) = \nabla\tau(\mathscr{F})$. Then formula (1.1) yields

$$(\nabla\tau(\mathscr{F}))_x = -(\operatorname{Tr} \tilde{\nabla}^2\pi)_x + S(\pi)_x$$

where

$$-\operatorname{Tr} \tilde{\nabla}^2\pi = \operatorname{Tr} \tilde{\nabla}\alpha = -d_\nabla^* \alpha \in \Gamma \operatorname{Hom}(TM, Q).$$

The restriction of the last expression to $\operatorname{Hom}(L, Q)$ is $\tau(\gamma)$. Therefore we obtain the identity (4.3).

The specific form (4.4) for $S(\pi)$ for the special framings described follows from the vanishing of $R_\nabla(X, Y)$ for $X, Y \in \Gamma L$, and formula (2.4). ∎

If $\mathscr{F}$ is a Riemannian foliation, and ∇ the unique torsion-free and metric connection on Q, then R_∇ is basic [6, 1.13]. Therefore, the terms $R_\nabla(e_\alpha, X)$ vanish for $X \in \Gamma L$. Note that $\nabla_X s$ for $X \in \Gamma L$ is the canonical (partial) Bott connection $\mathring{\nabla}_X s$ for $s \in \Gamma Q$. This yields the following result.

4.5 COROLLARY. *Let $\mathscr{F}$ be a Riemannian foliation of codimension q on a Riemannian manifold M. For the tension $\tau(\gamma)$ of the Gauss section γ of $\mathscr{F}$ we have the formula*

$$\tau(\gamma)_x(X) = \mathring{\nabla}_X \tau(\mathscr{F}) - \pi(\rho_{\nabla M}(X))$$

for $x \in M$ and $X \in L_x$.

If M is Ricci-flat, this reduces to a formula of Ruh-Vilms type for a leaf $\mathscr{L} \subset M$, implying that the Gauss section γ is harmonic iff the tension of $\mathscr{F}$ (i.e., the mean curvature) is parallel along $\mathscr{L}$ [15].

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