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LYAPOUNOV NUMBERS FOR THE ALMOST PERIODIC SCHRODINGER EQUATION

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1. Introduction

We consider the almost periodic Schrödinger operator

$$L = \frac{-d^2}{dt^2} + q(t). \tag{1.1}$$

where $q(t)$ is continuous and Bohr almost periodic. Associated to (1.1) is a rotation number $\alpha(\lambda)$ ($\lambda \in \mathbf{R}$), where

$$\alpha(\lambda) = -\lim_{t\to\infty} \frac{\theta(t)}{t}, \quad \theta(t) = \arg\,(\phi(t) + i\phi'(t)),$$

and $\phi \neq 0$ satisfies $L\phi = \lambda\phi$. It is known that $\alpha(\lambda)$ is independent of the solution ϕ, that α is continuous and monotone increasing in λ, and that α increases exactly on the essential spectrum F of L [7]. In addition,

$$\alpha(\lambda) = \lim_{\varepsilon\to 0} w(\lambda + i\varepsilon),$$

where $w(z)$ is holomorphic in the upper half plane $H^+ = \{z \mid \operatorname{Im} z > 0\}$, and $\operatorname{Im} w(z)$ measures the "complex rotation" of certain solutions of $L\phi = z\phi$. Moreover, $w(z)$ provides information about the higher-order $K\,dV$ equations with almost-periodic initial data [7].

In this paper, we consider the *real* part $-\operatorname{Re} w(z)$, and its boundary value $\beta(\lambda)$ ($\lambda \in \mathbf{R}$). It will be easy to see that

$$\operatorname{Re} w(z) = \lim_{t\to\infty} \frac{1}{2t} \ln\,[\psi(t)^2 + \psi'(t)^2]$$

where $L\psi = z\psi$ and $\psi \in L^2(0, \infty)$ ($z \in H^+$). Thus $\operatorname{Re} w(z)$ measures the exponential decay of solutions which are in $L^2(0, \infty)$. We will see that the boundary value of $\operatorname{Re} w$ also measures exponential decay of solutions. In fact, we *define*

$$\beta(\lambda) = \sup_{\phi\neq 0} \left\{ \overline{\lim_{t\to\infty}} \frac{1}{2t} \ln\,[\phi(t)^2 + \phi'(t)^2] \right\}, \tag{1.2}$$

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where $L\phi = \lambda\phi$; actually the sup is taken not just over solutions of $L\phi = \lambda\phi$, but over solutions to all equations

$$L_\omega \phi = \left(\frac{-d^2}{dt^2} + \omega(t)\right)\phi = \lambda\phi, \tag{1.3}$$

where $\omega(t)$ is in the *hull* of q (see Section 2). In 1.2, we allow λ to take on real and complex values. It then turns out that $\beta(\lambda) \geq 0$ everywhere, that $\mathrm{Re}\, w(\lambda) = -\beta(\lambda)$ if $\mathrm{Im}\, \lambda > 0$, and that

$$\lim_{\varepsilon \to 0} (-\mathrm{Re}\, w(\lambda + i\varepsilon)) = \beta(\lambda) \quad \text{for } all\ \lambda \in \mathbf{R}.$$

Moreover, if λ is real and $\beta(\lambda) = 0$, then all solutions of all equations $L_\omega \phi = \lambda\phi$ satisfy

$$\lim_{t \to \infty} \frac{1}{2t} \ln [|\phi(t)|^2 + |\phi'(t)|^2] = 0;$$

if $\beta(\lambda) > 0$, then for almost all ω, (1.3) admits a unique (up to constant multiple) solution ψ with

$$\lim_{t \to \infty} \frac{1}{2t} \ln [\psi(t)^2 + \psi'(t)^2] = -\beta(\lambda).$$

In particular, $\psi \in L^2(0, \infty)$.

The function β has several other properties; we prove two. First, it is harmonic on the resolvent set $\mathbf{C}\backslash F$ of the operator L (we give a simple proof based on [7]). This fact is used to prove that, if I is an open interval such that $F \cap I \neq \emptyset$, then $F \cap I$ has positive logarithmic capacity. Second, it is one-sided continuous at an endpoint λ of a spectral gap: if $\lambda_n \in \mathbf{R}\backslash F$ and $\lambda_n \to \lambda$, then $\beta(\lambda_n) \to \beta(\lambda)$.

To throw more light on the function $\beta(\lambda)$ ($\lambda \in \mathbf{R}$), we consider a class of examples, modeled on the example of [6] (in that example, $\beta(\lambda_0) > 0$ for at least one point λ_0 in F, namely the leftmost point in F). We assume

$$q(t) = \lim_{n \to \infty} q_n(t), \quad q_n(t + T_n) = q_n(t) \tag{1.4}$$

where the limit is *uniform* and the period T_{n+1} of q_{n+1} is an integer multiple of $T_n (n \geq 1)$; we also put various other conditions on the q_n.

We prove that $\beta(\lambda_0) > 0$ for the left endpoint $\lambda_0 \in F$, and that β is discontinuous at λ_0: in fact, $\beta(\lambda_n) \to 0$ for a sequence $\lambda_n \to \lambda_0$. Now,

$$\lim_{\varepsilon \to 0} w(\lambda + i\varepsilon) = -\beta(\lambda) + i\alpha(\lambda),$$

hence β is the Hilbert transform of the continuous function $-\alpha(\lambda)$. Hence β has the mean value property [14]. So β must oscillate wildly near λ_0.

2. Preliminaries

We first introduce the hull Ω of q. For $\tau \in \mathbf{R}$, the translate q is given by $q_\tau(t) = q(t + \tau)$ $(t \in \mathbf{R})$; then $\Omega = \mathrm{cl}\,\{q_\tau \mid \tau \in \mathbf{R}\}$, where the closure is taken in the uniform topology. Thus q is a point in Ω; we denote q also by ω_0. A flow $(\Omega, \mathbf{R})$ is defined by translation:

$$(\omega \cdot t)(s) = \omega(t + s) \quad (\omega \in \Omega).$$

We give Ω the structure of a compact, abelian topological group, as follows. If

$$\omega_1 = \lim_{n \to \infty} \omega_0 \cdot t_n, \quad \omega_2 = \lim_{n \to \infty} \omega_0 \cdot s_n,$$

then

$$\omega_1 \omega_2 = \lim_{n \to \infty} \omega_0 \cdot (t_n + s_n) \quad \text{and} \quad \omega_1^{-1} = \lim_{n \to \infty} \omega_0 \cdot (-t_n) \text{ [11]}.$$

Note that ω_0 is the identity of Ω. We may view $\mathbf{R}$ as a dense subgroup of Ω via the map $t \to \omega_0 \cdot t$.

We "extend q to Ω" in the natural way: define $Q(\omega) = \omega(0)$ $(\omega \in \Omega)$; then Q is continuous, and $Q(\omega_0 \cdot t) = q_t(0) = q(t)$. Thus q is regained from Q by evaluation along the orbit through $q = \omega_0$. We will consider the equations

$$L_\omega \phi = \left(\frac{-d^2}{dt^2} + Q(\omega \cdot t)\right)\phi = \lambda \phi \quad (\omega \in \Omega), \tag{2.1$_\omega$}$$

and the associated two-dimensional systems

$$u' = \begin{pmatrix} 0 & 1 \\ \lambda + Q(\omega \cdot t) & 0 \end{pmatrix} u, \quad u = \begin{pmatrix} \phi \\ \phi' \end{pmatrix} \qquad (\omega \in \Omega). \tag{2.2$_\omega$}$$

When it is necessary to avoid confusion, we will write $(2.1)_{\omega,\lambda}$ and $(2.2)_{\omega,\lambda}$ instead of $(2.1)_\omega$ and $(2.2)_\omega$.

2.3 Definition. Fix $\lambda \in \mathbf{C}$. Define

$$\beta(\lambda) = \sup \left\{ \overline{\lim_{t \to \infty}} \frac{1}{t} \ln \|u(t)\| : \ u(t) \text{ is a non-zero solution of some equation } (2)_\omega \right\}.$$

The sup is taken over all $u(t)$ and all $\omega \in \Omega$.

Fix $\lambda \in \mathbf{C}$. It is convenient to introduce the *projective flow* defined by equations $(2)_\omega$. Call a complex 1-dimensional subspace of $\mathbf{C}^2$ a complex line. For each $\omega \in \Omega$, equation $(2)_\omega$ is linear, so the fundamental matrix solution $\Phi_\omega(t)$ (with $\Phi_\omega(0) = I$) maps complex lines to complex lines. If l is a complex line in $\mathbf{C}^2$, let $l(\tau) = \Phi_\omega(t) \cdot l$ denote its image after time t. Letting $\mathbf{P}^1(\mathbf{C})$ be the usual space of all complex lines in $\mathbf{C}^2$, we define a flow on $\Sigma = \Omega \times \mathbf{P}^1(\mathbf{C})$ as follows:

$$(\omega, l) \cdot t = (\omega \cdot t, l(t)) \quad (\omega \in \Omega, l \in \mathbf{P}^1(\mathbf{C})).$$

The point of introducing $(\Sigma, \mathbf{R})$ is the following. Write

$$A_\lambda(\omega) = \begin{pmatrix} 0 & 1 \\ -\lambda + Q(\omega) & 0 \end{pmatrix} \quad (\omega \in \Omega).$$

Define

$$f_\lambda \colon \Sigma \to \mathbf{R} \colon (\omega, l) \to \operatorname{Re} \frac{\langle A_\lambda(\omega) u_0, u_0 \rangle}{\langle u_0, u_0 \rangle}, \tag{2.4}$$

where $0 \neq u_0$ is any vector in l. Then if $u(t)$ satisfies equation $(2)_\omega$ with $u(0) = u_0$, one has

$$\frac{1}{t} [\ln \|u(t)\| - \ln \|u(0)\|] = \frac{1}{t} \int_0^t f_\lambda((\omega, l) \cdot s)\, ds. \tag{2.5}$$

Thus the exponential growth of $u(t)$ is determined by a time average of f_λ. We will use the ergodic theory of the flow $(\Sigma, \mathbf{R})$ to study these time averages.

We remark that a flow $(\Sigma, \mathbf{R})$ is defined for each $\lambda \in \mathbf{C}$. When confusion can arise, we write $(\Sigma, \mathbf{R})_\lambda$ for the flow defined by equations $(2.2)_{\omega,\lambda}$.

If λ is *real*, we obtain also a flow on $\Sigma_{\mathbf{Re}} = \Omega \times \mathbf{P}^1(\mathbf{R})$, where $\mathbf{P}^1(\mathbf{R})$ is the space of (real) one-dimensional subspaces of $\mathbf{R}^2$. We will call such subspaces *lines* (as opposed to complex lines). It is convenient to view $\mathbf{P}^1(\mathbf{R})$ as a subset of $\mathbf{P}^1(\mathbf{C})$, and hence $\Sigma_{\mathbf{Re}}$ as a subset of Σ. To do this, we use the usual identification of the Riemann number sphere S^2 with $\mathbf{P}^1(\mathbf{C})$: if $[a, b]$ denotes the complex line on which the non-zero complex vector (a, b) lies, then we define Ident: $S^2 \to \mathbf{P}^1(\mathbf{C})$ by

$$z \to [1, z], \text{ if } z \neq \infty; \quad \text{Ident}(\infty) = [0, 1].$$

Then $\mathbf{P}^1(\mathbf{R})$ is identified with $\mathbf{R} \cup \{\infty\} \subset S^2$.

We also need to consider the singular boundary value problems

$$L_\omega \phi = \left(\frac{-d^2}{dt^2} + Q(\omega \cdot t)\right)\phi = \lambda\phi, \ \phi(0) = 0, \ \phi \in L^2(0, \infty) \quad (\omega \in \Omega). \tag{2.6$_\omega$}$$

Fix $\omega \in \Omega$. Since equation $(2.6)_\omega$ is of limit point type [3], there is a function $M_\omega(\lambda)$, defined and holomorphic for $\operatorname{Im} \lambda \neq 0$, satisfying

$$\frac{\operatorname{Im} M_\omega(\lambda)}{\operatorname{Im} \lambda} > 0,$$

such that if $\phi(t) \neq 0$ satisfies $L_\omega \phi = \lambda\phi$, then $\phi \in L^2(0, \infty)$ iff $\phi'(0) = M_\omega(\lambda)\phi(0)$. For fixed λ with $\operatorname{Im} \lambda \neq 0$, let $\psi^+(t)$ satisfy

$$L\psi^+ = \lambda\psi^+ \quad \text{and} \quad \psi^{+\prime}(0) = M_\omega(\lambda)\psi^+(0).$$

It is not hard to show that

$$\text{(2.7)} \quad \psi^+(t) = \psi^+(0) \exp\left(\int_0^t M_{\omega\cdot s}(\lambda)\, ds\right),$$

(2.8) $M_\omega(\lambda)$ is jointly continuous in ω and λ ($\operatorname{Im} \lambda \neq 0$).

Problem $(2.6)_\omega$ admits a monotone increasing *spectral function* $\rho_\omega(t)$ [3]; the points in the spectrum of the singular problem $(2.6)_\omega$ are the points of increase of ρ_ω. The function ρ_ω is unique if it is chosen to be right-continuous with $\rho_\omega(0) = 0$. We have

$$\operatorname{Im} M_\omega(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} \operatorname{Im} \frac{d\rho_\omega(t)}{t - \lambda} \quad (\operatorname{Im} \lambda > 0).$$

We note that the essential spectrum of $(2.6)_\omega$ is independent of ω, and equals the spectrum F on $L^2(-\infty, \infty)$ of each and every operator

$$L_\omega = \frac{-d^2}{dt^2} + Q(\omega \cdot t)$$

(see [14]; viewed on $L^2(-\infty, \infty)$, the L_ω's all have the same spectrum F, and it is always essential).

2.9 Definition [7]. For $\operatorname{Im} \lambda \neq 0$, define $w(\lambda) = \int_\Omega M_\omega(\lambda)\, d\omega$, where $d\omega$ is normalized Haar measure on the compact topological group Ω.

Using 2.8, one shows that $w(\lambda)$ is holomorphic for $\operatorname{Im} \lambda \neq 0$. Since $d\omega$ is the only measure on Ω invariant with respect to the flow $(\Omega, \mathbf{R})$, we have for fixed ω, λ

$$\operatorname{Re} w(\lambda) = \lim_{t\to\infty} \frac{1}{t}\int_0^t M_{\omega\cdot s}(\lambda)\, ds,$$

so using 2.7,

$$\operatorname{Re} w(\lambda) = \lim_{t\to\infty} \frac{1}{2t} \ln\left(|\psi^+(t)|^2 + |\psi^{+\prime}(t)|^2\right).$$

Thus Re $w(\lambda)$ is the exponential rate of decay of solutions

$$u(t) = (\psi^+(t), \psi^{+\prime}(t))$$

of $(2.2)_\omega$ for which $\psi^+ \in L^2(0, \infty)$. Note Re $w(\lambda) \leq 0$, and since it is harmonic, Re $w(\lambda) < 0$ if Im $\lambda \neq 0$.

For each $\omega \in \Omega$, there is also a holomorphic function $M_\omega^-(\lambda)$ (Im $\lambda \neq 0$), satisfying

$$\frac{\operatorname{Im} M_\omega^-(\lambda)}{\operatorname{Im} \lambda} < 0,$$

such that $\psi^-(t) = \exp\left(\int_0^t M_{\omega\cdot s}^-(\lambda)\, ds\right)$ is in $L^2(-\infty, 0)$; $M_\omega^-(\lambda)$ is also jointly continuous in ω and λ. It is proved in [7] that

$$w(\lambda) = -\int_\Omega M_\omega^-(\lambda)\, dw \quad (\operatorname{Im} \lambda \neq 0).$$

Using these facts, one can show that

$$\beta(\lambda) = -\operatorname{Re} w(\lambda) \quad (\operatorname{Im} \lambda \neq 0), \tag{2.10}$$

where β is as defined in 2.3.

Now introduce polar coordinates (r, θ) in equations $(2.2)_\omega$, where λ is *real* and fixed. Then θ satisfies

$$\dot{\theta} = \sin^2 \theta + (-\lambda + Q(\omega \cdot t)) \cos^2 \theta. \tag*{$(2.11)_\omega$}$$

In [7], the rotation number $\alpha(\lambda)$ is defined as follows:

$$\alpha(\lambda) = -\lim_{t\to\infty} \frac{\theta(t)}{t}, \tag{2.12}$$

where a choice of $\omega \in \Omega$ and $\theta(0) = \theta_0$ is made.

2.13 THEOREM [7]. *The rotation number $\alpha(\lambda)$ is independent of ω and θ_0, and the convergence in* (2.12) *is uniform in t, ω, θ_0. Also α is continuous and monotone increasing in $\lambda \in \mathbf{R}$. One has that α increases exactly on the spectrum F of the L_ω, and*

$$\lim_{\varepsilon\to 0+} \operatorname{Im} w(\lambda + i\varepsilon) = \alpha(\lambda) \quad (\lambda \in \mathbf{R}). \tag{2.14}$$

Next we recall some results from the theory of almost periodic linear systems, as applied to equations $(2.2)_\omega$.

2.15. Let $\lambda \in \mathbf{R}$, and let f_λ be defined in 2.4. If $\beta(\lambda) = 0$, then $\int_\Sigma f_\lambda\, d\mu = 0$ for every invariant measure [10] μ on Σ. Every time average,

$$\lim_{|b-a|\to\infty} \frac{1}{b-a} \int_a^b f_\lambda(\sigma \cdot s)\, ds \quad (\sigma \in \Sigma),$$

is 0 and the convergence is uniform in a, b, σ (this uses the proof of Lemma 3.5 in [7]). On the other hand, if $\beta(\lambda) > 0$, then there are exactly two ergodic measures on Σ; one has $\mu_{\pm}(\Sigma_{\mathrm{Re}}) = 1$, and

$$\int_{\Sigma} f_\lambda \, d\mu_+ = \hat{\beta} > 0, \quad \int_{\Sigma} f_\lambda \, d\mu_- = -\hat{\beta}. \tag{2.16}$$

See [5]; $\hat{\beta}$ is in fact the right end-point of the *Sacker-Sell spectrum* [12] of equations $(2.2)_\omega$. (Note that $\beta(\lambda) \geq 0$ because a fundamental matrix solution of $(2.2)_\omega$ has constant determinant; so there are no other possibilities for $\beta(\lambda)$.)

2.17 PROPOSITION. *If $\lambda \in \mathbf{R}$ and $\beta(\lambda) > 0$, then $\beta(\lambda) = \hat{\beta}$. (It follows directly from* 2.3, 2.5, *and* 2.15 *that, if $\beta(\lambda) = 0$, then $\hat{\beta} = 0$.)*

Proof. Let $\varepsilon > 0$. Since $\beta(\lambda) > 0$, we can find $\omega \in \Omega$, a sequence $(t_n) \to \infty$, and a line $l \in \mathbf{P}^1(\mathbf{R})$ such that, if $u(t) \neq 0$ is a solution of $(2)_\omega$ with $(u(0), u'(0)) \in l$, then $(\sigma = (\omega, l) \in \Sigma)$

$$\lim_{n\to\infty} \frac{1}{t_n} \ln \|u(t_n)\| = \lim_{n\to\infty} \frac{1}{t_n} \int_0^{t_n} f_\lambda(\sigma \cdot s) \, ds > \beta(\lambda) - \varepsilon.$$

Using the classical Krylov–Bogoliubov argument as in the proof of [7, Lemma 3.5], we can find an invariant measure η on Σ so that

$$\int_{\Sigma} f_\lambda \, d\eta > \beta(\lambda) - \varepsilon.$$

From 2.15, there are non-negative numbers a, b such that

$$a + b = 1 \quad \text{and} \quad \eta = a\mu_+ + b\mu_-.$$

From (2.16), we see that $\hat{\beta} \geq \beta(\lambda) - \varepsilon$, and hence $\hat{\beta} \geq \beta(\lambda)$. On the other hand, 2.15 and the Birkhoff ergodic theorem give us a point $\sigma = (\omega, l) \in \Sigma$ for which

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t f_\lambda(\sigma \cdot s) \, ds = \hat{\beta}.$$

Let $u(t)$ be a solution of $(2.2)_\omega$ with $u(0) \neq 0$ on the line l. Then

$$\lim_{t\to\infty} \frac{1}{t} \ln \|u(t)\| = \hat{\beta},$$

and hence $\beta(\lambda) \geq \hat{\beta}$. So $\beta(\lambda) = \hat{\beta}$.

3. Facts about β

We show that

$$\lim_{\varepsilon \to 0^+} -\operatorname{Re} w(\lambda + i\varepsilon) = \beta(\lambda) \quad \text{for all } \lambda \in \mathbf{R}$$

(a more precise statement will be proved), and derive some properties of β.

3.1. THEOREM. (a) *If $\lambda \in \mathbf{R}$ and $z \to \lambda$ non-tangentially* (n.t.) *for $z \in H^+$, the upper half-plane, then $-\operatorname{Re} w(z) \to \beta(\lambda)$. If $\beta(\lambda) = 0$, then β is continuous at λ, and $-\operatorname{Re} w(z) \to \beta(\lambda)$ whenever $z \to \lambda$, non-tangentially or not.*

(b) *β is upper semi-continuous on $\mathbf{R}$.*

(c) *On $\mathbf{R}$, β is non-negative, of first Baire class, and has the mean value property.*

Proof. First consider (c). We noted in Section 2 that $\beta(\lambda) \geq 0$ for all λ. That β is of first Baire class follows from (a) or (b). Also, β has the mean value property because it is the Hilbert transform of the continuous function $-\alpha$ (part (a), 2.14, and [15]). So we need only prove (a) and (b).

Let us prove the first statement in (a). Consider the functions $M_\omega(z)$ discussed in Section 2 ($z \in H^+$). For fixed $z \in H^+$, define a measure μ_z on Σ as follows:

$$\int_\Sigma g \, d\mu_z = \int_\Omega g(\omega, M_\omega(z)) \, d\omega$$

whenever $g: \Sigma \to \mathbf{R}$ is continuous. Using 2.7, we see that

$$(\omega, M_\omega(z)) \cdot t = (\omega \cdot t, M_{\omega \cdot t}(z)),$$

where the dot on the left-hand side refers to the flow $(\Sigma, \mathbf{R})_z$. It follows that μ_z is invariant under this flow. In fact, μ_z is *ergodic*, because the set $A_z = \{(\omega, M_\omega(z)) \mid \omega \in \Omega\} \subset \Sigma$ is an invariant set which is flow isomorphic to $(\Omega, \mathbf{R})$ via the projection $\pi: \Sigma \to \Omega: (\omega, l) \to \omega$.

Let ψ^+ be a non-zero solution of $(2.1)_\omega$ with $\psi^{+\prime}(0) = M_\omega(z)\psi^+(0)$. Writing $u(t) = (\psi^+(t), \psi^{+\prime}(t))$, recalling that

$$\operatorname{Re} w(z) = \lim_{t \to \infty} \frac{1}{2t} \ln \left(|\psi^+(t)|^2 + |\psi^{+\prime}(t)|^2 \right),$$

and using (2.5) and (2.10), we get

$$\begin{aligned} -\beta(z) &= \lim_{t \to \infty} \frac{1}{t} \ln \|u(t)\| \\ &= \lim_{t \to \infty} \frac{1}{t} \int_0^t f_z((\omega, M_\omega(z)) \cdot s) \, ds \\ &= \lim_{t \to \infty} \frac{1}{t} \int_0^t f_z((\omega \cdot s, M_{\omega \cdot s}(z)) \, ds, \end{aligned}$$

and by the Birkhoff ergodic theorem the last limit equals

$$\int_{\Omega} f_z(\omega, M_\omega(z))\, d\omega.$$

Hence we have

$$\beta(z) = -\int_{\Sigma} f_z\, d\mu_z \quad (z \in H^+). \tag{3.2}$$

Now fix $\lambda \in \mathbf{R}$, and suppose first that $\beta(\lambda) = 0$. Let $z_n \in H^+$, $z_n \to \lambda$, and suppose $\beta(z) = -\operatorname{Re} w(z_n)$ does not tend to zero. We may assume

$$\beta(z_n) \to \delta > 0.$$

Write $\mu_n = \mu_{z_n}$, $f_n = f_{z_n}$, and note that $f_n \to f_\lambda$ uniformly on Σ. The sequence $\{\mu_n\}$ of measures has a weakly convergent subsequence $\{\mu_k\}$. Suppose $\mu_k \to \eta$. Then η is invariant with respect to $(\Sigma, \mathbf{R})_\lambda$, and hence $\int_\Sigma f_\lambda\, d\eta = 0$ (2.15). However, it is clear that this contradicts 3.2. Hence $\beta(z_n) \to 0 = \beta(\lambda)$.

Suppose next that $\beta(\lambda) > 0$. Let μ_- be the measure on $\Sigma_{\mathbf{Re}}$ given by 2.15. For μ_--a.a. $(\omega, l) \in \Sigma_{\mathbf{Re}}$, any solution $u(t) \neq 0$ of $(2.2)_{\omega,\lambda}$ with $u(0) \in l$ satisfies

$$\lim_{t \to \infty} \frac{1}{t} \ln \|u(t)\| = -\beta(\lambda).$$

Given $\omega \in \Omega$, there can be at most one line l_ω in $\mathbf{R}^2$ with this property. We conclude that, for $d\omega$-a.a. $\omega \in \Omega$, there is a unique $l_\omega \in \mathbf{P}^1(\mathbf{R})$ such that, if ψ is a solution of $(2.1)_{\omega,\lambda}$ such that

$$(\psi(0), \psi'(0))$$

lies on l_ω, then $\psi \in L^2(0, \infty)$. Let

$$\Omega_0 = \{\omega \in \Omega \mid \text{there exists } l_\omega \text{ as above}\}.$$

We now claim that if $\omega \in \Omega_0$, then $M_\omega(z) \to l_\omega$ in $\mathbf{P}^1(\mathbf{C})$ whenever $z \to \lambda$ n.t. To see this, let

$$\theta_\omega \in (-\pi/2, \pi/2)$$

be the angle l_ω makes with the positive ϕ-axis in (ϕ, ϕ')-space $\mathbf{R}^2$, and consider the singular boundary-value problem

$$L_\omega \phi = \left(\frac{-d^2}{dt^2} + Q(\omega \cdot t)\right)\phi = \nu\phi \tag*{$(3.3)_\omega$}$$

$$\phi(0) = \cos\theta_\omega, \quad \phi'(0) = \sin\theta_\omega, \quad \phi \in L^2(0, \infty).$$

This problem admits a spectral function ρ_ω^θ, which has a jump discontinuity at $\nu \in \mathbf{R}$ if and only if $(3.3)_\omega$ has eigenvalue ν [3]. There is also a function

$M_\omega^\theta(z)$, holomorphic for $\text{Im}\, z \neq 0$, such that:

(i) the solution ψ^+ of $L_\omega \phi = z\phi$ satisfying

$$\phi^+(0) = \sin\theta_\omega + M_\omega^\theta(z)\cos\theta_\omega, \quad \psi^{+\prime}(0) = \cos\theta_\omega + M_\omega^\theta(z)\sin\theta_\omega$$

is in $L^2(0, \infty)$;

(ii) $\text{Im}\, M_\omega^\theta(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \text{Im}\, \frac{d\rho_\omega^\theta(t)}{t-z}$ for $\text{Im}\, z > 0$.

Now, $(3.3)_\omega$ has, for each $\omega \in \Omega_0$, an eigenvalue at the point λ under consideration. Hence $M_\omega^\theta(z) \to \infty$ if $z \to \lambda$ n.t. and $z \in H^+$ [15]. But, using (i), 2.7, and uniqueness of ψ^+ up to constant multiple, we seen that $M_\omega(z) \to \tan\theta_\omega$; i.e. $M_\omega(z) \to l_\omega$ in $\mathbf{P}^1(\mathbf{C})$. This is what we wanted to show.

Next let $\{z_n\}$ be any sequence in H^+ such that $z_n \to \lambda$ and $M_\omega(z_n) \to l_\omega$ for all $\omega \in \Omega_0$. Clearly if $\omega \in \Omega_0$, then $\omega \cdot t \in \Omega_0$ for all $t \in \mathbf{R}$, and $M_{\omega\cdot t}(z_n) \to l_{\omega\cdot t}$. Also

$$-\beta(z_n) = \int_\Omega f_{z_n}(\omega, M_{z_n}(\omega))\, d\omega \to \int_\Omega f_\lambda(\omega, l_\omega)\, d\omega$$

by bounded convergence. However, it is easily seen that the map

$$\omega \to (\omega, l_\omega) \colon \Omega \to \Sigma$$

defines a $d\omega$-measurable, invariant section of the sphere bundle Σ; i.e.,

$$(\omega, l_\omega) \cdot t = (\omega \cdot t, l_\omega \cdot t) \quad \text{for all } t \in \mathbf{R} \quad \text{and} \quad \omega \in \Omega_0.$$

Hence we can define an *ergodic* measure μ_0 on Σ by

$$\int_\Sigma g\, d\mu_0 = \int_\Omega g(\omega, l_\omega)\, d\omega$$

when $g \colon \Sigma \to \mathbf{R}$ is continuous; μ_0 is ergodic because the projection $\pi \colon \Sigma \to \Omega$ restricts to a measurable bijection from $\{(\omega, l_\omega) \mid \omega \in \Omega\}$ to Ω_0. So, by 2.15,

$$\int_\Sigma f_\lambda\, d\mu_0 = \pm\beta(\lambda),$$

and since $0 \geq -\beta(z_n) \to \int_\Sigma f_\lambda\, d\mu_0$, we must have $\int_\Sigma f_\lambda\, d\mu_0 = -\beta(\lambda)$. So

$$\beta(z_n) \to \beta(\lambda) \quad \text{if } z_n \to \lambda \text{ n.t.}$$

We have proved the first statement in part (a) of the theorem.

There remains to prove that $\beta(\lambda) = 0$ implies β is continuous at λ, and that β is upper semi-continuous on $\mathbf{R}$. Since $\beta \geq 0$ everywhere, it suffices to prove that β is upper semi-continuous on $\mathbf{R}$.

Suppose for contradiction that there is a sequence $\lambda_n \to \lambda \in \mathbf{R}$ such that $\lim_{n\to\infty} \beta(\lambda_n) > \beta(\lambda)$. Then we can assume that $\beta(\lambda_n) \geq \delta > 0$. By 2.15, the flow $(\Sigma, \mathbf{R})_{\lambda_n}$ admits an ergodic measure μ_n with

$$\int_\Sigma f_{\lambda_n}\, d\mu_n \geq \delta \quad \text{for all } n.$$

We can assume that $\mu_n \to \mu$ weakly; one checks that μ is invariant with respect to $(\Sigma, \mathbf{R})_\lambda$. Since $f_{\lambda_n} \to f_\lambda$ uniformly,

$$\beta(\lambda_n) \to \int_\Sigma f_\lambda\, d\mu.$$

By 2.15, $\int_\Sigma f_\lambda\, d\mu \leq \beta(\lambda)$. This is a contradiction; we have proved that β is upper semi-continuous. This completes the proof of 3.1.

Next we use the results of [7] to prove that β is harmonic on the resolvent set of $L = -d^2/dt^2 + q(t)$ acting on $L^2(-\infty, \infty)$. The result is also a special case of a more general proposition proved in [4].

3.4 PROPOSITION. *Let F be the spectrum of $L = -d^2/dt^2 + q(t)$ acting on $L^2(-\infty, \infty)$. Then β is harmonic on the resolvent set $\mathbf{C}\backslash F$.*

Proof. Consider the function $w(\lambda) = \int_\Omega M_\omega(\lambda)\, d\omega$ introduced in Section 2; then

$$\beta(\lambda) = -\mathrm{Re}\, w(\lambda) \quad (\mathrm{Im}\ \lambda \neq 0).$$

Recall that $M_\omega(\bar{\lambda}) = \overline{M_\omega(\lambda)}$, hence $w(\bar{\lambda}) = \overline{w(\lambda)}$. Now if I is an interval in $\mathbf{R}\backslash F$, then

$$-\beta(\lambda) + i\alpha(\lambda) = \lim_{\varepsilon \to 0^+} w(\lambda + i\varepsilon),$$

and $\alpha(\lambda)$ is constant for $\lambda \in I$, say α_I [7]. We also have

$$\lim_{\varepsilon \to 0^-} w(\lambda + i\varepsilon) = \beta(\lambda) - \alpha_I \quad (\lambda \in I).$$

So if we define

$$w^*(\lambda) = \begin{cases} w(\lambda), & \mathrm{Im}\ \lambda > 0, \\ \beta(\lambda) + i\alpha_I, & \lambda \in I, \\ w(\lambda) + 2i\alpha_I, & \mathrm{Im}\ \lambda < 0, \end{cases}$$

then w^* is holomorphic on $\{\mathrm{Im}\ \lambda \neq 0\} \cup I$ by the reflection principle. It follows that β is harmonic on the resolvent set.

As a corollary, we prove that F cannot be too small.

3.5 COROLLARY. *Let $I \subset \mathbf{R}$ be an open interval such that $I \cap F \neq \emptyset$. Then $F \cap I$ has positive logarithmic capacity* [11].

Proof. Suppose $F \cap I \neq \emptyset$ but the logarithmic capacity is zero. By 3.4, β extends harmonically to the entire open disc D with diameter I (since β is clearly bounded on D). Let h be that harmonic conjugate of β on D such that $h = \operatorname{Im} w$ on $\{\lambda \in D \mid \operatorname{Im} \lambda > 0\}$. By 2.10 and 2.13, α is the restriction of h to I; hence α is continuously differentiable on I. Now by 2.13, $\alpha'(\lambda) = 0$ except when $\lambda \in F$. Since $F \cap I$ has capacity zero, it has Lebesgue measure zero. So α' is identically zero on I, and α is constant on I. So by 2.13, $F \cap I = \phi$. This is a contradiction, so $F \cap I$ has positive logarithmic capacity.

Finally, we consider the behaviour of β at endpoints of spectral gaps.

3.6 PROPOSITION. *Let $\lambda_0 \in \mathbf{R}$ be an endpoint of a spectral gap I (i.e. I is a maximal interval in $\mathbf{R} \backslash F$). If $\lambda_n \to \lambda_0$ and $\lambda_n \in I$, then $\beta(\lambda_n) \to \beta(\lambda_0)$.*

Proof. Recall the functions $M_\omega(z)$, $M_\omega^-(z)$ discussed in Section 2; for each $\omega \in \Omega$, these are defined and holomorphic for $\operatorname{Im} z \neq 0$. Since the spectrum F of L_ω is independent of ω, each $M_\omega(z)$ extends meromorphically through I ($\omega \in \Omega$), and so does each $M_\omega^-(z)$ [3]. In addition, either $\operatorname{Im} M_\omega(\lambda) = 0$ or $M_\omega^-(\lambda) = \infty$ ($\lambda \in I$), and the same holds for each $M_\omega^-(\lambda)$.

Now, the vector $(1, M_\omega(\lambda)) \in \mathbf{C}^2$ defines a line $l_\omega^+(\lambda)$ in $\mathbf{P}^1(\mathbf{R})$ for each $\lambda \in I$; if $M_\omega(\lambda) = \infty$, then $l_\omega^+(\lambda)$ is the line containing the vector $(0, 1)$. Similarly, $(1, M_\omega^-(\lambda))$ defines a line $l_\omega^-(\lambda)$ ($\lambda \in I$). We coordinatize the circle $\mathbf{P}^1(\mathbf{R})$ with the usual polar coordinate θ, $-\pi/2 \leq \theta \leq \pi/2$, where $\theta = -\pi/2$ and $\theta = \pi/2$ are identified. Orient $\mathbf{P}^1(\mathbf{R})$ in the direction of increasing θ.

Fix $\omega \in \Omega$. It is remarked in [6] that, if λ increases through I, then $M_\omega(\lambda)$ and $M_\omega^-(\lambda)$ move in opposite directions on $\mathbf{P}^1(\mathbf{R})$ (the remark is just [2, Problem 9, p. 257]. It can also be shown that $M_\omega(\lambda)$ and $M_\omega^-(\lambda)$ can never coincide if $\lambda \in I$ [6].

It is clear from these two remarks that, as $\lambda_n \to \lambda_0$ in I, the limits

$$\lim_{n\to\infty} l_\omega^\pm(\lambda_n)$$

exist in $\mathbf{P}^1(\mathbf{R})$. Call these limits $l_\omega^\pm$. The sets

$$S^\pm = \{(\omega, l_\omega^\pm) \mid \omega \in \Omega\} \subset \Sigma_{\mathbf{Re}}$$

are measurable sections of $\Sigma = \Omega \times \mathbf{P}^1(\mathbf{R})$. Hence they define ergodic measures $\mu^\pm$ on Σ via the formulas

$$\int_\Sigma g \, d\mu^\pm = \int_\Omega g(\omega, l_\omega^\pm) \, d\omega$$

for continuous $g\colon \Sigma \to \mathbf{R}$. We have

$$-\beta(\lambda_n) = \int_\Sigma f_{\lambda_n}(\omega, M_\omega(\lambda_n)) \, d\omega \to \int_\Sigma f_{\lambda_0}(\omega, l_\omega^+) \, d\omega = \int_\Sigma f_{\lambda_0} \, d\mu^+.$$

Using 2.15, we have $\int_\Sigma f_{\lambda_0} \, d\mu^+ = -\beta(\lambda_0)$. This completes the proof of 3.6.

4. Discontinuity of β

We construct a.p. Schrödinger operators for which β is discontinuous. Note that such examples cannot be periodic, for β is always continuous for a periodic Schrödinger operator, and in fact the essential spectrum is determined by the condition $\beta = 0$.

We begin with some general remarks. Let $\{q_n\}$ be a sequence of almost periodic function such that $q_n(t) \to q(t)$ uniformly on $\mathbf{R}$. Then $q(t)$ is almost periodic. Consider the operators

$$L(q_n) = \frac{-d^2}{dt^2} + q_n(t), \quad L(q) = \frac{-d^2}{dt^2} + q(t).$$

From [7, see Section 2 above], we obtain corresponding functions $w(q_n, \lambda)$, $w(q, \lambda)$, holomorphic for $\operatorname{Im} \lambda > 0$, such that

$$\lim_{\varepsilon \to 0^+} w(q_n, \lambda + i\varepsilon) = -\beta(q_n, \lambda) + i\alpha(q_n, \lambda),$$
$$\lim_{\varepsilon \to 0^+} w(q, \lambda + i\varepsilon) = -\beta(q, \lambda) + i\alpha(q, \lambda) \quad (\lambda \in \mathbf{R}).$$

Here β is defined in 2.3, and α is the rotation number. From now on, it will be convenient to indicate the potential q in the arguments of β and α.

From now on, we use the term "resolvent of $L(q)$" to mean the operator-theoretic resolvent of $L(q)$, viewed as a self-adjoint operator on $L^2(-\infty, \infty)$.

4.1 Proposition. *If $q_n \to q$ uniformly on $\mathbf{R}$, then*

$$\alpha(q_n, \lambda) \to \alpha(q, \lambda),$$

uniformly on compact subsets of $\mathbf{R}$. If $I = (a, b) \subset \mathbf{R}$ is a subset of the resolvent of $L(q)$, then $\beta(q_n, \lambda) \to \beta(q, \lambda)$ for all $\lambda \in I$.

Proof. The first statement is proved in [7, Theorem 6.2]. To prove the second statement, we use 6.3 and 6.4 of [7] to conclude that $w(q, z)$ is continuous as a function of q for fixed z, $\operatorname{Im} z > 0$ (in fact, it is differentiable). Hence $w(q_n, z) \to w(q, z)$ if $\operatorname{Im} z > 0$. Now, if $\lambda \in I$, then some interval I_1 containing λ is in the resolvent of $L(q_n)$ for sufficiently large n, say $n \geq N$. Since the rotation number is constant on intervals in the resolvent [7, Theorem 4.7], we can extend $w(q_n, \lambda)$ and $w(q, \lambda)$ holomorphically through I_1 if $n \geq N$, and it follows easily that $\beta(q_n, \lambda) \to \beta(q, \lambda)$.

We remark that, if λ is in the resolvent of $L(q)$, then $\alpha(q_n, \lambda)$ is eventually equal to $\alpha(q, \lambda)$ if $q_n \to q$ uniformly. This uses [7, Theorem 4.7].

Now we borrow two facts from [6]. First, we fix a constant δ:

$$\delta = 2^{-10} = 1/1024. \tag{4.2}$$

(I) There is a periodic function $q_0(t)$, of period $T_0 > 10$, with

$$1 \geq q_0(t) \geq -3 \quad (t \in \mathbf{R}), \tag{4.3}$$

such that the Hill equation

$$H(q_0)\colon u' = \begin{pmatrix} 0 & 1 \\ q_0(t) & 0 \end{pmatrix} u, \quad u = \begin{pmatrix} \phi \\ \phi' \end{pmatrix},$$

admits two solutions, $u_\pm$, (t), with the following properties. First,

$$\frac{1}{T_0} \ln \|u_\pm(T_0)\| = \pm\beta_0, \quad \beta_0 > 2/3. \tag{4.4}$$

If we introduce polar coordinates

$$r = (\phi^2 + \phi'^2)^{1/2}, \quad \theta = \arg(\phi + i\phi'),$$

then $H(q_0)$ becomes

$$\mathscr{R}(q_0)\colon r'/r = (1 + q_0(t)) \cos\theta \sin\theta,$$

$$\Theta(q_0)\colon \theta = -\sin^2\theta + q_0(t)\cos^2\theta.$$

Writing $u_\pm(t) = r_\pm(t) \exp i\theta_\pm(t)$, we have

$$-\pi/4 < \theta_-(0) = \theta_-(T_0) < \theta_+(0) = \theta_+(T_0) < -\pi/4 + 2\delta; \tag{4.5}$$

$$0 < \theta_+(0) - \theta_-(0) < 2^{-12} = 1/4096. \tag{4.6}$$

Using Floquet theory, it is trivial to see that $\beta_0 = \beta(q_0, 0)$. Observe also the rotation number $\alpha(q_0, \lambda)$ satisfies

$$\alpha(q_0, 0) = 0, \quad \alpha(q_0, 2) \geq 1, \tag{4.7}$$

since $q_0(t) \leq 1$ for all t.

(II) Suppose a T_N-periodic function $q_N(t)$ is given ($N \geq 0$), with the following properties.

(i) The Hill equation $H(q_N)$ admits two solutions, $u_\pm(q_N, t)$, with

$$\frac{1}{T_N} \ln \|u_+(q_N, T_N)\| > 2/3, \quad \frac{1}{T_N} \ln \|u_-(q_N, T_N)\| < -2/3. \tag*{(4.4)$_N$}$$

(ii) If $u_\pm(q_N, t) = r_\pm(q_N, t) \exp i\theta_\pm(q_N, t)$, then

$$\begin{aligned} -\pi/4 < \theta_-(q_N, 0) &= \theta_-(q_N, T_N) \\ &< \theta_+(q_N, 0) \\ &= \theta_+(q_N, T_N) \\ &< -\pi/4 + 2\delta; \end{aligned} \tag*{(4.5)$_N$}$$

$$0 < \theta_+(q_N, 0) - \theta_-(q_N, 0) < 2^{-N-12}. \tag*{(4.6)$_N$}$$

Then, there is an integer l_N, which may be chosen as large as desired, and a non-negative function $\eta_N(t)$, periodic of period $T_{N+1} = l_N T_N$, satisfying

$$(4.8)_{N+1} \qquad 0 \leq \eta_N(t) \leq 3/2\ 2^{-N-2} < 2^{-N-1},$$

$$\text{Support } (\eta_N) \cap [0, T_{N+1}] \subset [T_{N+1} - \delta, T_{N+1}],$$

such that, if $p_{N+1} = q_N - \eta_N$, then the Hill equation $H(p_{N+1})$ admits solutions

$$u_\pm(p_{N+1}, t) = r_\pm(p_{N+1}, t) \exp i\theta_\pm(p_{N+1}, t)$$

for which $(4.4)_{N+1}$, $(4.5)_{N+1}$, and $(4.6)_{N+1}$ hold. In addition,

$$(4.9)_{N+1} \qquad \theta_-(q_N, 0) < \theta_-(p_{N+1}, 0) < \theta_+(p_{N+1}, 0) < \theta_+(q_N, 0).$$

We remark that (I) and (II) can be achieved by replacing the integer n in (6, Section 5] by $N = n - 12$, $n \geq 12$.

We now construct, by induction, a sequence $\{q_N\}$ of periodic functions such that q_N converges uniformly to an almost periodic function q. We will show that q has the following properties: $\lambda = 0$ is the left endpoint of the spectrum of $L(q)$; also

$$\beta(q, 0) \geq 2/3; \ \lim_{\lambda \to 0^+} \inf \beta(q, \lambda) = 0.$$

Let us begin by letting $q_0(t)$ be a T_0-periodic function satisfying the conditions of (I). Let $T_1 = m_0 T_0$, $\eta_0(t)$, and $p_1 = q_0 - \eta_0$ be as in (II). By a theorem of Moser (10, Proposition 1; note $\Phi_1 > 0$], we can find a non-negative, continuous, T_1-periodic function $\sigma_0(t)$, such that

$$0 \leq \eta_0 + \sigma_0 < 2^{-2} = 1/4 \quad \text{and} \quad \text{Support } (\sigma_0) \cap [0, T_1) \subset [T_1 - \delta, T_1],$$

with the following additional property. If $q_1 = p_1 - \sigma_0$, then, corresponding to each such integer $k \geq 1$, there is a non-empty open interval $I_1(k)$ in the resolvent of

$$L(q_1) = \frac{-d^2}{dt^2} + q_1(t)$$

such that the rotation number $\alpha(q_1, \lambda)$ equals $k\pi/T_1$ on $I_1(k)$. We take $I_1(k)$ to be the maximal open interval with this property. Since we can choose σ_0 as small as we please, we can also ensure that conditions $(4.4)_1$, $(4.5)_1$, $(4.6)_1$, and $(4.9)_1$ hold.

Consider an interval $I_1(k) = (a, b)$ $(1 \leq k < T_1/\pi)$. Using (4.3) and the fact that $q_1 \leq q_0$, we see that $(a, b) \subset (0, 2)$. For each $\lambda \in (a, b)$, consider the Hill equation

$$H(-\lambda + q_1)\colon u' = \begin{pmatrix} 0 & 1 \\ -\lambda + q_1(t) & 0 \end{pmatrix} u, \quad u = \begin{pmatrix} \phi \\ \phi' \end{pmatrix}.$$

This equation has solutions

$$u_\pm(-\lambda + q_1, t) = r_\pm(-\lambda + q_1, t) \exp i\theta_\pm(-\lambda + q_1, t)$$

for which

$$\frac{1}{T_1} \ln \|u_\pm(-\lambda + q_1, T_1)\| = \pm\beta(q_1, \lambda) \neq 0,$$

and for which

$$\theta_\pm(-\lambda + q_1, 2T_1) = \theta_\pm(-\lambda + q_1, 0) \pmod{2\pi}.$$

Note that, by [2, probs. 8 and 9, p. 257] $\theta_\pm$ may be chosen to be differentiable in λ, and when this is done,

$$\frac{\partial\theta_+}{\partial\lambda} < 0, \quad \frac{\partial\theta_-}{\partial\lambda} > 0$$

for each fixed t. This fact, and analysis of the discriminant [8] $\Delta(\lambda)$ of

$$L(q_1) = \frac{-d^2}{dt^2} + q_1(t),$$

show that one can further assume

$$\lim_{\lambda\to a^+} \theta_+(-\lambda + q_1, t) - \theta_-(-\lambda + q_1, t) = \pi,$$

$$\lim_{\lambda\to b^-} \theta_+(-\lambda + q_1, t) - \theta_-(-\lambda + q_1, t) = 0$$

for all $t \in \mathbf{R}$. Hence we can choose a closed subinterval $J_1(k) \subset I_1(k)$, with int $J_1(k) \neq \phi$, such that if $\lambda \in J_1(k)$, then

$$\pi > \theta_+(-\lambda + q_1, 0) - \theta_-(-\lambda + q_1, 0) > \pi/2 - 1/2.$$

Now suppose that we have constructed functions $q_1 \geq q_2 \geq \cdots \geq q_N$ such that q_i has period $T_i = m_{i-1} T_{i-1}$ for some even integer $m_{i-1} \geq 6$ ($i = 0, 1, N - 1$). Suppose that $(4.4)_i$–$(4.6)_i$ hold for all i, $1 \leq i \leq N$. Suppose moreover that the following conditions hold.

$(4.10)_N$ $\quad 0 \leq q_i - q_{i+1} \leq 2^{-i-1}(1 \leq i \leq N - 1);$

$(4.11)_N$ $\quad 1 \geq q_i \geq -3 - \sum_{l=1}^{i} 2^{-l}(1 \leq i \leq N);$

$(4.12)_N$ for each $1 \leq i \leq N$, there is a non-empty open interval $I_i(k)$ ($1 \leq k < \infty$) such that $I_i(k)$ is in the resolvent of $L(q_i)$, and $\alpha(q_i, \lambda) = k\pi/T_i$ for all $\lambda \in I_i(k)$.

Using (4.7) and the fact that q_i decreases with i, we see that $I_i(k) \subset (0, 2)$ for $1 \leq k < T_i/\pi$, $i \leq i \leq N$. We assume that $I_i(k)$ is the *maximal* interval with the property stated in $(4.12)_N$, and call it a *spectral gap*.

Now, in each spectral gap $I_i(k) = (a, b)$, we can choose differentiable families $\theta_\pm(-\lambda + q_i, t)$ of $2T_i$-periodic solutions (mod 2π) of $\Theta(-\lambda + q_i)$ such that $\partial\theta_+/\partial\lambda < 0$, $\partial\theta_-/\partial\lambda > 0$, and

$$\lim_{\lambda\to a} \theta_+(-\lambda + q_i, t) - \theta_-(-\lambda + q_i, t) = \pi,$$

$$\lim_{\lambda\to b} \theta_+(-\lambda + q_i, t) - \theta_-(-\lambda - q_i, t) = 0 \quad (\lambda \in I_i(k), t \in \mathbf{R}).$$

We assume the following three conditions.

$(4.13)_N$ In each spectral gap $I_i(k)$ $(1 \leq k < T_i/\pi)$, there is a closed subinterval $J_i(k)$, with int $J_i(k) \neq \phi$, such that, if $\lambda \in J_i(k)$, then

$$\pi > \theta_+(-\lambda + q_i, 0) - \theta_-(-\lambda + q_i, 0) > \pi/2 - \sum_{l=r}^{i} 2^{-l},$$

where r is the smallest integer such that $k\pi/T_i = h\pi/T_r$ for some integer h $(1 \leq i \leq N)$.

$(4.14)_N$ If $k/T_i = h/T_r$ for some $r < i$ and some integer h, then $J_r(h) = J_i(k)$, and $\alpha(q_r, \lambda) = \alpha(q_i, \lambda)$ for all $\lambda \in J_r(h)$ $(1 \leq i \leq N)$.

$(4.15)_N$ If $1 \leq r \leq i \leq N$ and $\lambda \in J_r(4)$, then

$$\beta(q_r, \lambda) < 2^{-r} \quad \text{and} \quad \beta(q_i, \lambda) < \sum_{l=r}^{i} 2^{-l}.$$

We will construct a $T_{N+1} = m_N T_N$-periodic function q_{N+1} with $m_N \geq 6$ such that $(4.2)_{N+1}$–$(4.4)_{N+1}$ hold, and so that $(4.10)_{N+1}$–$(4.15)_{N+1}$ hold.

Begin by choosing a number $\tau_0 \geq T_N$ such that, if $u(t)$ is any solution of $H(-\lambda + q_N)$ $(0 \leq \lambda \leq 2)$ satisfying $\|u(0)\| = 1$, then

$$\frac{1}{t} \ln \|u(t)\| < \beta(q_N, \lambda) + 2^{-N-2}(t \geq \tau_0). \tag{4.16}$$

For completeness, we include a proof that τ_0 can be so chosen in an appendix.

Next, fix a number $\gamma \in (0, 1)$, which will be more precisely determined later. Choose some interval $J_N(k)$, $1 \leq k < T_N/\pi$. Let $\lambda \in J_N(k)$. Let

$$\theta_1 = \theta_-(-\lambda + q_N, 0) < \theta_+(-\lambda + q_n, 0) = \theta_2 < \theta_1 + \pi.$$

Here $\theta_\pm$ are given by the discussion preceding $(4.13)_N$. Let $\tau_1 = cT_N$, where c is a positive *even* integer to be determined. Define

$$H_\lambda\colon [\theta_1, \theta_2] \to [\theta_1, \theta_2)$$

by

$$H_\lambda(\bar{\theta}) = \theta(\tau_1) - \theta(0) + \theta_1 + \tau_1 \cdot \frac{\pi k}{T_N}$$

(recall $\underline{\alpha}(q_N, \lambda) = \pi k/T_N$). Here $\theta(t)$ is the solution of $\Theta(-\lambda + q_N)$ satisfying $\theta(0) = \bar{\theta}$. We see that H_λ measures the rotation of $\theta(t)$ with respect to $\theta_-(-\lambda + q_N, t)$. Note that $H_\lambda(\theta_1) = \theta_1$, $H_\lambda(\theta_2) = \theta_2$.

Since $\beta(q_N; \lambda) > 0$, one has

$$\lim_{m \to \infty} \theta(2m \cdot T_N) = \theta_+(-\lambda + q_N, 0) = \theta_2 \quad \text{for all } \bar{\theta},\ \theta_1 < \bar{\theta} < \theta_2.$$

Since the families $\theta_\pm$ are continuous in λ, and since the right-hand side in equation $\Theta(-\lambda + q_N)$ depends continuously on λ, we can find τ_1 so that

$$H_\lambda(\theta) > \theta_2 - \gamma \quad \text{if} \quad \theta_1 + \gamma \le \theta < \theta_2 \quad \text{and} \quad \lambda \in \bigcup_{k=1}^{A} J_N(k), \tag{4.17}$$

where A is the greatest integer less than T_N/π.

Now fix $\gamma < 2^{-N-2}$. Choose $m_N \ge 6$ such that m_N is even and

$$T_{N+1} = m_N \cdot T_N > \max(\tau_0, \tau_1).$$

Choose a T_{N+1}-periodic function $\eta_N(t)$, for which $(4.7)_{N+1}$ holds, in such a way that $(4.4)_{N+1}$–$(4.6)_{N+1}$ hold for $p_{N+1} = q_N - \eta_N$. Then, use the Moser theorem [10, Proposition 1] to find a non-negative function $\sigma_N(t)$, with period T_{N+1}, such that

$$\text{Support}(\sigma_N) \cap [0, T_{N+1}] \subset [T_{N+1} - \delta, T_{N+1}] \quad \text{and} \quad 0 \le \eta_N + \sigma_N < 2^{-N-1},$$

so that the following conditions hold.

(i) Conditions $(4.4)_{N+1}$–$(4.6)_{N+1}$ hold with $q_{N+1} = p_{N+1} - \sigma_N$ in place of p_{N+1}.

(ii) The operator $L(q_{N+1})$ admits a spectral gap $I_{N+1}(k)$ such that, if $\lambda \in I_{N+1}(k)$, then

$$\alpha(q_{N+1}, \lambda) = \frac{k\pi}{T_{N+1}} \quad (1 \le k < \infty).$$

Observe now that $(4.10_{N+1}$, $(4.11)_{N+1}$, and $(4.12)_{N+1}$ hold. We show that $(4.13)_{N+1}$, $(4.14)_{N+1}$, and $(4.15)_{N+1}$ hold with our choices of m_N and q_{N+1}.

First, fix k, $1 \le k < \pi/T_{N+1}$. If k is not a multiple of m_N, i.e., if $k/T_{N+1} \ne h/T_r$ for all integers h and all $r < N + 1$, then let $J_{N+1}(k)$ be any closed subinterval of $I_{N+1}(k)$ with non-empty interior on which

$$\theta_+(-\lambda + q_{N+1}, 0) - \theta_-(-\lambda + q_{N+1}, 0) > \pi/2 - \tfrac{1}{2}.$$

Here we shoose $\theta_\pm$ as in the discussion preceding $(4.13)_N$.

Next, suppose $k/T_{N+1} = h/T_N$ for some integer h. We examine $J_N(h)$. Let $\lambda \in J_N(h)$, and consider again the interval $[\theta_1, \theta_2]$, where

$$\theta_1 = \theta_-(-\lambda + q_N, 0), \quad \theta_2 = \theta_+(-\lambda + q_N, 0).$$

Define the map

$$H_\lambda\colon [\theta_1, \theta_2] \to [\theta_1, \theta_2]$$

as above, with T_{N+1} replacing T_N. Let $\bar{\theta} \in [\theta_1, \theta_2]$. Let $\theta_N(t)$, resp. $\theta_{N+1}(t)$, be the solution of $\Theta(-\lambda + q_n)$, resp. $\Theta(-\lambda + q_{N+1})$, with

$$\theta_N(0) = \theta_{N+1}(0) = \bar{\theta}.$$

Then $\theta_N(t) = \theta_{N+1}(t)$ on $[0, T_{N+1} - \delta]$. By Gronwall's inequality applied to the θ-equation, and using $|-\lambda + q_N| < 6$, $|-\lambda + q_{N+1}| < 6$ (this uses (4.7)), we have

$$0 < \theta_N(T_{N+1}) - \theta_{N+1}(T_{N+1}) < \delta 2^{-N-1} e^{6\delta} < 2^{-N-2}. \tag{4.18}$$

Let us now define

$$R(\bar{\theta}) = \theta_N(T_{N+1}) - \theta_{N+1}(T_{N+1}) \quad \text{for } \bar{\theta} \in [\theta_1, \theta_2].$$

Compare the graphs of H_λ and R. Using $\gamma < 2^{-N-2}$, we see that these graphs have exactly two points of intersection, defined by points $\psi_\pm$ in $[\theta_1, \theta_2]$; moreover

$$0 < \theta_+ - \psi_+ < 2^{-N-2}, \quad 0 < \psi_- \theta_- < 2^{-N-1}.$$

There are no more than two points of intersection, because: (i) any point $\bar{\theta} \in (\theta_1 + \gamma, \theta_2 - 2^{-N-1})$ satisfies $H_\lambda(\bar{\theta}) > R(\bar{\theta})$; (ii) if there were three points of intersection, then that fundamental matrix solution $\Phi(t)$ of $H(-\lambda + q_{N+1})$ satisfying $\Phi(0) = I$ would preserve three driections in $\mathbf{R}^2$ at $t = T_{N+1}$; hence $\Phi(T_{N+1})$ would be the identity since $\det \Phi(t) = 1$; this would contradict (i).

Let $\theta_\pm(-\lambda + q_{N+1}, t)$ be the solutions of $\Theta(-\lambda + q_{N+1})$ satisfying

$$\theta_\pm(-\lambda + q_{N+1}, 0) = \psi_\pm.$$

Then $\exp i\theta_\pm$ are T_{N+1}-periodic, and $\theta_\pm + \pi$ are (mod 2π) the only other solutions of $\Theta(-\lambda + q_{N+1})$ with this property. It follows from Floquet theory that λ is in the resolvent of $L(q_{N+1})$. Clearly

$$\alpha(q_{N+1}, \lambda) = \alpha(q_N, \lambda).$$

Now set $J_{N+1}(k) = J_N(h)$ (recall $k/T_{N+1} = h/T_N$). From all that we have said, $(4.13)_{N+1}$ and $(4.14)_{N+1}$ hold.

We must still consider $(4.15)_{N+1}$. First, let λ_N be the left endpoint of the spectrum of $L(q_N)$. Note $\alpha(q_N, \lambda_N) = 0$. We claim that $\alpha(q_{N+1}, \lambda_N) \leq \pi/T_{N+1}$. To see this, note $\Theta(-\lambda_N + q_N)$ has a T_N-periodic solution $\psi_N(t)$. Let $\psi_{N+1}(t)$ satisfy

$$\Theta(-\lambda_N + q_{N+1}) \quad \text{with } \psi_{N+1}(0) = \psi_N(0).$$

From (4.7), $0 < \lambda_N < 2$; hence we can apply Gronwall's inequality to the Θ-equation and obtain

$$0 < \psi_N(T_{N+1}) - \psi_{N+1}(T_{N+1}) < \delta 2^{-N-1} e^{6\delta} < 2^{-N}.$$

Since $q_N = q_{N+1}$ on $[T_{N+1}, 2T_{N+1} - \delta]$, we see that

$$\psi_N(t) - \pi < \psi_{N+1}(t) < \psi_N(t) \quad \text{for } t \in [T_{N+1}, 2T_{N+1} - \delta];$$

hence applying Gronwall again, we get

$$-\pi - 2^{-N} + \psi_N(0) < \psi_{N+1}(2T_{N+1}) < \psi_N(0) = \psi_N(2T_{N+1}).$$

Applying a similar argument to all succeeding periods, we obtain

$$0 < \psi_N(0) - \psi_{N+1}(l \cdot T_{N+1}) < (l-1)\pi + 2^{-N} \quad \text{for } l = 1, 2, 3, \ldots,$$

Hence $\alpha(q_{N+1}, \lambda_N) \leq \pi/T_{N+1}$ (by Theorem 2.13).

Now, on the other hand, if $\lambda \in J_N(k)$, then

$$\alpha(q_{N+1}, \lambda) = \alpha(q_N, \lambda) \quad (1 \leq k < \pi/T_N).$$

In particular, if $\lambda \in J_N(1)$, then $\alpha(q_{N+1}, \lambda) = \pi/T_N$. Combining this fact with the preceding paragraph, and recalling that $m_{N+1} \geq 6$, we see that, if $\lambda \in J_{N+1}(4)$ (i.e. if $\alpha(q_{N+1}, \lambda) = 4\pi/T_{N+1}$), then $\beta(q_N, \lambda) = 0$.

Let us fix $\lambda \in J_{N+1}(4)$, and let

$$\xi_{N+1}(t) = \theta_+(-\lambda + q_{N+1}, t).$$

Let $r_{N+1}(t)$ be obtained by solving equation $\mathscr{R}(-\lambda + q_{N+1})$ with ξ_{N+1} replacing θ, with initial condition $r(0) = 1$. If

$$u_{N+1}(t) = r_{N+1}(t) \exp i\xi_{N+1}(t),$$

then

$$\frac{1}{T_{N+1}} \ln \|u_{N+1}(T_{N+1})\| = \beta(q_{N+1}, \lambda).$$

Let $\xi_N(t)$ be the solution of $\Theta(-\lambda + q_N)$ satisfying $\xi_N(0) = \xi_{N+1}(0)$. Let

$$u_N(t) = r_N(t) \exp i\xi_N(t),$$

where r_N is obtained as above from $\mathscr{R}(-\lambda + q_N)$. Applying Gronwall's inequality to estimate $\xi_N(t) - \xi_{N+1}(t)$, then comparing $\mathscr{R}(-\lambda + q_N)$ and $\mathscr{R}(-\lambda + q_{N+1})$, we obtain

$$\frac{1}{T_{N+1}} |\ln r_N(T_{N+1}) - \ln r_{N+1}(T_{N+1})| < 2^{-N-2}.$$

Using 4.16, we get $\beta(q_{N+1}, \lambda) < 2^{-N-1}$(recall $T_{N+1} > \tau_1$).

By a similar argument, we get

$$\beta(q_{N+1}, \lambda) < \sum_{l=r}^{N+1} 2^{-l} \quad \text{if } \lambda \in J_r(4),\ r \le N+1.$$

Hence $(4.15)_{N+1}$ is finally verified.

By induction, construct a sequence $\{q_N\}$ satisfying $(4.2)_N$–$(4.4)_N$ and $(4.10)_N$–$(4.15)_N$ for each $N \ge 1$. Then $q_N \to q$ uniformly on $\mathbf{R}$, where q is almost periodic. Let $\rho_N(t)$, resp. $\rho(t)$, be the spectral function of the singular boundary value problem

$$L(q_N)\phi = \lambda\phi, \quad \text{resp. } L(q)\phi = \lambda\phi,$$

with boundary conditions

$$\phi(0) = 0,\ \phi \in L^2(0, \infty).$$

Using the Helly theorem, one can show that $\rho_N \to \rho$ at all continuity points of ρ.

The arguments of [6, Section 5] show that $\lambda = 0$ is the left-most point in the spectrum of $L(q)$, and also that $\beta(q, 0) \ge 2/3$. Now, ρ_N is constant on each $J_N(k)$ $(1 \le k < T_N/\pi)$, except perhaps for a single isolated jump discontinuity (a discontinuity occurs if and only if $\theta_-(-\lambda + q_N, 0) = \pi/2$ or $3\pi/2 \bmod 2\pi$). By $(4.14)_N$, ρ is also constant on int $J_N(k)$, except perhaps for an isolated jump discontinuity. Hence int $J_N(k)$ is in the resolvent of $L(q)$ [3]. From 4.1 and $(4.15)_N$, we see that $\beta(q, \lambda) < 2^{-N+1}$ on $J_N(4)$. Since $\alpha(q_N, \lambda) \to \alpha(q, \lambda)$, and since $\alpha(q_N, \lambda) = 4\pi/T_N$ for $\lambda \in J_N(4)$, we conclude from $(4.14)_N$ that $\alpha(q, \lambda) = 4\pi/T_N$ for $\lambda \in J_N(4)$. Since $\alpha(q, \lambda)$ is continuous and $\alpha(q, \lambda) > 0$ for $\lambda > 0$ (see Theorem 2.13), we see that

$$\liminf_{\lambda \to 0^+} \beta(q, \lambda) = 0.$$

We have proved everything that we set out to prove.

Appendix

We prove statement 4.16. Let q be any continuous periodic function, and consider the equations

$$H(-\lambda + q)\colon u' = \begin{pmatrix} 0 & 1 \\ -\lambda + q(t) & 0 \end{pmatrix} u,$$

where λ ranges over some compact interval $I \subset \mathbf{R}$. Introduce the hull Ω of q; since q is periodic, Ω is a circle. We denote the element q of Ω by ω_0. Let $\Sigma_{\mathbf{Re}}$ be the projective bundle $\Sigma_{\mathbf{Re}} = \Omega \times \mathbf{P}^1(\mathbf{R})$. As in Section 2, each Hill equation $H(-\lambda + q)$ defines a flow $(\Sigma_{\mathbf{Re}}, \mathbf{R})_\lambda$ $(\lambda \in I)$.

Suppose for contradiction that there exist $\gamma > 0$ and sequences $\lambda_n \in I$, $t_n \in \mathbf{R}$, $\bar{u}_n \in \mathbf{R}^2$, with $t_n \to \infty$ and $\|\bar{u}_n\| = 1$, so that

$$\frac{1}{t_n} \ln \|u_n(t_n)\| \geq \beta(q, \lambda_n) + \gamma;$$

here $u_n(t)$ satisfies $H(-\lambda + q)$ with $u_n(0) = \bar{u}_n$. Let l_n be the line in $\mathbf{R}^2$ containing $\bar{u}_n$. Let $f_\lambda\colon \Sigma \to \mathbf{R}$ be the function defined in 2.4 ($\lambda \in I$).

Using the Cantor diagonal process as in [11, Theorem 9.05], we can find measures μ_n on Σ such that $\|\mu_n\| = 1$, and

$$\int_\Sigma f_{\lambda_n}\, d\mu_n = \frac{1}{t_n} \int_0^{t_n} f_{\lambda_n}((\omega_0, l_n) \cdot s)\, ds,$$

where $(\omega_0, l_n) \cdot s$ is computed using the flow $(\Sigma, \mathbf{R})_{\lambda_n}$.

Then

$$\int_\Sigma f_{\lambda_n}\, d\mu_n = \frac{1}{t_n} \ln \|u_n(t_n)\| \geq \beta(q, \lambda_n) + \gamma.$$

We can assume that $\lambda_n \to \lambda_0$, and that $\mu_n \to \mu$ in the weak topology on measures. Since $t_n \to \infty$, μ is *invariant* with respect to $(\Sigma, \mathbf{R})_\lambda$.

Now, $f_{\lambda_n} \to f_\lambda$ uniformly on Σ. Hence

$$\int_\Sigma f_\lambda\, d\mu = \lim_{n\to\infty} \int_\Sigma f_{\lambda_n}\, d\mu_n \geq \overline{\lim_{n\to\infty}}\, \beta(q, \lambda_n) + \gamma.$$

Since $\beta(q, \lambda_n) \geq 0$ and $\beta(q, \lambda) \geq \int_\Sigma f_\lambda\, d\mu$ (this uses (2.16)), we see that $\beta(q, \lambda) > 0$. But then, by Floquet theory, λ is in the resolvent of

$$L(q) = \frac{-d^2}{dt^2} + q(t);$$

see [8]. By Proposition 4.1,

$$\beta(q, \lambda_n) \to \beta(q, \lambda) \geq \overline{\lim_{n\to\infty}}\, \beta(q, \lambda_n) + \gamma.$$

This is a contradiction; (4.16) is proved.

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