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MANIFOLDS THAT ADMIT PARALLEL VECTOR FIELDS[1]

BY

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1. Introduction

In a survey paper on G-structures in differential geometry [2], S.S. Chern posed the question of when there is a non-zero vector field on a compact manifold, that is parallel with respect to a Riemmanian metric. He observes that the first two betti numbers must satisfy

$$b_1 \geq 1 \quad \text{and} \quad b_2 \geq b_1 - 1,$$

and then conjectures that these conditions are not sufficient.

Further conditions on the betti numbers were given by Leon Karp [4], where he also gave an example of a manifold that satisfied Chern's criterion above, plus Bott's condition that the Pontryagin number vanish, yet admitted no parallel vector fields under any metric.

Let M be a compact, connected manifold. The main aim of this paper is to describe topologically those M which carry a nontrivial vector field that is parallel with respect to a Riemannian metric. The simplest examples of such manifolds are tori. The next simplest are Cartesian products of tori with arbitrary manifolds. The principal result is that up to a finite cover, these are all the possibilities. Sections 2 and 3 are devoted to proving this theorem:

THEOREM 1. *Let M be a compact, connected manifold. Then the following are equivalent*:

(a) *M has a vector field that is parallel with respect to some Riemannian metric.*

(b) *Under a suitable metric, M has a Killing vector field v and a harmonic 1-form α such that $\alpha(v) \neq 0$.*

(c) *M is a fibre bundle over a torus, with finite structural group.*

We have $(a) \Rightarrow (b)$ of course, since parallel vector fields are precisely those that are both Killing and harmonic [9]. Section 2 shows that $(b) \Rightarrow (c)$, and

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[1] This paper is an excerpt from the author's doctoral dissertation done at the University of Notre Dame, under the supervision of Professor Tadashi Nagano.

Section 3 proves that $(c) \Rightarrow (a)$. Section 4 discusses ramifications, especially in regards to the cohomology. Section 5 considers the Riemannian case.[2]

2. Construction of the fibre bundle

Let M be a connected, compact Riemannian manifold. It is a well known fact that $\alpha(v)$ is a constant function of M, whenever v is a Killing vector field and α is a harmonic 1-form. Indeed,

$$0 = \mathscr{L}_v \alpha = d \circ \iota_v(\alpha) + \iota_v \circ d\alpha = d(\alpha(v)),$$

where $\mathscr{L}_v$ is the Lie derivative.

Notation. $H_R(X)$ (or just H_R when $X = M$) will denote the vector space of harmonic 1-forms on the manifold X. Similar notation, with Z replacing R, will be used for the subset whose elements yield integral values when integrated on closed curves.

LEMMA 1. *H_Z is a lattice group of rank* $=$ dim $H_R < \infty$.

Proof. If M is orientable, H_R can be identified with the deRham cohomology, and the lemma holds. If M is unorientable, there is a double cover

$$\overline{M} \xrightarrow{\zeta} M$$

which is orientable. ζ is a local isometry, once $\overline{M}$ is given the pullback metric. Thus ζ^* can be thought of as a map from $H_R M$ to $H_R \overline{M}$. Clearly, it is an injection. Hence $H_R M$ is finite dimensional. Furthermore, $\zeta^* H_Z M \subset H_Z \overline{M}$.

Let τ be the nontrivial deck transformation of the cover. τ^2 is the identity map on M, and so decomposes $H_R \overline{M}$ into the direct sum of the $+1$ eigenspace $H_R^+ \overline{M}$ and the -1 eigenspace $H_R^- \overline{M}$. Note that $H_R^+ \overline{M} = \zeta^* H_R M$, i.e. the harmonic 1-forms on $\overline{M}$ that are invariant under τ are precisely those that are pullbacks of 1-forms on M.

It does not follow that $H_Z \overline{M}$ admits such a decomposition. However, one can consider the sublattice H_2, consisting of all elements of the form 2α, where α is in $H_Z \overline{M}$. Write

$$2\alpha = (\alpha + \tau^*\alpha) + (\alpha - \tau^*\alpha).$$

[2] I would like to thank the referee for pointing out that a weaker formulation of (a) $\Rightarrow$ (c) follows from a result of D. Tischler (Topology, vol. 9, pp. 153–154).

The first term lies in $H_Z^+ := H_R^+\overline{M} \cap H_Z\overline{M}$, and the second term lies in $H_Z^- := H_R^-\overline{M} \cap H_Z\overline{M}$.

H_2 has the same rank as $H_Z\overline{M}$, hence H_Z^+ is a lattice of maximal rank in $H_R^+\overline{M}$. Furthermore, $H_Z^+ = \zeta^* H_Z M$. Thus

$$\text{rank } H_Z M = \text{rank } H_Z^+ = \dim H_R^+\overline{M} = \dim H_R M. \qquad \blacksquare$$

For the rest of this section, assume the existence of a killing vector field v, so that $\alpha(v)$ is non-zero for some harmonic 1-form α. P will denote the 1-parameter group of isometries generated by v. The Riemannian metric on M will be denoted by $\langle \ , \ \rangle$. As usual, $I_0(M)$ is the identity component of the group of isometries of M.

Let A be the usual Albanese torus H_R/H_Z. The Albanese map may be defined as follows: choose a basis $\alpha_1, \ldots, \alpha_n$ for H_Z. Then any 1-form $\Sigma a^j\alpha_j$ in H_R can be identified with a n-tuple $(a^1, \ldots, a^n)$. For brevity, this may be denoted $(a^j)_{j=1,\ldots n}$ or even $(a^j)_j$. The set of these n-tuples may be considered modulo Z^n. On the form level, this corresponds to taking sums modulo $\alpha_1, \ldots, \alpha_n$.

Now fix a point e in M. Then given an arbitrary point y in M, let η be a path from e to y. Consider the n-tuple $(a^j)_j$ where

$$a^j = \int_\eta \alpha_j, \quad j = 1, \ldots, n.$$

The n-tuple only depends on the homotopy class of η with fixed endpoints. However, these classes differ by closed curves, on which the α_j give integral values when integrated. So the n-tuple is uniquely determined mod Z^n. Define $f_1(y) := (a^j)_j$. For the sake of convenience, the following notations are used:

$$f_1(y) = \left(\int_\eta \alpha_j\right)_{j=1,\ldots,n} \equiv \left(\int_e^y \alpha_j\right)_j \equiv \Sigma\left(\int_\eta \alpha_j\right)\alpha_j (\text{mod } \alpha_j).$$

DEFINITION. C is the closure in $I_0(M)$ of the group P. C is a torus since the isometry group of a compact manifold is itself compact.

Notation. $h_1(c) := f_1(c(e))$, where c is in C.

LEMMA 2. *$h_1(\exp v) = (\alpha_j(v))_j$, v in the Lie algebra of C.*

Proof. Let η be the path in M whose value at time t is $\exp(tv)(e)$, so $d\eta/dt = v(\eta(t))$. Note that v is a killing vector field. Then

$$\begin{aligned} h_1(\exp v) &= \left(\int_\eta \alpha_j\right)_{j=1,\dots,n} \\ &= \left(\int_0^1 \eta^*\alpha_j\right)_j \\ &= \left(\alpha_j(d\eta/dt)\right)_j \\ &= \alpha_j(v). \end{aligned}$$ ■

From the lemma it follows easily that h_1 is a homomorphism. Moreover, its restriction to P is locally injective, i.e., with discrete kernel. In fact the kernel consists precisely of those elements v so that $\alpha_j(v)$ is an integer for all j. In particular, the image of C is non-trivial.

LEMMA 3. *h_1 defines an action of C on A so that f_1 is C-equivariant, where A is identified with its translation group.*

Proof.

$$\begin{aligned} f_1(c(y)) &= \left(\int_e^{c(y)} \alpha_j\right)_{j=1,\dots,n} \\ &= \left(\int_e^{c(e)} \alpha_j\right)_j + \left(\int_{c(e)}^{c(y)} \alpha_j\right)_j \\ &= h_1(c) + f_1(y). \end{aligned}$$ ■

Notation. Ω is the image of h_1. From now on, h_1 will be considered as a map into Ω instead of the full Albanese torus.

This is our desired torus. Now the sought fibration can be defined. If Ω is of dimension t, then it defines a subspace E of H_R of dimension t. $E \cap H_Z$ is a sublattice of H_Z of rank t. Let $\beta_1, \dots, \beta_t$ be a basis for this sublattice.

Now define a map $f: M \to \Omega$ in the same way that the Albanese map was defined, using the basis $\beta_1, \dots, \beta_t$ and t-tuples instead:

$$f(y) = \left(\int_e^y \beta_i\right)_{i=1,\dots,t} \equiv \sum \left(\int_e^y \beta_i\right)\beta_i.$$

This is our desired fibration, as will be shown.

Notation. $h(c) = f(c(e))$.

Similar to before, h defines an action of C on Ω, for which f is C-equivariant. The context will make clear which action of C on Ω is referred to: h is associated to f, and h_1 is associated to f_1. As before, we can express h as follows: $h(\exp v) = (\beta_i(v))_{i=1,\dots,t}$.

PROPOSITION 1. *f is a submersion.*

Proof. Let $w_1, \dots, w_t$ be vector fields dual to the β_i. It is convenient to let ∂ denote $d/dt|_{t=0}$. Let y be an arbitrary point in M. For an arbitrary index j, consider a path $\eta : [0, 1] \to M$ whose initial tangent vector $\partial\eta = w_j$. Let $\eta_t := \eta|[0, t]$. Then

$$\begin{aligned} f_*(w_j)_y &= \left(\partial \int_{\eta_t} \beta_i\right)_{i=1,\dots,t} = \left(\partial \int_0^t \eta^* \beta_i\right)_i = (\beta_i(\partial\eta)_y)_i \\ &= \left(\beta_i(w_j)_y\right)_i = (\langle w_i, w_j \rangle_y)_i. \end{aligned}$$

f is a submersion if the matrix $(\langle w_i, w_j\rangle_y)_{i,j}$ is non-singular for all y in M. This is true if the $(\beta_i)_y$ are linearly independent at each point y in M. A priori, they are only linearily independent as forms on M.

Suppose $\beta = \Sigma b^i \beta_i$ vanishes at some y in M. Then $\beta(v) = 0$ for all Killing vector fields v on M, since β is harmonic. On the other hand, one can also write $\beta = \Sigma a^j \alpha_j$, where there is a Killing vector field v in the Lie algebra of C so that $a^j = \alpha_j(v)$. This follows from observing that h_1 is a submersion from C onto Ω, which is generated by the β_i, and then by applying lemma 2. Hence we have

$$0 = \beta(v) = \Sigma\left(\alpha_j(v)\right) \cdot \alpha(v).$$

Thus $\alpha_j(v) = 0$ for all j; i.e., β vanishes identically so all the b^i are zero, Q.E.D.

Remark. The above proof also shows that the β_i are orthogonal to the fibre of f, since $f_*(v) = \Sigma\beta_i(v)\beta_i$, which is zero if and only if $\beta_i(v) = 0$ for all i. This fact is used in Proposition 2, §5.

Notation. H_1 is the identity component of $\ker h_1$; H is the identity component of $\ker h$.

LEMMA 4. $H = H_1$.

Proof. For v sufficiently small, one can write

$$h_1(\exp v) = \Sigma\alpha_j(v)\alpha_j = \Sigma b^i\beta_i \quad \text{for some } b^i,$$
$$h(\exp v) = \Sigma\beta_i(v)\beta_i.$$

Large v need not be considered, since H and H_1 are the same if they share a neighborhood of the identity.

If $h_1(\exp v) = 0$, then $\alpha_j(v) = 0$ for all j. Hence $\beta_i(v)$ is zero for all i, since the β_i are linear combinations of the α_j. Thus $h(\exp v) = 0$.

On the other hand, if $h(\exp v) = 0$, then $\beta_i(v) = 0$ for all i. This implies that $0 = \Sigma b^i\beta_i(v) = \Sigma\alpha_j(v)\alpha_j(v)$ and so $\alpha_j(v) = 0$ for all j; i.e., $h_1(\exp v) = 0$. ∎

COROLLARY 1. *h is a surjection.*

LEMMA 5. *There is a Lie subalgebra $\mathscr{T} \subset LC$ so that*
(a) *$LC = \mathscr{T} \oplus LH$ as Lie algebras, and*
(b) *$T := \exp\mathscr{T}$ is a subtorus of C.*

Proof. Let K be the kernel of the exponential map $LC \to C$. This is a lattice of maximal rank, say k, in LC. $K \cap LH$ is a sublattice of rank at most $k - t$. So there exists t linearily independent elements $z_1, \ldots, z_t$ of K that do not lie in $K \cap LH$. Let T be the real span of these elements. $LC = \mathscr{T} \oplus LH$ as vector spaces, indeed as Lie algebras since C is abelian.

$T := \exp\mathscr{T}$ is an abelian group, of dimension t. Consider the 1-parameter subgroups generated by the z_i. These are closed since the z_i lie in K, and they generate T. T is then a torus, and in fact can be expressed as a quotient group of $\{\exp sz_1\} \times \ldots \times \{\exp sz_t\}$. ∎

Notation. $G := H \cap T \equiv \ker(h|T)$.

G is finite since both T and H are compact. The exact sequence of groups $0 \to G \to T \to \Omega \to 0$, where the first map is an inclusion and the second is $h|T$, also represents T as a principle bundle over Ω with fibre G. Equivalently, we can say that T is a regular finite covering of Ω with deck transformation group G.

Note that the action of T on M is almost free, i.e., with discrete isotropy groups, since it is almost free on Ω and f is C-equivariant. Now we are in a position to describe M. Let F be the fibre of f containing e. G fixes the fibres of f, and so acts on F, say on the right.

Notation.

$$T \times_G F \xrightarrow{\phi} \Omega$$

is the fibre bundle with fibre F and group G associated to the principle bundle $T \to \Omega$. Recall that $T \times_G F$ is the quotient space of $T \times F$ modulo the equivalence relation that identifies (s, y) with (gs, yg^{-1}), where g is in G. ϕ is the quotient map.

THEOREM 1A. *There is a diffeomorphism Ψ such that the following diagram commutes*:

$$\begin{array}{ccc} T \times_G F & \xrightarrow{\Psi} & M \\ \phi \searrow & & \swarrow f \\ & \Omega & \end{array}$$

Proof. Consider the evaluation map from $T \times F$ to M; i.e., the pair (s, y) is mapped to ys. Note that $ys = y_0 s_0$ if and only if $y = y_0 s_0 s^{-1}$. But $s_0 s^{-1}$ lies in G if and only if y and y_0 lie in F. Thus (s, y) is equivalent to (s_0, y_0) if and only if they have the same image in M.

The evaluation map then descends to an injective map of $T \times_G F$ into M; call it Ψ. It is differentiable, since its lift, the evaluation map, is the restriction of the action of T on M. Ψ clearly carries the fibres of ϕ into the fibres of f, whereupon we have the commutative diagram. Finally, to see that it is a diffeomorphism, it suffices to note that Ψ is an immersion when restricted to each factor, and furthermore the image of each factor is transversal to the other, Q.E.D.

3. The converse

The proof of Theorem 1 will be complete once the following converse is proved:

THEOREM 1B. *Suppose M (not necessarily compact) is a fibre bundle over a torus with finite structural group. Then under a suitable metric, M admits as many parallel vector fields as the dimension of the torus.*

Proof. Let F denote an arbitrary fibre of the bundle, and G its structural group. The bundle $M \to \Omega$ over the t-torus Ω can be associated with a principle bundle $G \to T \to \Omega$. Since G is finite, T is also a torus, and Ω is the quotient space T/G. Indeed, T is a covering torus of Ω.

M can be expressed as $T \times_G F$, and so it suffices to work with the latter. Put a flat metric on Ω; this induces a flat metric on T, which is invariant under G. Since G is finite, one can put a Riemannian metric on F that is invariant under G. Give $T \times F$ the product metric. The action of G on $T \times F$ defined by $g(s, y) := (gs, yg^{-1})$ is then an isometric action.

The quotient map $\Psi : T \times F \to T \times_G F$ is a covering map, and so pushes the metric on $T \times F$ down to $T \times_G F$, making Ψ a local isometry. Let $\{v_i\}$ be t linear independent vector fields on Ω. They lift to vector fields $\{v_i'\}$ on $T \times F$ and $\{v_k''\}$ on $T \times_G F$. These are parallel since both quotient maps are local isometries, Q.E.D.

4. The cohomology of M

Because of Theorem 1, the study of compact manifolds which carry a parallel vector field, under a suitable metric, is reduced to the study of fibre bundles over tori with finite structural group. This result can be restated in terms of a finite cover, which immediately yields a description of the cohomology.

THEOREM 1'. *A compact manifold M admits a parallel vector field under same metric if and only if M is diffeomorphic to $(T \times F)/G$ where T is a torus, F compact, and G a finite subgroup of $T \times$ Diff(F) such that the first projection on G is injective.*

THEOREM 2. *Let M, T, G, F be as in Theorem* 1*a or* 1'. *Then*

$$H^*M \simeq H^*(T \times F)^G \simeq (H^*T) \otimes (H^*F)^G.$$

Here $(H^*F)^G$ denotes that part of the cohomology that is fixed by G, where the action of G on $T \times F$ induces an action of G on F. The first isomorphism is true in greater generality [3, Chapter 5]. The second is just the Kunneth formula (for example, see [7]) along with the fact that translations do not affect cohomology.

The group G in the theorem is also the holonomy group of the bundle. In other words, the holonomy group is the structure group of the bundle. Another observation is that M must contain the real cohomology of a torus. Finally, Theorem 2 yields inequalities on the betti numbers of M.

COROLLARY 2 (LEON KARP). *If a compact manifold admits a parallel vector field, then its betti numbers satisfy $b_1 \geq 1$; $b_k \geq b_{k-1} - b_{k-2}$, $k > 1$.*

5. The Riemannian situation

Theorem 1 produces all compact differentiable manifolds which admit a metric that carries a parallel vector field, but the construction does not yield all such Riemannian manifolds. To see this, note that the resulting parallel vector fields form a toral group of isometries. As a counter example, one can construct a compact Riemannian manifold with precisely one parallel vector field, up to linear independence, and whose integral curves are not all closed.

In fact, let M' be the Riemannian product of $\mathbf{R}$ and S^2, and let L be the group of isometries generated by $\zeta \times \rho$, where ζ is translation of $\mathbf{R}$ by some constant, and ρ is an irrational rotation of the sphere; i.e., ρ^n is not the identity for all $n \neq 0$. Then $M := M'/L$, the orbit space of L acting on M, is naturally a Riemannian manifold. This is the desired counterexample.

The theorem can be used to characterize compact Riemannian manifolds that carry a parallel vector field. However, the deRham decomposition (see [5; V, §5, 6]) gives a more direct approach that requires only completeness instead of compactness. The next theorem states the characterization and is followed by a sketch of the proof, since the details are fairly straightforward [8]. Note that Euclidean space is identified with its translation group.

THEOREM 3. *A complete, connected Riemannian manifold M admits p linearly independent parallel vector fields if and only if there is a Riemannian manifold M_2, and a group $L \subset \mathbf{R}^p \times I(M_2)$ such that*

(a) *the first projection $pr|L$ is injective and*

(b) *the orbits of L in $\mathbf{R}^p \times M_2$ are discrete, so that M is isometric to $(\mathbf{R}^p \times M_2)/L$.*

Sketch of proof. Assume M has p linearily independent vector fields. It suffices to assume p is maximal. The universal cover $\tilde{M}$ factors into $M_0 \times M_1$, where M_0 is isometric to Euclidean space. $M_0 \simeq E_1 \times E_2$ where $E_1 \simeq \mathbf{R}^p$ corresponds to the lifts of parallel vector fields on M. It is hard not to see that $\pi_1 M$ is contained in $I(E_1) \times I(E_2) \times I(M_1)$, since the tangent spaces to E_1, E_2 and M_1 are holonomy invariant.

Define K to be the kernel of the first projection restricted to $\pi_1 M$. Then $\tilde{M}/K$ is a covering space of M, with deck transformation group

$$L = \pi_1 M/K \simeq \text{image of } \pi_1 M \text{ in } I(E_1).$$

Indeed, the image is in E_1, where Euclidean space is identified with its translation group. Then $\tilde{M}/K$ factors into a Riemannian product of E_1 with some other manifold M_2. Furthermore, L satisfies conditions (a) and (b).

As for the converse, note that any group of isometries acting freely with discrete orbits, acts in fact properly discontinuously. ∎

The following is a list of observations pertaining to the above theorem (details in [8]).

(1) $y \times M_2$ is immersed injectively into M, orthogonal and transverse to parallel vectors on M, for each $y \in \mathbf{R}^p$.

(2) If M is compact, rank $L \geq p$.

(3) When M is compact, the following are equivalent:
  (a) the image of the first projection $pr|L$ is discrete;
  (b) the immersion in (1) is an embedding;
  (c) M_2 is compact.

(4) The immersion $M_2 \to M$ induces an exact sequence $1 \to \pi_1 M_2 \to \pi_1 M \to L \to 1$ where L is a free $\mathbf{Z}$-module. If in addition M is compact,

$$\text{then rank } L = \text{codim } M_2 \text{ if and only if } M_2 \text{ is compact.}$$

PROPOSITION 2. *Let M be a compact, connected Riemannian manifold, all of whose harmonic* 1*-forms are parallel (e.g., M of positive semi-definite curvature, like a sphere). Then M admits p parallel vector fields if and only if there exists a manifold* M_2 *and a group L, so M is isometric to* $(\mathbf{R}^p \times M_2)/L$, *where L lies in* $\mathbf{R}^p \times I(M_2)$, *and its first projection carries L injectively into a discrete lattice of* $\mathbf{R}^p$. *The quotient space is always a Riemmanian manifold if the rank of* $L = p$.

Proof. Let A be the appropriate Albanese Torus for M, constructed in the proof of Theorem 1. All the harmonic forms are parallel, so $p = \dim A$. From the remark after the proposition in Section 2, it follows that the parallel vector fields are orthogonal to the fibres of the Albanese map. Hence M_2 is the fiber of the Albanese map, and hence compact. Apply observation (3) above. ■

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