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TWISTOR SPACES OVER 6-DIMENSIONAL RIEMANNIAN MANIFOLDS

BY

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Dedicated to Professor Wilhelm Stoll on the occasion of his sixty-first birthday

Introduction

Let E be an oriented $2n$-dimensional vector space over $\mathscr{R}$ with inner product $\langle\ ,\ \rangle$. The space of almost complex structures on E compatible with the inner product and orientation is isomorphic to the rank one symmetric space $\mathrm{SO}(2n)/U(n)$. On an oriented $2n$-dimensional Riemannian manifold M we thus obtain a bundle $\mathscr{J}$ whose fiber over each point x of M consists of all almost complex structures on the tangent space T_xM compatible with the metric and orientation. The bundle $\mathscr{J}$ is equipped with a natural almost complex structure defined as follows. The connection on $\mathscr{J}$ induced by the Riemannian connection on M defines a splitting of the tangent space at each point $(x, J) \in \mathscr{J}$ into a direct sum of the vertical subspace and the horizontal subspace. Along the vertical subspace the almost complex structure is defined by the standard invariant complex structure on $\mathrm{SO}(2n)/U(n)$. On the horizontal subspace $(\approx T_xM)$ at T_xM the almost complex structure is defined simply as J. We are interested in the following two problems concerning $\mathscr{J}$.

Problem I. Find necessary and sufficient conditions (on the curvature of M) for the integrability of the canonical almost complex structure on $\mathscr{J}$.

Problem II. If the canonical almost complex structure is not integrable on all of $\mathscr{J}$, find necessary and sufficient conditions for the integrability of some (if any) natural almost complex submanifolds of $\mathscr{J}$.

The four dimension case ($n = 2$) has been studied extensively. In this case the almost complex structures are parametrized by the positive (resp. negative) spinors. It was shown (cf. Atiyah-Hitchen-Singer [2]) that the canonical almost

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complex structure on $\mathscr{P}(V_+)$ (resp. $\mathscr{P}(V_-)$) the projective positive (resp. negative) spinor bundle is integrable if and only if W_+ (resp. W_-) the self-dual (resp. anti-self-dual) component of the Weyl tensor vanishes. Classifications of those twistor spaces which are compact Kähler are also known (this happens iff M is compact Einstein with positive scalar curvature and $W_- = 0$; cf. Friedrich-Kurke [10], Friedrich [9] and Hitchen [13]). For further information on twistors, its relationship with the Yang-Mills equation, stable vector bundles etc., we refer the reader to A-H-S [2], Atiyah [1], Douady-Verdier [8], Hartshorne [11], Trautmann [18], Donaldsons [22], [23], Taubes [25], [26] and Kobayashi [24].

In this paper we study the six dimensional case ($n = 3$). This is the first case where Problem II is meaningful. The six dimensional case is still somewhat special for the following reasons. First of all the Lie algebras $\mathbf{so}(6)$ and $\mathbf{su}(4)$ are isomorphic via the positive (and negative) spinor representation. Secondly, it is still possible to parametrize the almost complex structures by $\mathscr{P}(V_+)$ (or $\mathscr{P}(V_-)$). (This is not true if $n \geq 4$.) Our first result formally generalize the integrability theorem of A-H-S for four manifolds. It is shown that $\mathscr{P}(V_+)$ (resp. $\mathscr{P}(V_-)$) is integrable if and only if W_+ (resp. W_-) vanishes where W_+ (resp. W_-) is the positive (resp. negative) spinor representation of the Weyl tensor. Notice that in the four dimensional case the self-dual (resp. anti-self-dual) component of W is the same as the positive (resp. negative) representation of W. However in the six dimensional case $W_+ = 0$ iff $W_- = 0$ iff $W = 0$ and the two spaces $\mathscr{P}(V_+)$ and $\mathscr{P}(V_-)$ are essentially the same space. We give the proof for both as it requires essentially no extra effort and more importantly the proof is designed so that it gives at the same time A-H-S's theorem for four manifolds.

Our second main result provides answers to Problem II assuming that M is a complex 3-dimensional Kähler manifold. There is a natural almost complex submanifold $\mathscr{P}(F_\pm) \subset \mathscr{P}(V_\pm)$ which is integrable if and only if M is Kähler Einstein with vanishing Bochner tensor (cf. Theorem 9, §4). The fiber $\mathscr{P}(F_\pm)_x$ over a point $x \in M$ is isomorphic to $\mathbf{C}P^2$ (note that $\mathscr{P}(V_\pm)_x \simeq \mathrm{SO}(6)/U(3) \simeq \mathbf{C}P^3$).

The paper is organized as follows. In §1 we describe explicitly the parametrization of almost complex structures by spinors (unlike the 4-dimensional case, this now requires proof). This is used in §2 to construct local basis of $(1,0)$-forms on $\mathscr{P}(V_\pm)$. By differentiating this basis we show that the Frobenius integrability condition is equivalent to the condition that the curvature $R_\pm$ of the spinor bundle be of a very special form. Section 3 begins with a review of the decomposition of a Riemannian curvature into invariant (under the action of the orthogonal group) components one of which is the Weyl tensor (cf. Kulkarni [14], Polombo [15]). From this we obtain a decomposition of $R_\pm$ which is readily seen to be of the special form of §2 iff the component $W_\pm$ corresponding to the Weyl tensor vanishes. In §4 we decompose (locally) $V_\pm$ into a direct sum of subspaces $F_\pm \oplus G_\pm$. The crucial observation here is that the connection (hence also the curvature) of $V_\pm$ splits according to the splitting $F_\pm \oplus G_\pm$ provided that M is Kähler. Upon examination of the integrability condition of $\mathscr{P}(F_\pm)$, it is discovered that the $F_\pm$ components of $R_\pm$ must assume a special form. The Kählerian curvature admits also a decomposition into invariant (under the action of the unitary group) components, one of which is the Bochner tensor (cf. Sitaramaya [17], Tricerri-Vanhecke [19]). We get from this a decomposition of the $F_\pm$ component of $R_\pm$ and that it assumes the special form mentioned above iff the components corresponding to the Bochner tensor vanishes. In §5 we give some examples, in particular it is pointed out that the twistor space over S^6 is the complex 6-dimensional hyperquadric Q^6 and the submanifold $\mathscr{P}(F_-)$ of the twistor $\mathscr{P}(V_-)$ over $\mathbf{C}P^3$ is the complex homogeneous manifold $U(4)/U(1) \times U(1) \times U(2)$.

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1. Almost complex structures and spinors

Let E be a $2n$-dimensional real vector space with inner product $\langle\ ,\ \rangle$. Let $e_1, \ldots, e_{2n}$ be an oriented orthonormal basis for E. The complexification $E \otimes \mathscr{C}$ splits into a direct sum $E' \oplus E''$ where E' (resp. E'') is spanned by

$$\xi_1 = (e_1 - ie_2)/2, \ldots, \xi_n = (e_{2n-1} - ie_{2n})/2 \quad \left(\text{resp. } \bar{\xi}_1, \ldots, \bar{\xi}_n\right).$$

Denote by $\tilde{C}_{2n}$ the complex Clifford algebra generated by $e_1, \ldots, e_{2n}$ with respect to the inner product $-\langle\ ,\ \rangle$. Let $\tilde{C}_{2n}^+$ (resp. $\tilde{C}_{2n}^-$) be the complex vector subspace generated by elements of the form $e_{i_1}, \ldots, e_{i_p}$ with p even (resp. odd). The subalgebra of $\tilde{C}_{2n}$ generated by E' is called the *spinor algebra* and is denoted by V. Elements of the complex vector subspace $V^\pm = V \cap \tilde{C}_{2n}^\pm$ are called the positive (resp. negative) spinors. Note that V is isomorphic to the exterior algebra $\Lambda E'$, hence $\dim_{\mathbf{C}} V = 2^n$ and

$$V^+ \simeq \bigoplus_{k \text{ even}} \Lambda^k E', \quad V^- \simeq \bigoplus_{k \text{ odd}} \Lambda^k E'$$

so that $\dim_{\mathbf{C}} V^+ = \dim_{\mathbf{C}} V^- = 2^{n-1}$.

It is well known that $\tilde{C}_{2n}$ is isomorphic to the matrix algebra $\operatorname{End}_{\mathbf{C}} V$. The isomorphism ρ is defined as follows. For $\xi \in E' \subset \tilde{C}_{2n}$, multiplication by ξ on the left, $v \mapsto \xi v$, is an endormorphism of V. On the other hand, for $\bar{\xi} \in E''$, interior product by $\bar{\xi}$, $v \mapsto \iota(\bar{\xi})v$, also defines an endormorphism of V. Now every element $x \in E$ is of the form $\xi + \bar{\xi}$ or $i(\xi - \bar{\xi})$ with $\xi \in E'$. Thus ρ is defined on E and satisfies $\rho(x)^2 = -\langle x, x\rangle 1$. This is precisely the condition that ρ can be extended uniquely to all of $\tilde{C}_{2n}$ (cf. Dieudonne [7], p. 148).

If we fix $0 \neq \xi \in V_+$ then the map $x \mapsto \rho(x) \cdot \xi$ is an isomorphism over $\mathbf{R}$ of E onto a subspace of V_-. From now on we will write $x \cdot \xi$ for $\rho(x) \cdot \xi$. Similarly an element $\bar{\xi}$ of V_- defines an isomorphism (over $\mathbf{R}$) of E onto a subspace of V_+. For $n = 2$, $\dim_{\mathbf{R}} E = \dim_{\mathbf{R}} V_+ = \dim_{\mathbf{R}} V_- = 4$, so that each ξ of V_+ (resp. V_-) gives rise to an isomorphism between E and V_- (resp. V_+) and we endow E with the complex structure of V_- (resp. V_+). Furthermore any two elements of V_+ give rise to the same complex structure if and only if they differ by a non-zero multiple $\lambda \in \mathbf{C}^*$.

We now examine the case $n = 3$ in detail. For $n = 3$,

$$E = \langle e_1, e_2, e_3, e_4, e_5, e_6 \rangle_{\mathbf{R}} \simeq \mathbf{R}^6$$

$$V_+ = \langle \xi_1\xi_2, \xi_1\xi_3, \xi_3\xi_1, 1 \rangle_{\mathbf{C}} \simeq \mathbf{C}^4$$

$$V_- = \langle \xi_1, \xi_2, \xi_3, \xi_1\xi_2\xi_3 \rangle_{\mathbf{C}} \simeq \mathbf{C}^4$$

where $\xi_\alpha = (e_{2\alpha-1} - ie_{2\alpha})/2$.

Here is the multiplication table for V_+:

	1	$\xi_1\xi_2$	$\xi_2\xi_3$	$\xi_3\xi_1$
e_1	ξ_1	ξ_2	$\xi_1\xi_2\xi_3$	$-\xi_3$
e_2	$i\xi_1$	$-i\xi_2$	$i\xi_1\xi_2\xi_3$	$i\xi_3$
e_3	ξ_2	$-\xi_1$	ξ_3	$\xi_1\xi_2\xi_3$
e_4	$i\xi_2$	$i\xi_1$	$-i\xi_3$	$i\xi_1\xi_2\xi_3$
e_5	ξ_3	$\xi_1\xi_2\xi_3$	$-\xi_2$	ξ_1
e_6	$i\xi_3$	$i\xi_1\xi_2\xi_3$	$i\xi_2$	$-i\xi_1$

Remark. The upper left corner (double-lined) of the above table is the multiplication table for V_+ for the case $n = 2$.

For any $\xi \in V_+$ with $\xi = a + b\xi_1\xi_2 + c\xi_2\xi_3 + d\xi_3\xi_1$, a, b, c, $d \in \mathbf{C}$, we obtain from the table above,

$$\begin{aligned}
e_1 \cdot \xi &= a\xi_1 + b\xi_2 - d\xi_3 + c\xi_1\xi_2\xi_3, \\
e_2 \cdot \xi &= ia\xi_1 - ib\xi_2 + id\xi_3 + ic\xi_1\xi_2\xi_3, \\
e_3 \cdot \xi &= -b\xi_1 + a\xi_2 + c\xi_3 + d\xi_1\xi_2\xi_3, \\
e_4 \cdot \xi &= ib\xi_1 + ia\xi_2 - ic\xi_3 + id\xi_1\xi_2\xi_3, \\
e_5 \cdot \xi &= d\xi_1 - c\xi_2 + a\xi_3 + b\xi_1\xi_2\xi_3, \\
e_6 \cdot \xi &= -id\xi_1 + ic\xi_2 + ia\xi_3 + ib\xi_1\xi_2\xi_3.
\end{aligned}$$

With respect to the basis $e_1, e_2, e_3, e_4, e_5, e_6$ for E and $\xi_1, i\xi_1, \xi_2, i\xi_2, \xi_3, i\xi_3, \xi_1\xi_2\xi_3$ and $i\xi_1\xi_2\xi_3$ for V_-, the matrix ν_ξ representing the real isomorphism of E *into* V_- defined by $\xi \in V_+$ is given by

$$\nu_\xi = \begin{bmatrix} a_1 & -a_2 & -b_1 & -b_2 & d_1 & d_2 \\ a_2 & a_1 & -b_2 & b_1 & d_2 & -d_1 \\ b_1 & b_2 & a_1 & -a_2 & -c_1 & -c_2 \\ b_2 & -b_1 & a_2 & a_1 & -c_2 & c_1 \\ -d_1 & -d_2 & c_1 & c_2 & a_1 & -a_2 \\ -d_2 & d_1 & c_2 & -c_1 & a_2 & a_1 \\ c_1 & -c_2 & d_1 & -d_2 & b_1 & -b_2 \\ c_2 & c_1 & d_2 & d_1 & b_2 & b_1 \end{bmatrix}$$

where a_1, a_2 are the real and imaginary parts of a etc.

It can be verified directly that the image $\nu_\xi(E)$ is a complex vector subspace of V_-. The corresponding complex structure J_ξ on E is defined so that the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{\nu_\xi} & V_- \\ {\scriptstyle J_\xi}\downarrow & & \downarrow{\scriptstyle i} \\ E & \xrightarrow[\nu_\xi]{} & V_- \end{array}$$

Observe that ${}^t\nu_\xi \circ \nu_\xi = (|a|^2 + |b|^2 + |c|^2 + |d|^2) \cdot \mathrm{Id}_E$ (this can be verified directly from the matrix representation of ν_ξ). Thus $J_\xi = {}^t\nu_\xi \circ i \circ \nu_\xi$. The complex structure i on V_- relative to the basis $\xi_1, i\xi_1, \xi_2, i\xi_2, \xi_3, i\xi_3, \xi_4, i\xi_4$ is given by the matrix

$$\begin{bmatrix} 0 & -1 & & & & & & \\ 1 & 0 & & & & & & \\ & & 0 & -1 & & & & \\ & & 1 & 0 & & & & \\ & & & & 0 & -1 & & \\ & & & & 1 & 0 & & \\ & & & & & & 0 & -1 \\ & & & & & & 1 & 0 \end{bmatrix}$$

By a direct calculation we have the following matrix representation of J_ξ:

$$J_\xi = \frac{\begin{array}{cccc||cc} 0 & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} \\ A_{21} & 0 & A_{23} & A_{24} & A_{25} & A_{26} \\ A_{31} & A_{32} & 0 & A_{34} & A_{35} & A_{36} \\ A_{41} & A_{42} & A_{43} & 0 & A_{45} & A_{46} \\ \hline\hline A_{51} & A_{52} & A_{53} & A_{54} & 0 & A_{56} \\ A_{61} & A_{62} & A_{63} & A_{64} & A_{65} & 0 \end{array}}{\left(|a|^2 + |b|^2 + |c|^2 + |d|^2\right)}$$

The matrix above is skew symmetric and the entries are given by

$$\begin{aligned}
A_{12} &= -|a|^2 + |b|^2 - |c|^2 + |d|^2, \\
A_{13} &= 2[(a_1b_2 - a_2b_1) - (c_1d_2 - c_2d_1)], \\
A_{14} &= -2[(a_1b_1 + a_2b_2) + (c_1d_1 + c_2d_2)], \\
A_{15} &= -2[(a_1d_2 - a_2d_1) - (b_1c_2 - b_2c_1)], \\
A_{16} &= 2[(a_1d_1 + a_2d_2) - (b_1c_1 + b_2c_2)], \\
A_{23} &= -2[(a_1b_1 + a_2b_2) - (c_1d_1 + c_2d_2)], \\
A_{24} &= -2[(a_1b_2 - a_2b_1) + (c_1d_2 - c_2d_1)], \\
A_{25} &= 2[(a_1d_1 + a_2d_2) + (b_1c_1 + b_2c_2)], \\
A_{26} &= 2[(a_1d_2 - a_2d_1) + (b_1c_2 - b_2c_1)], \\
A_{34} &= -|a|^2 + |b|^2 + |c|^2 - |d|^2, \\
A_{35} &= 2[(b_1d_2 - b_2d_1) + (a_1c_2 - c_2a_1)], \\
A_{36} &= -2[(b_1d_1 + b_2d_2) + (a_1c_1 + a_2c_2)], \\
A_{45} &= 2[(b_1d_1 + b_2d_2) - (a_1c_1 + a_2c_2)], \\
A_{46} &= 2[(b_1d_2 - b_2d_1) - (a_1c_2 - a_2c_1)], \\
A_{56} &= -|a|^2 - |b|^2 + |c|^2 + |d|^2.
\end{aligned}$$

Remark. (1) By setting $c = d = 0$ in the upper left hand corner (double-lined) above we obtain the almost complex structure J_ξ for $n = 2$, $\xi \in V_+$.

(2) If M is a complex manifold with complex structure J and basis $e_1, e_2 = Je_1, e_3, e_4 = Je_3, e_5, e_6 = Je_5$; then $J = J_\xi$ with $\xi = 1 \in V_+$, i.e., $a = 1$, $b = c = d = 0$.

Observe that ξ, ξ' in V_+ defines the same almost complex structure iff $\xi = \lambda\xi'$ with $\lambda \in \mathbf{C}^*$. Furthermore $J_\xi \in \mathrm{SO}(6)$, that is J_ξ is compatible with the metric $\langle\ ,\ \rangle$ and the orientation on E. The space of almost complex structures on E compatible with the metric and orientation is $\mathrm{SO}(6)/U(3) \cong \mathbf{C}P^3$. Since $\dim_{\mathbf{C}} V_+ = 4$ we see that $\mathscr{P}(V_+)$ parametrizes all such complex structures.

The case of V_- is similar. The multiplication table for V_- is given below:

	ξ_1	ξ_2	ξ_3	$\xi_1\xi_2\xi_3$
e_1	1	$\xi_1\xi_2$	$\xi_1\xi_3$	$\xi_2\xi_3$
e_2	$-i$	$i\xi_1\xi_2$	$i\xi_1\xi_3$	$-i\xi_2\xi_3$
e_3	$-\xi_1\xi_2$	1	$\xi_2\xi_3$	$-\xi_1\xi_3$
e_4	$-i\xi_1\xi_2$	$-i$	$i\xi_2\xi_3$	$i\xi_1\xi_3$
e_5	$-\xi_1\xi_3$	$-\xi_2\xi_3$	1	$\xi_1\xi_2$
e_6	$-i\xi_1\xi_3$	$-i\xi_2\xi_3$	$-i$	$-i\xi_1\xi_2$

Remark. The upper left corner is the multiplication table for $n = 2$.
For $\xi = a\xi_1 + b\xi_2 + c\xi_3 + d\xi_1\xi_2\xi_3 \in V_-$ we have

$$
\begin{aligned}
e_1 \cdot \xi &= a + b\xi_1\xi_2 + c\xi_1\xi_3 + d\xi_2\xi_3,\\
e_2 \cdot \xi &= -ia + ib\xi_1\xi_2 + ic\xi_1\xi_3 - id\xi_2\xi_3,\\
e_3 \cdot \xi &= b - a\xi_1\xi_2 + c\xi_2\xi_3 - d\xi_1\xi_3,\\
e_4 \cdot \xi &= -ib - ia\xi_1\xi_2 + ic\xi_2\xi_3 + id\xi_1\xi_3,\\
e_5 \cdot \xi &= c + d\xi_1\xi_2 - b\xi_2\xi_3 - a\xi_3\xi_1,\\
e_6 \cdot \xi &= -ic - id\xi_1\xi_2 - ib\xi_2\xi_3 - ia\xi_3\xi_1.
\end{aligned}
$$

With respect to the basis $e_1, e_2, e_3, e_4, e_5, e_6$ for E and $1, i, \xi_1\xi_2, i\xi_1\xi_2, \xi_2\xi_3, i\xi_2\xi_3, \xi_3\xi_1, i\xi_3\xi_1$ for V_+, the matrix ν_ξ representing the real isomorphism of E into V_+ defined by $\xi \in V_-$ is given by

$$
\nu_\xi = \begin{bmatrix}
a_1 & a_2 & b_1 & b_2 & c_1 & c_2\\
a_2 & -a_1 & b_2 & -b_1 & c_2 & -c_1\\
b_1 & -b_2 & -a_1 & a_2 & d_1 & d_2\\
b_2 & b_1 & -a_2 & -a_1 & d_2 & -d_1\\
c_1 & -c_2 & -d_1 & -d_2 & -a_1 & a_2\\
c_2 & c_1 & -d_2 & d_1 & -a_2 & -a_1\\
d_1 & d_2 & c_1 & -c_2 & -b_1 & b_2\\
d_2 & -d_1 & c_2 & c_1 & -b_2 & -b_1
\end{bmatrix}
$$

and $J_\xi = {}^t\nu_\xi \circ i \circ \nu_\xi$ is given by

$$J_\xi = \frac{\left[\begin{array}{cccc|cc} 0 & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} \\ A_{21} & 0 & A_{23} & A_{24} & A_{25} & A_{26} \\ A_{31} & A_{32} & 0 & A_{34} & A_{35} & A_{36} \\ A_{41} & A_{42} & A_{43} & 0 & A_{45} & A_{46} \\ \hline A_{51} & A_{52} & A_{53} & A_{54} & 0 & A_{56} \\ A_{61} & A_{62} & A_{63} & A_{64} & A_{65} & 0 \end{array}\right]}{\left(|a|^2 + |b|^2 + |c|^2 + |d|^2\right)^6}$$

The matrix is skew symmetric with

$$\begin{aligned}
A_{12} &= |a|^2 - |b|^2 - |c|^2 + |d|^2, \\
A_{13} &= -2[(a_1b_2 - a_2b_1) - (c_1d_2 - c_2d_1)], \\
A_{14} &= +2[(a_1b_1 + a_2b_2) - (c_1d_1 + c_2d_2)], \\
A_{15} &= -2[(a_1c_2 - a_2c_1) + (b_1d_2 - b_2d_1)], \\
A_{16} &= 2[(a_1c_1 + a_2c_2) + (b_1d_1 + b_2d_2)], \\
A_{23} &= -2[(a_1b_1 + a_2b_2) + (c_1d_1 + c_2d_2)], \\
A_{24} &= -2[(a_1b_2 - a_2b_1) + (c_1d_2 - c_2d_1)], \\
A_{25} &= -2[(a_1c_1 + a_2c_2) - (b_1d_1 + b_2d_2)], \\
A_{26} &= -2[(a_1c_2 - a_2c_1) - (b_1d_2 - b_2d_1)], \\
A_{34} &= -|a|^2 + |b|^2 - |c|^2 + |d|^2, \\
A_{35} &= 2[(a_1d_2 - a_2d_1) - (b_1c_2 - b_2c_1)], \\
A_{36} &= -2[(a_1d_1 + a_2d_2) - (b_1c_1 + b_2c_2)], \\
A_{45} &= -2[(a_1d_1 + a_2d_2) + (b_1c_1 + b_2c_2)], \\
A_{46} &= -2[(a_1d_2 - a_2d_1) + (b_1c_2 - b_2c_1)], \\
A_{56} &= -|a|^2 - |b|^2 + |c|^2 + |d|^2.
\end{aligned}$$

Remark. (1) The upper left-hand corner with $c = d = 0$ is the almost complex structure J_ξ for $\xi = a\xi_1 + b\xi_2$ when $n = 2$, $\xi \in V_-$.

(2) If M is a complex manifold with complex structure and basis $e_1, e_2 = Je_1, e_3, e_4 = Je_3, e_5$ and $e_6 = Je_5$, then J_ξ with $\xi = \xi_1\xi_2\xi_3 \in V_-$ (i.e., with $a = b = c = 0$, $d = 1$) is the conjugate of J. That is $J_\xi e_{2\alpha-1} = -e_{2\alpha}$.

In §2 we will have to compute the curvature of the spinor bundles, for that purpose we need the explicit isomorphism between the Lie algebras **so**(6) and **su**(4). Given an element $(a^\alpha_\beta) \in$ **so**(6) the corresponding element in **su**(4) obtained via the *positive spinor representation* is given in Fig. A whereas the one obtained via the *negative spinor representation* is given in Fig. B.

$$\left(\frac{1}{2}\right)^6 \left[\begin{array}{cc|cc} i\left(a_2^1 + a_4^3 + a_6^5\right) & -\left(a_3^1 - a_4^2\right) + i\left(a_4^1 + a_3^2\right) & -\left(a_5^1 - a_6^2\right) + i\left(a_6^1 + a_5^2\right) & -\left(a_5^3 - a_6^4\right) + i\left(a_6^3 + a_5^4\right) \\ \left(a_3^1 - a_4^2\right) + i\left(a_4^1 + a_3^2\right) & -i\left(a_2^1 + a_4^3 - a_6^5\right) & \left(a_5^3 + a_6^4\right) - i\left(a_6^3 - a_5^4\right) & -\left(a_5^1 + a_6^2\right) + i\left(a_6^1 - a_5^2\right) \\ \hline \left(a_5^1 - a_6^2\right) + i\left(a_6^1 + a_5^2\right) & -\left(a_5^3 + a_6^4\right) - i\left(a_6^3 - a_5^4\right) & -i\left(a_2^1 - a_4^3 + a_6^5\right) & \left(a_3^1 + a_4^2\right) - i\left(a_4^1 - a_3^2\right) \\ \left(a_5^3 - a_6^4\right) + i\left(a_6^3 + a_5^4\right) & \left(a_5^1 + a_6^2\right) + i\left(a_6^1 - a_5^2\right) & -\left(a_3^1 + a_4^2\right) - i\left(a_4^1 - a_3^2\right) & i\left(a_2^1 - a_4^3 - a_6^5\right) \end{array}\right]$$

FIG. A

$$\left(\frac{1}{2}\right)^6 \left[\begin{array}{cc|cc} -i\left(a_2^1 - a_4^3 - a_6^5\right) & \left(a_3^1 + a_4^2\right) - i\left(a_4^1 - a_3^2\right) & \left(a_5^1 + a_6^2\right) - i\left(a_6^1 - a_5^2\right) & -\left(a_5^3 - a_6^4\right) + i\left(a_6^3 + a_5^4\right) \\ -\left(a_3^1 + a_4^2\right) - i\left(a_4^1 - a_3^2\right) & i\left(a_2^1 - a_4^3 + a_6^5\right) & \left(a_5^3 + a_6^4\right) - i\left(a_6^3 - a_5^4\right) & \left(a_5^1 - a_6^2\right) - i\left(a_6^1 + a_5^2\right) \\ \hline -\left(a_5^1 + a_6^2\right) - i\left(a_6^1 - a_5^2\right) & -\left(a_5^3 + a_6^4\right) - i\left(a_6^3 - a_5^4\right) & i\left(a_2^1 + a_4^3 - a_6^5\right) & -\left(a_3^1 - a_4^2\right) + i\left(a_4^1 + a_3^2\right) \\ \left(a_5^3 - a_6^4\right) + i\left(a_6^3 + a_5^4\right) & -\left(a_5^1 - a_6^2\right) - i\left(a_6^1 + a_5^2\right) & \left(a_3^1 - a_4^2\right) + i\left(a_4^1 + a_3^2\right) & -1\left(a_2^1 + a_4^3 + a_6^5\right) \end{array}\right]$$

FIG. B

Remark. Setting $a_6^5 = 0$ in the upper left-hand corner of the above matrices we obtain the corresponding $\pm$ spinor representations for $\mathbf{so}(4) \simeq \mathbf{su}(2) \oplus \mathbf{su}(2)$.

Since these formulas will be used quite a few times in later sections, we digress a little here to explain how these formulas were derived. The Lie algebra $\mathbf{so}(2n)$ is spanned by $\{e_\alpha \wedge e_\beta : 1 \le \alpha < \beta \le 2n\}$ where $e_\alpha \wedge e_\beta$ is identified with the matrix $E_{\alpha\beta}$ whose entries are all zero except the (β, α) and (α, β) entries which are $+1$ and -1 respectively. The exterior products $e_\alpha \wedge e_\beta$ is identified with $e_\alpha \cdot e_\beta$ in the Clifford algebra whose elements are identified with endormorphisms of spinors. In fact $e_\alpha e_\beta$ takes $\pm$spinors to $\pm$spinors. For instance, take the element

$$e_1 \cdot e_2 = (\xi_1 + \bar{\xi}_1) \cdot i(\xi_1 - \bar{\xi}_1) = i(\xi_1 + \bar{\xi}_1)(\xi_1 - \bar{\xi}_1);$$

then

$$\begin{aligned} e_1 \cdot e_2 \cdot 1 &= i(\xi_1 + \bar{\xi}_1)\xi_1 &&\text{(because } \iota(\bar{\xi}_1)1 = 0\text{)} \\ &= i &&\text{(because } \xi_1 \cdot \xi_1 = 0\text{)}, \\ e_1 \cdot e_2 \cdot \xi_1\xi_2 &= i(\xi_1 + \bar{\xi}_1)(-\xi_2) = -i\xi_1\xi_2, \\ e_1 \cdot e_2 \cdot \xi_2\xi_3 &= i(\xi_1 + \bar{\xi}_1)(\xi_1\xi_2\xi_3) = i\xi_2\xi_3, \\ e_1 \cdot e_2 \cdot \xi_3\xi_1 &= i(\xi_1 + \bar{\xi}_1)(\xi_3) = -i\xi_3\xi_1. \end{aligned}$$

Thus with respect to the basis 1, $\xi_1\xi_2$, $\xi_2\xi_3$, $\xi_3\xi_1$, the endomorphism on V_+ corresponding to $e_1 \cdot e_2$ is given by

$$\begin{bmatrix} i & & & \\ & -i & & \\ & & i & \\ & & & -i \end{bmatrix}$$

The corresponding endomorphisms are listed in Figs. 1–3.

2. Integrability of the spinor bundles

Let M be an oriented Riemannian manifold of dimension 6. At each point $x \in M$, $E_x = T_xM$ is an even dimensional oriented real vector space with inner product given by the Riemannian metric. As explained in §1, there associates to E_x the spaces of positive and negative spinors $V_x^\pm$. In general there is no consistent way of defining $V^\pm$ globally over M, unless M is spin (i.e., the second Whitney class $w_2(M) = 0$). However $\mathscr{P}(V^\pm)$ the projective spinor bundles are globally defined and we denote the projection by $\pi : \mathscr{P}(V^\pm) \to M$. There is a canonical almost complex structure on $\mathscr{P}(V^\pm)$,

	$\mathrm{End}(V_+)$	$\mathrm{End}(V_-)$
$e_1 \cdot e_2$	$\left[\begin{array}{cc\|cc} i & & & \\ & -i & & \\ \hline & & i & \\ & & & -i \end{array}\right]$	$\left[\begin{array}{cc\|cc} -i & & & \\ & i & & \\ \hline & & i & \\ & & & -i \end{array}\right]$
$e_1 \cdot e_3$	$\left[\begin{array}{cc\|cc} & -1 & & \\ 1 & & & \\ \hline & & & 1 \\ & & -1 & \end{array}\right]$	$\left[\begin{array}{cc\|cc} & 1 & & \\ -1 & & & \\ \hline & & & -1 \\ & & 1 & \end{array}\right]$
$e_1 \cdot e_4$	$\left[\begin{array}{cc\|cc} & i & & \\ i & & & \\ \hline & & & i \\ & & i & \end{array}\right]$	$\left[\begin{array}{cc\|cc} & -i & & \\ -i & & & \\ \hline & & & i \\ & & i & \end{array}\right]$
$e_1 \cdot e_5$	$\left[\begin{array}{cc\|cc} & & & 1 \\ & & -1 & \\ \hline & 1 & & \\ -1 & & & \end{array}\right]$	$\left[\begin{array}{cc\|cc} & & 1 & \\ & & & 1 \\ \hline -1 & & & \\ & -1 & & \end{array}\right]$
$e_1 \cdot e_6$	$\left[\begin{array}{cc\|cc} & & & -i \\ & & i & \\ \hline & i & & \\ -i & & & \end{array}\right]$	$\left[\begin{array}{cc\|cc} & & -i & \\ & & & -i \\ \hline -i & & & \\ & -i & & \end{array}\right]$

FIG. 1

defined as follows. The Riemannian connection on TM induces a connection on $\mathscr{P}(V^{\pm})$ via the spinor representations. At a point $\xi \in \mathscr{P}(V^{\pm})$, the tangent space of $\mathscr{P}(V^{\pm})$ at ξ decomposes into the direct sum of the vertical space and horizontal space. The vertical subspace is tangent to the fiber $= \mathscr{P}(V^{\pm}) \approx \mathbf{C}P^3$ with canonical complex structure whereas the horizontal subspace H_ξ is isomorphic to TM_x where $x = \pi(\xi)$, and has a natural complex structure J_ξ defined by the spinor ξ (cf. §1).

The problem we are interested in is to derive a necessarily and sufficient condition for the integrability of the canonical almost complex structure defined above. Later we will also study the integrability of certain submanifolds of $\mathscr{P}(V^{\pm})$.

Since the integrability is a local problem, we will work directly with $V^{\pm}$. Now locally,

$$(2.1) \qquad V_- = \left\{ z^1\xi_1 + z^2\xi_2 + z^3\xi_3 + z^4\xi_1\xi_2\xi_3 : z^\alpha \in \mathscr{C} \right\}$$

where $\xi_\alpha = (e_{2\alpha-1} - ie_{2\alpha})/2$ and $e_1, \ldots, e_6$ is an oriented orthonormal frame

	$\mathrm{End}(V_+)$	$\mathrm{End}(V_-)$
$e_2 \cdot e_3$	$\left[\begin{array}{cc\|cc} & i & & \\ i & & & \\ \hline & & & -i \\ & & -i & \end{array}\right]$	$\left[\begin{array}{cc\|cc} & i & & \\ i & & & \\ \hline & & & i \\ & & i & \end{array}\right]$
$e_2 \cdot e_4$	$\left[\begin{array}{cc\|cc} & 1 & & \\ -1 & & & \\ \hline & & & 1 \\ & & -1 & \end{array}\right]$	$\left[\begin{array}{cc\|cc} & 1 & & \\ -1 & & & \\ \hline & & & 1 \\ & & -1 & \end{array}\right]$
$e_2 \cdot e_5$	$\left[\begin{array}{cc\|cc} & & & -i \\ & & -i & \\ \hline & -i & & \\ -i & & & \end{array}\right]$	$\left[\begin{array}{cc\|cc} & & i & \\ & & & -i \\ \hline i & & & \\ & -i & & \end{array}\right]$
$e_2 \cdot e_6$	$\left[\begin{array}{cc\|cc} & & & -1 \\ & & -1 & \\ \hline & 1 & & \\ 1 & & & \end{array}\right]$	$\left[\begin{array}{cc\|cc} & & 1 & \\ & & & -1 \\ \hline -1 & & & \\ & 1 & & \end{array}\right]$
$e_3 \cdot e_4$	$\left[\begin{array}{cc\|cc} i & & & \\ & -i & & \\ \hline & & -i & \\ & & & i \end{array}\right]$	$\left[\begin{array}{cc\|cc} i & & & \\ & -i & & \\ \hline & & i & \\ & & & -i \end{array}\right]$
$e_3 \cdot e_5$	$\left[\begin{array}{cc\|cc} & & -1 & \\ & & & -1 \\ \hline i & & & \\ & i & & \end{array}\right]$	$\left[\begin{array}{cc\|cc} & & & -1 \\ & & 1 & \\ \hline & -1 & & \\ 1 & & & \end{array}\right]$

FIG. 2

for TM with dual coframes $\theta^1, \ldots, \theta^6$. To check integrability we first describe a basis of $(1,0)$ forms on V_- and by explicit calculation of their derivatives we will show that the Frobenius conditions reduce to conditions on the curvature of the spinor bundle.

The following is a basis of *vertical* $(1,0)$-forms:

$$\nabla z^\alpha = : dz^\alpha + \sum_{\beta=1}^{4} z^\beta (\omega_-)^\alpha_\beta, \quad \alpha = 1,2,3,4, \tag{2.2}$$

where $(w_-)^\alpha_\beta$ are the connection 1-forms of the negative spinor bundle induced by the Riemannian connection on TM via the negative spinor representation. Notice that the horizontal tangent space is given by $\{\nabla z^\alpha = 0,\ \alpha = 1,2,3,4\}$ and along the fibers $\nabla z^\alpha = dz^\alpha$.

	$\mathrm{End}(V_+)$	$\mathrm{End}(V_-)$
$e_3 \cdot e_6$	$\left[\begin{array}{cc\|cc} & & i & \\ & & & i \\ \hline i & & & \\ & i & & \end{array}\right]$	$\left[\begin{array}{cc\|cc} & & & i \\ & & -i & \\ \hline & -i & & \\ i & & & \end{array}\right]$
$e_4 \cdot e_5$	$\left[\begin{array}{cc\|cc} & & i & \\ & & & -i \\ \hline i & & & \\ & -i & & \end{array}\right]$	$\left[\begin{array}{cc\|cc} & & & i \\ & & i & \\ \hline & i & & \\ i & & & \end{array}\right]$
$e_4 \cdot e_6$	$\left[\begin{array}{cc\|cc} & & 1 & \\ & & & -1 \\ \hline -1 & & & \\ & 1 & & \end{array}\right]$	$\left[\begin{array}{cc\|cc} & & & 1 \\ & & 1 & \\ \hline & -1 & & \\ -1 & & & \end{array}\right]$
$e_5 \cdot e_6$	$\left[\begin{array}{cc\|cc} i & & & \\ & i & & \\ \hline & & -i & \\ & & & -i \end{array}\right]$	$\left[\begin{array}{cc\|cc} i & & & \\ & i & & \\ \hline & & -i & \\ & & & -i \end{array}\right]$

FIG. 3

There are many ways to choose the horizontal (1,0)-forms, we make the following seemingly complicated choice but it turns out to be easier to differentiate than the more obvious choices.

Let $z^1 = a_1 + ia_2$, $z^2 = b_1 + ib_2$, $z^3 = c_1 + ic_2$, $z^4 = d_1 + id_2$ where a_α, b_α, c_α, d_α are real and let

$$\begin{aligned} v_1 &= a_1e_1 + a_2e_2 + b_1e_3 + b_2e_4 + c_1e_5 + c_2e_6, \\ v_2 &= b_1e_1 - b_2e_2 - a_1e_3 + a_2e_4 + d_1e_5 + d_2e_6, \\ v_3 &= c_1e_1 - c_2e_2 - d_1e_3 - d_2e_4 - a_1e_5 + a_2e_6, \\ v_4 &= d_1e_1 + d_2e_2 + c_1e_3 - c_2e_4 - b_1e_5 + b_2e_6. \end{aligned}$$

Recall that the complex structure J_ξ defined by

$$\xi = z^1\xi_1 + z^2\xi_2 + z^3\xi_3 + z^4\xi_1\xi_2\xi_3 \in V_-$$

is given explicitly in §1.

By a straightforward calculation,

$$\begin{aligned} J_\xi v_1 &= a_2e_1 - a_1e_2 + b_2e_3 - b_1e_4 + c_2e_5 - c_1e_6, \\ J_\xi v_2 &= b_2e_1 + b_1e_2 - a_2e_3 - a_1e_4 + d_2e_5 - d_1e_6, \\ J_\xi v_3 &= c_2e_1 + c_1e_2 - d_2e_3 + d_1e_4 - a_2e_5 - a_1e_6, \\ J_\xi v_4 &= d_2e_1 - d_1e_2 + c_2e_3 + c_1e_4 - b_2e_5 - b_1e_6, \end{aligned}$$

The vector fields $(v_\alpha - iJv_\alpha)/2$, $\alpha = 1,2,3,4$ are of type $(1,0)$. Their dual are then forms of type $(1,0)$. Explicitly they are given by

$$\begin{aligned}\eta^1 &= a_1\theta^1 + a_2\theta^2 + b_1\theta^3 + b_2\theta^4 + c_1\theta^5 + c_2\theta^6 \\ &\quad + i\big(a_2\theta^1 - a_1\theta^2 + b_2\theta^3 - b_1\theta^4 + c_2\theta^5 - c_1\theta^6\big) \\ &= (a_1 + ia_2)(\theta^1 - i\theta^2) + (b_1 + ib_2)(\theta^3 - i\theta^4) + (c_1 + ic_2)(\theta^5 - i\theta^6) \\ &= z^1\overline{\Theta}^1 + z^2\overline{\Theta}^2 + z^3\overline{\Theta}^3\end{aligned}$$

and similarly,

$$\begin{aligned}\eta^2 &= -z^1\Theta^2 + z^2\Theta^1 + z^4\overline{\Theta}^3 \\ \eta^3 &= -z^1\Theta^3 + z^3\Theta^1 - z^4\overline{\Theta}^2 \\ \eta^4 &= -z^2\Theta^3 + z^3\Theta^2 + z^4\overline{\Theta}^1\end{aligned}$$

Remark. For $n = 2$, we need only two horizontal forms, η_1 and η_2 given as above with $z_3 = z_4 = 0$.

It is convenient to express these forms as a matrix:

$$\begin{bmatrix}\eta^1\\ \eta^2\\ \eta^3\\ \eta^4\end{bmatrix} = \left[\begin{array}{cc|cc}\overline{\Theta}^1 & \overline{\Theta}^2 & \overline{\Theta}^3 & 0\\ -\Theta^2 & \Theta^1 & 0 & \overline{\Theta}^3\\ \hline -\Theta^3 & 0 & \Theta^1 & -\overline{\Theta}^2\\ 0 & -\Theta^3 & \Theta^2 & \overline{\Theta}^1\end{array}\right]\begin{bmatrix}z^1\\ z^2\\ z^3\\ z^4\end{bmatrix} \tag{2.3}$$

where $\Theta^\alpha = \theta^{2\alpha-1} + i\theta^{2\alpha}$, $\alpha = 1,2,3$. The horizontal forms η^1, η^2, η^3, η^4 are not linearly independent but at any point, three of them form a basis for $(1,0)$-forms on the horizontal tangent space. (For example, if $z^4 = 0$ then η^1, η^2 and η^3 form a basis whereas if $z^1 = 0$ then η^2, η^3 and η^4 form a basis.) We now compute their derivatives:

$$d\eta^1 = dz^1 \wedge \overline{\Theta}^1 + dz^2 \wedge \overline{\Theta}^2 + dz^3 \wedge \overline{\Theta}^3 + z^1\, d\overline{\Theta}^1 + z^2\, d\overline{\Theta}^2 + z^3\, d\overline{\Theta}^3$$

where

$$dz^\alpha \wedge \overline{\Theta}^\alpha = \nabla z^\alpha \wedge \overline{\Theta}^\alpha - \sum_{\beta=1}^{4} z^\beta(\omega_-)^\alpha_\beta \wedge \overline{\Theta}^\alpha$$

Denoting by (ω^α_β) the Riemannian connection forms on M which takes values in **so**(6) and by the explicit isomorphism between **so**(6) and **su**(4) via the negative spinor representation (cf. §1) we have

$$
\begin{aligned}
dz^1 \wedge \overline{\Theta}^1 = \nabla z^1 \wedge \overline{\Theta}^1 + \frac{1}{2}\big\{ & iz^1(\omega_2^1 - \omega_4^3 - \omega_6^5) - z^2(\omega_3^1 + \omega_4^2) \\
& + iz^2(\omega_4^1 - \omega_3^2) - z^3(\omega_5^1 + \omega_6^2) + iz^3(\omega_6^1 - \omega_5^2) \\
& + z^4(\omega_5^3 - \omega_6^4) - iz^4(\omega_6^3 + \omega_5^4)\big\} \wedge (\theta^1 - i\theta^2), \\
dz^2 \wedge \overline{\Theta}^2 = \nabla z^2 \wedge \overline{\Theta}^2 + \frac{1}{2}\big\{ & z^1(\omega_3^1 + \omega_4^2) + iz^1(\omega_4^1 - \omega_3^2) \\
& - iz^2(\omega_2^1 - \omega_4^3 + \omega_6^5) - z^3(\omega_5^3 + \omega_6^4) + iz^3(\omega_6^3 - \omega_5^4) \\
& - z^4(\omega_5^1 - \omega_6^2) + iz^4(\omega_6^1 + \omega_5^2)\big\} \wedge (\theta^3 - i\theta^4), \\
dz^3 \wedge \overline{\Theta}^3 = \nabla z^3 \wedge \overline{\Theta}^3 + \frac{1}{2}\big\{ & z^1(\omega_5^1 + \omega_6^2) + iz^1(\omega_6^1 - \omega_5^2) + z^2(\omega_5^3 + \omega_6^4) \\
& + iz^2(\omega_6^3 - \omega_5^4) - iz^3(\omega_2^1 + \omega_4^3 - \omega_6^5) \\
& + z^4(\omega_3^1 - \omega_4^2) - iz^4(\omega_4^1 + \omega_3^2)\big\} \wedge (\theta^5 - i\theta^6).
\end{aligned}
$$

We also have

$$
\begin{aligned}
z^1\, d\overline{\Theta}^1 &= z^1(d\theta^1 - i d\theta^2) = z^1\Big\{\sum \theta^\beta \wedge \big(\omega_\beta^1 - i\omega_\beta^2\big)\Big\}, \\
z^2\, d\overline{\Theta}^2 &= z^2\Big\{\sum \theta^\beta \wedge \big(\omega_\beta^3 - i\omega_\beta^4\big)\Big\}, \\
z^3\, d\overline{\Theta}^3 &= z^3\Big\{\sum \theta^\beta \wedge \big(\omega_\beta^5 - i\omega_\beta^6\big)\Big\}.
\end{aligned}
$$

Summing up all the above expressions we get $d\eta^1$. To see what the result is we look at all the terms involving z^1:

$$
\begin{aligned}
z^1\Big\{ & \frac{1}{2}(\omega_2^1 - \omega_4^3 - \omega_6^5) \wedge \theta^2 + \frac{1}{2}(\omega_3^1 + \omega_4^2) \wedge \theta^3 \\
& + \frac{1}{2}(\omega_4^1 - \omega_3^2) \wedge \theta^4 + \frac{1}{2}(\omega_5^1 + \omega_6^2) \wedge \theta^5 \\
& + \frac{1}{2}(\omega_6^1 - \omega_5^2) \wedge \theta^6 - \omega_2^1 \wedge \theta^2 - \omega_3^1 \wedge \theta^3 - \omega_4^1 \wedge \theta^4 - \omega_5^1 \wedge \theta^5 - \omega_6^1 \wedge \theta^6 \\
& + \frac{i}{2}(\omega_2^1 - \omega_4^3 - \omega_5^5) \wedge \theta^1 + \frac{i}{2}(\omega_4^1 - \omega_3^2) \wedge \theta^3 - \frac{i}{2}(\omega_3^1 + \omega_4^2) \wedge \theta^4 \\
& + \frac{i}{2}(\omega_6^1 - \omega_5^2) \wedge \theta^5 - \frac{i}{2}(\omega_5^1 + \omega_6^2) \wedge \theta^6 + i\big(\omega_1^2 \wedge \theta^1 + \omega_3^2 \wedge \theta^3 \\
& + \omega_4^2 \wedge \theta^4 + \omega_5^2 \wedge \theta^5 + \omega_6^2 \wedge \theta^6\big)\Big\}
\end{aligned}
$$

$$= \frac{z^1}{2}\left\{-\left(\omega_2^1+\omega_4^3+\omega_6^5\right)\wedge\theta^2-\left(\omega_3^1-\omega_4^2\right)\wedge\theta^3-\left(\omega_4^1+\omega_3^2\right)\wedge\theta^4\right.$$
$$-\left(\omega_5^1-\omega_6^2\right)\wedge\theta^5-\left(\omega_6^1+\omega_5^2\right)\wedge\theta^6-i\left(\omega_2^1+\omega_4^3+\omega_6^5\right)\wedge\theta^1$$
$$+i\left(\omega_4^1+\omega_3^2\right)\wedge\theta^3-i\left(\omega_3^1-\omega_4^2\right)\wedge\theta^4+i\left(\omega_6^1+\omega_5^2\right)\wedge\theta^5$$
$$\left.-i\left(\omega_5^1-\omega_6^2\right)\wedge\theta^6\right\}$$
$$= -\frac{z^1}{2}\left\{i\left(\omega_2^1+\omega_4^3+\omega_6^5\right)\wedge\left(\theta^1-i\theta^2\right)\right.$$
$$+\left[\left(\omega_3^1-\omega_4^2\right)-i\left(\omega_4^1+\omega_3^2\right)\right]\wedge\left(\theta^3+i\theta^4\right)$$
$$\left.+\left[\left(\omega_5^1-\omega_6^2\right)-i\left(\omega_6^1+\omega_5^2\right)\right]\wedge\left(\theta^5+i\theta^6\right)\right\}$$
$$= z^1\left\{-(\omega_+)_1^1\wedge\overline{\Theta}^1+(\omega_+)_2^1\wedge\Theta^2+(\omega_+)_3^1\wedge\Theta^3\right\}$$

where $(\omega_+)_\beta^\alpha$ are the connection forms for the positive spinor bundle. Similarly, terms involving z^2 and respectively z^3, z^4 are given by

$$z^2\left\{-(\omega_+)_2^1\wedge\Theta^1-(\omega_+)_1^1\wedge\overline{\Theta}^2+(\omega_+)_4^1\wedge\Theta^3\right\},$$
$$z^3\left\{-(\omega_+)_3^1\wedge\Theta^1-(\omega_+)_4^1\wedge\Theta^2-(\omega_+)_1^1\wedge\overline{\Theta}^3\right\},$$
$$z^4\left\{-(\omega_+)_4^1\wedge\overline{\Theta}^1+(\omega_+)_3^1\wedge\overline{\Theta}^2-(\omega_+)_2^1\wedge\overline{\Theta}^3\right\}.$$

The above calculations show that $d\eta^1$ is given by

$$d\eta^1 = -\sum_{\beta=1}^{4}(\omega_+)_\beta^1\wedge\eta^\beta-\sum_{\mu=1}^{3}\overline{\Theta}^\mu\wedge\nabla z^\mu. \tag{2.4}$$

Analogously, we get

$$d\eta^2 = -\sum_{\beta=1}^{4}(\omega_+)_\beta^2\wedge\eta^\beta+\Theta^2\wedge\nabla z^1-\Theta^1\wedge\nabla z^2-\overline{\Theta}^3\wedge\nabla z^4,$$
$$d\eta^3 = -\sum_{\beta=1}^{4}(\omega_+)_\beta^3\wedge\eta^\beta+\Theta^3\wedge\nabla z^1-\Theta^1\wedge\nabla z^3+\overline{\Theta}^2\wedge\nabla z^4, \tag{2.5}$$
$$d\eta^4 = -\sum_{\beta=1}^{4}(\omega_+)_\beta^4\wedge\eta^\beta+\Theta^3\wedge\nabla z^2-\Theta^3\wedge\nabla z^1-\overline{\Theta}^1\wedge\nabla z^4.$$

Remark. For $n = 2$, we need only the first two equations with $\eta^3 = \eta^4 = \nabla z^3 = \nabla z^4 = 0$.
We abbreviate these formulas simply by

$$d\eta = -\omega_+\wedge\eta-\Theta\wedge\nabla z \tag{2.6}$$

where Θ is the matrix

$$
(2.7)\qquad \left[\begin{array}{cc||cc}
\bar{\Theta}^1 & \bar{\Theta}^2 & \bar{\Theta}^3 & 0 \\
-\Theta^2 & \Theta^1 & 0 & \bar{\Theta}^3 \\
\hline\hline
-\Theta^3 & 0 & \Theta^1 & -\bar{\Theta}^2 \\
0 & -\Theta^3 & \Theta^2 & \bar{\Theta}^1
\end{array}\right].
$$

As for the vertical forms, we get

$$
\begin{aligned}
d\nabla z^\alpha &= d\Big\{dz^\alpha + \sum_\beta z^\beta(\omega_-)^\alpha_\beta\Big\} \\
&= \sum_\beta \big\{dz^\beta \wedge (\omega_-)^\alpha_\beta + z^\beta d(\omega_-)^\alpha_\beta\big\} \\
&= \sum_\beta \Big\{\Big(\nabla z^\beta - \sum_\alpha z^\gamma(\omega_-)^\beta_\gamma\Big) \wedge (\omega_-)^\alpha_\beta + z^\beta d(\omega_-)^\alpha_\beta\Big\} \\
&= -\sum_\beta (\omega_-)^\alpha_\beta \wedge \nabla z^\beta + \sum_\beta (R_-)^\alpha_\beta z^\beta
\end{aligned}
$$

where

$$
(R_-)^\alpha_\beta = d(\omega_-)^\alpha_\beta - \sum_\gamma (\omega_-)^\gamma_\beta \wedge (\omega_-)^\alpha_\gamma
$$

is the curvature forms of the negative spinor bundle. Again we abbreviate these equations as

$$
(2.8)\qquad d\nabla z = -\omega_- \wedge \nabla z + R_- z.
$$

We thus arrived at the following integrability conditions:

THEOREM 1. *The canonical almost complex structure on the negative spinor bundle $\mathscr{P}(V_-)$ is integrable if and only if its curvature satisfies $R_- z = A \wedge \eta$, i.e.,*

$$
\sum_{\beta=1}^{4} (R_-)^\alpha_\beta z^\beta = \sum_{b=1}^{4} A^\alpha_\beta \wedge \eta^\beta \quad \textit{for } 1 \le \alpha \le 4
$$

where R_- is the curvature of the negative spinor bundle and the η^β's are horizontal forms of type (1,0) *given by* (2.3).

Remark. The conditions above is equivalent to

$$
(2.9)\qquad (R_-)^\alpha_\beta = \sum_\gamma A^\alpha_\gamma \wedge \Theta^\gamma_\beta
$$

for $1 \leq \alpha, \beta \leq 4$, where Θ^{γ}_{β} are the components of the matrix of 1-forms Θ given as in (2.7). We abbreviate by writing

$$R_{-} = A \wedge \Theta. \tag{2.10}$$

We shall examine the meaning of these conditions in the next section.

As for the positive spinor bundle V_{+}, we choose

$$\begin{aligned}
v_1 &= a_1e_1 - a_2e_2 - b_1e_3 - b_2e_4 - c_1e_5 - c_2e_6,\\
v_2 &= b_1e_2 + b_2e_1 + a_1e_3 - a_2e_4 - d_1e_5 - d_2e_6,\\
v_3 &= c_1e_1 + c_2e_2 + d_1e_3 + d_2e_4 + a_1e_5 - a_2e_6,\\
v_4 &= d_1e_1 - d_2e_2 - c_1e_3 + c_2e_4 + b_1e_5 - b_2e_6,
\end{aligned}$$

and applying the almost complex structure J_ξ defined by the positive spinor ξ, we get

$$\begin{aligned}
J_\xi v_1 &= a_2e_1 + a_1e_2 - b_2e_3 + b_1e_4 - c_2e_5 + c_1e_6,\\
J_\xi v_2 &= b_2e_1 - b_1e_2 + a_2e_3 + a_1e_4 - d_2e_5 + d_1e_6,\\
J_\xi v_3 &= c_2e_1 - c_1e_2 + d_2e_3 - d_1e_4 + a_2e_5 + a_1e_6,\\
J_\xi v_4 &= d_2e_1 + d_1e_2 - c_2e_3 - c_1e_4 + b_2e_5 + b_1e_6,
\end{aligned}$$

and the horizontal form η^α of type $(1,0)$ dual to $v_\alpha - iJ_\xi v_\alpha$ are given by

$$\begin{aligned}
\eta^1 &= z^1\Theta^1 - z^2\overline{\Theta}^2 - z^3\overline{\Theta}^3,\\
\eta^2 &= z^1\Theta^2 + z^2\overline{\Theta}^1 - z^4\overline{\Theta}^3,\\
\eta^3 &= z^1\Theta^3 + z^3\overline{\Theta}^1 + z^4\overline{\Theta}^2,\\
\eta^4 &= + z^2\Theta^3 - z^3\Theta^2 + z^4\Theta^1.
\end{aligned} \tag{2.11}$$

We also have a basis for vertical $(1,0)$ forms:

$$\nabla z^\alpha = dz^\alpha + \sum_{\beta=1}^{4} z^\beta(\omega_+)^\alpha_\beta.$$

Calculating as in the case of V_{-}, we get

$$\begin{aligned}
d\eta^1 &= -\sum_\beta(\omega_-)^1_\beta \wedge \eta^\beta - \Theta^1 \wedge \nabla z^1 + \overline{\Theta}^2 \wedge \nabla z^2 + \overline{\Theta}^3 \wedge \nabla z^3,\\
d\eta^2 &= -\sum_\beta(\omega_-)^2_\beta \wedge \eta^\beta - \Theta^2 \wedge \nabla z^1 - \overline{\Theta}^1 \wedge \nabla z^2 + \overline{\Theta}^3 \wedge \nabla z^4,\\
d\eta^3 &= -\sum_\beta(\omega_-)^3_\beta \wedge \eta^\beta - \Theta^3 \wedge \nabla z^1 - \overline{\Theta}^1 \wedge \nabla z^3 - \overline{\Theta}^2 \wedge \nabla z^4,\\
d\eta^4 &= -\sum_\beta(\omega_-)^4_\beta \wedge \eta^\beta - \Theta^3 \wedge \nabla z^2 + \Theta^2 \wedge \nabla z^3 - \Theta^1 \wedge \nabla z^4,
\end{aligned} \tag{2.12}$$

and

$$d\nabla z^\alpha = \sum_\beta (R_+)^\alpha_\beta z^\beta - \sum_\beta (\omega_+)^\alpha_\beta \wedge \nabla z^\beta. \tag{2.13}$$

We abbreviate these formulas as follows:

$$d\eta = -\omega_- \wedge \eta - \Psi \wedge \nabla z, \quad d\nabla z = R_+ z - \omega_+ \wedge \nabla z \tag{2.14}$$

where $\Psi = {}^t\overline{\Theta}$ is the matrix

$$\left[\begin{array}{cc|cc} \Theta^1 & -\overline{\Theta}^2 & -\overline{\Theta}^3 & 0 \\ \Theta^2 & \overline{\Theta}^1 & 0 & \overline{\Theta}^3 \\ \hline \Theta^3 & 0 & \overline{\Theta}^1 & \overline{\Theta}^2 \\ 0 & \Theta^3 & -\Theta^2 & \Theta^1 \end{array}\right] \tag{2.15}$$

THEOREM 2. *The integrability condition for the positive spinor bundle $\mathscr{P}(V_+)$ is given by $R_+ z = A \wedge \eta$, i.e., for $1 \le \alpha \le 4$,*

$$\sum_{\beta=1}^{4} (R_+)^\alpha_\beta \wedge z^\beta = \sum_{\beta=1}^{4} A^\alpha_\beta \wedge \eta^\beta$$

where R_+ is the curvature of the positive spinor bundle and η^β are horizontal forms of type $(1,0)$ *given by* (2.11).

Remark. From (2.11) the forms η^α are given by $\eta = {}^t\overline{\Theta} z$ so that the integrability condition above is equivalent to

$$(R_+)^\alpha_\beta = \sum_{\beta=1}^{4} B^\alpha_\gamma \wedge \Psi^\gamma_\beta, \quad 1 \le \alpha, \beta \le 4, \tag{2.16}$$

where Ψ^α_β are the components of the matrix of 1-forms $\Psi = {}^t\overline{\Theta}$. We abbreviate by writing

$$R_+ = B \wedge {}^t\overline{\Theta}. \tag{2.17}$$

3. Curvature of the spinor bundles

We begin by reviewing the decomposition of the Riemannian curvature tensor under the action of the orthogonal group. We then apply the spinor representation to get a decomposition of the curvature of the spinor bundle.

Let E be a real vector space of dimension n with inner product $\langle\ ,\ \rangle$. Denote by $\mathscr{S}\Lambda^2 E$ the vector space of symmetric endomorphisms on $\Lambda^2 E$.

Under the action of the orthogonal group $\mathscr{S}\Lambda^2 E$ decomposes into direct sums of four irreducible invariant subspaces:

$$\mathscr{S}\Lambda^2 E = \Lambda^4 E \oplus E_1 \oplus E_2 \oplus E_3.$$

An endomorphism R of $\Lambda^2 E$ is in $E_1 \oplus E_2 \oplus E_3$ if and only if it satisfies the Bianchi identity, i.e., R is a curvature tensor. The subspace E_1 is spanned by the endomorphism $I \wedge I$ where I is the identity on E, i.e.,

$$I \wedge I(x \wedge y) = \frac{1}{2}(Ix \wedge Iy - Iy \wedge Ix) = x \wedge y. \tag{3.1}$$

Note that

$$\langle I \wedge I(x \wedge y), u \wedge v\rangle = \langle x, u\rangle\langle y, v\rangle - \langle x, v\rangle\langle y, u\rangle;$$

hence we may identify $I \wedge I$ with the $(1, 3)$ tensor

$$L(x, y)z = \langle y, z\rangle x - \langle x, z\rangle y$$

which is characterized by the property that its sectional curvature is identically 1.

The subspace E_2 consists of curvature tensors of the form $R_0 \wedge I$ where $R_0 \in \mathscr{S}_0 E = \{\text{symmetric endomorphisms on } E \text{ with zero trace}\}$. The subspace E_3 is the orthogonal complement in $\mathscr{S}\Lambda^2 E$ of the other subspaces. The elements of E_3 are called Weyl tensors.

The dimensions of the various subspaces are

$$\begin{aligned}
\dim_{\mathbf{R}} \mathscr{S}(\Lambda^2 E) &= \frac{1}{8} n(n-1)(n^2 - n + 2),\\
\dim_{\mathbf{R}} \Lambda^4 E &= \frac{1}{4!} n(n-1)(n-2)(n-3),\\
\dim_{\mathbf{R}} E_1 &= 1,\\
\dim_{\mathbf{R}} E_2 &= \frac{1}{2} n(n+1) - 1,\\
\dim_{\mathbf{R}} E_3 &= \frac{1}{12}(n-3)n(n+1)(n+2),
\end{aligned}$$

where $\dim_{\mathbf{R}} E = n \geq 4$.

Let R be a curvature tensor and R_1, R_2, W its projections into the invariant subspaces E_1, E_2 and E_3 respectively, then $R = R_1 + R_2 + W$ with

$$R_1 = \frac{\sigma}{n(n-1)} I \wedge I, \tag{3.2}$$

$$R_2 = \frac{2}{n-2} S \wedge I - \frac{2\sigma}{n(n-2)} I \wedge I, \tag{3.3}$$

and

$$W = R - \frac{2}{n-2} S \wedge I + \frac{\sigma}{(n-1)(n-2)} I \wedge I. \tag{3.4}$$

Here S is the Ricci tensor of R and σ is the scalar curvature. Note that $W \equiv 0 \Leftrightarrow$ conformally flat and $R_2 = 0 \Leftrightarrow$ Einstein.

For more details concerning the decomposition of curvature tensors, we refer the reader to Kulkarni [14], Polombo [15].

For our purpose we prefer to think of the curvature as matrix of two forms. Let L be the curvature matrix corresponding to $I \wedge I$; then

$$L^\alpha_\beta = \theta^\alpha \wedge \theta^\beta \tag{3.5}$$

where $\theta^1, \ldots, \theta^n$ is an othnormal coframe. Let T be the $(1,3)$ tensor corresponding to $S \wedge I$, i.e.

$$\begin{aligned}\langle T(x,y)v, u\rangle &= \langle S \wedge I(x \wedge y), u \wedge v\rangle \\ &= \tfrac{1}{2}\langle Sx \wedge y + x \wedge Sy, u \wedge v\rangle \\ &= \tfrac{1}{2}\{\langle Sx, u\rangle\langle y, v\rangle - \langle Sx, v\rangle\langle y, u\rangle \\ &\qquad + \langle x, u\rangle\langle Sy, v\rangle - \langle x, v\rangle\langle Sy, u\rangle\}.\end{aligned}$$

In other words

$$T(x,y)v = \tfrac{1}{2}\{\langle Sy, v\rangle x - \langle Sx, v\rangle y + \langle y, v\rangle Sx - \langle x, v\rangle Sy\}. \tag{3.6}$$

The components of T,

$$T^\beta_\alpha = \langle Te_\alpha, e_\beta\rangle,$$

are two forms which can be expressed as follows:

$$\begin{aligned}T^\beta_\alpha(x,y) &= \tfrac{1}{2}\{\langle Sy, e_\alpha\rangle\langle x, e_\beta\rangle - \langle Sx, e_\alpha\rangle\langle y, e_\beta\rangle \\ &\qquad + \langle y, e_\alpha\rangle\langle Sx, e_\beta\rangle - \langle x, e_\alpha\rangle\langle Sy, e_\beta\rangle\} \\ &= \tfrac{1}{2}\{S^\alpha(y)\theta^\beta(x) - S^\alpha(x)\theta^\beta(y) + \theta^\alpha(y)S^\beta(x) - \theta^\alpha(x)S^\beta(y)\},\end{aligned}$$

that is

$$T^B_\alpha = \tfrac{1}{2}(S^\beta \wedge \theta^\alpha + \theta^\beta \wedge S^\alpha) \tag{3.7}$$

where S^β is the one form defined by $S^\beta(x) = \langle Sx, e_\beta\rangle$ where S is the Ricci tensor. In terms of the orthonormal frames e_α and its dual θ^α,

$$S^\lambda(x) = \langle Sx, e_\lambda\rangle = \left\langle \sum_\mu R(x, e_\mu)e_\mu, e_\lambda \right\rangle = \sum_{\mu,\alpha} R^\alpha_{\mu\alpha\mu} x^\alpha.$$

Thus $S^\lambda = \sum_{\mu,\alpha} R^\lambda_{\mu\alpha\mu}\theta^\alpha$. The following identity will be useful later:

$$\begin{aligned}\sum_\lambda S^\lambda \wedge \theta^\lambda &= \sum_{\mu,\alpha,\lambda} R^\lambda_{\mu\alpha\mu}\theta^\alpha \wedge \theta^\lambda = \sum_{\mu,\alpha,\lambda} R^\alpha_{\mu\lambda\mu}\theta^\alpha \wedge \theta^\lambda \\ &= -\sum_{\mu,\alpha,\lambda} R^\alpha_{\mu\lambda\mu}\theta^\lambda \wedge \theta^\alpha \\ &= -\sum_{\mu,\alpha,\lambda} R^\lambda_{\mu\alpha\mu}\theta^\alpha \wedge \theta^\lambda \\ &= -\sum_\lambda S^\lambda \wedge \theta^\lambda.\end{aligned}$$

Thus

$$\sum_\lambda S^\lambda \wedge \theta^\lambda = 0. \tag{3.8}$$

From the spinor representations we get

$$\begin{aligned}(L_-)^1_1 &= -\tfrac{i}{2}\left(L^1_2 - L^3_4 - L^5_6\right) \\ &= -\frac{1}{2}(\theta^1 \wedge \theta^2 - \theta^3 \wedge \theta^4 - \theta^5 \wedge \theta^6) \\ &= \frac{1}{4}\{\Theta^1 \wedge \overline{\Theta}^1 - \Theta^2 \wedge \overline{\Theta}^2 - \Theta^3 \wedge \overline{\Theta}^3\}\end{aligned}$$

where $\Theta^\alpha = \theta^{2\alpha-1} + i\theta^{2\alpha}$ and $\overline{\Theta}^\alpha$ its conjugate; similarly we get

$$\begin{aligned}
(L_-)^1_2 &= -\overline{(L_-)^2_1} = \tfrac{1}{2}\left\{L^1_3 + L^2_4 - i\left(L^1_4 - L^2_3\right)\right\} = \frac{1}{2}\Theta^1 \wedge \overline{\Theta}^2, \\
(L_-)^1_3 &= -\overline{(L_-)^3_1} = \frac{1}{2}\Theta^1 \wedge \overline{\Theta}^3, \\
(L_-)^1_4 &= -\overline{(L_-)^4_1} = -\frac{1}{2}\overline{\Theta}^2 \wedge \overline{\Theta}^3, \\
(L_-)^2_2 &= \frac{1}{4}\{-\Theta^1 \wedge \overline{\Theta}^1 + \Theta^2 \wedge \overline{\Theta}^2 - \Theta^3 \wedge \overline{\Theta}^3\} \\
(L_-)^2_3 &= -\overline{(L_-)^3_2} = \frac{1}{2}(\Theta^2 \wedge \Theta^3) \\
(L_-)^2_4 &= -\overline{(L_-)^4_2} = \frac{1}{2}(\overline{\Theta}^1 \wedge \overline{\Theta}^3) \\
(L_-)^3_3 &= \frac{1}{4}\{-\Theta^1 \wedge \overline{\Theta}^1 - \Theta^2 \wedge \overline{\Theta}^2 + \Theta^3 \wedge \overline{\Theta}^3\} \\
(L_-)^3_4 &= -\overline{(L_-)^4_3} = -\frac{1}{2}\overline{\Theta}^1 \wedge \overline{\Theta}^2 \\
(L_-)^4_4 &= \frac{1}{4}\{\Theta^1 \wedge \overline{\Theta}^1 + \Theta^2 \wedge \overline{\Theta}^2 + \Theta^3 \wedge \overline{\Theta}^3\}.
\end{aligned}$$

Thus we may represent $(I \wedge I)_-$ in the following form:

$$
\begin{aligned}
(R_1)_- &= \frac{\sigma}{n(n-1)}(I \wedge I)_- \\
&= \frac{\sigma}{4n(n-1)} \left[\begin{array}{cc|cc}
\Theta^1 & -\bar{\Theta}^2 & -\bar{\Theta}^3 & 0 \\
\Theta^2 & \bar{\Theta}^1 & 0 & -\bar{\Theta}^3 \\
\hline
\Theta^3 & 0 & \bar{\Theta}^1 & \bar{\Theta}^2 \\
0 & \Theta^3 & -\Theta^2 & \Theta^1
\end{array}\right] \\
&\quad \wedge \left[\begin{array}{cc|cc}
\bar{\Theta}^1 & \bar{\Theta}^2 & \bar{\Theta}^3 & 0 \\
-\Theta^2 & \Theta^1 & 0 & \bar{\Theta}^3 \\
\hline
-\Theta^3 & 0 & \Theta^1 & -\bar{\Theta}^2 \\
0 & -\Theta^3 & \Theta^2 & \bar{\Theta}^1
\end{array}\right].
\end{aligned}
$$

Analogously, we have

$$
\begin{aligned}
(T_-)_1^1 &= -\frac{i}{2}\left(T_2^1 - T_4^3 - T_6^5\right) \\
&= -\frac{i}{4}\{S^1 \wedge \theta^2 + \theta^1 \wedge S^2 - S^3 \wedge \theta^4 \\
&\qquad -\theta^3 \wedge S^4 - S^5 \wedge \theta^6 - \theta^5 \wedge S^6\} \\
&= \frac{1}{4}\{\mathscr{S}^1 \wedge \bar{\Theta}^1 - S^1 \wedge \theta^1 - S^2 \wedge \theta^2 \\
&\qquad +\bar{\mathscr{S}}^2 \wedge \Theta^2 - S^3 \wedge \theta^3 - S^4 \wedge \theta^4 \\
&\qquad +\bar{\mathscr{S}}^3 \wedge \Theta^3 - S^5 \wedge \theta^5 - S^6 \wedge \theta^6\} \\
&= \frac{1}{4}\{\mathscr{S}^1 \wedge \bar{\Theta}^1 + \bar{\mathscr{S}}^2 \wedge \Theta^2 + \bar{\mathscr{S}}^3 \wedge \Theta^3\}
\end{aligned}
$$

where $\mathscr{S}^\alpha = S^{2\alpha-1} + iS^{2\alpha}$ and $\bar{\mathscr{S}}^\alpha$ its conjugate. We have also use equation (3.8). Proceeding in a similar way, we get

$$
\begin{aligned}
(T_-)_2^1 &= -\overline{(T_-)_1^2} = \frac{1}{2}\left\{T_3^1 + T_4^2 - i\left(T_4^1 - T_3^2\right)\right\} \\
&= \frac{1}{4}\{\mathscr{S}^1 \wedge \bar{\Theta}^2 - \bar{\mathscr{S}}^2 \wedge \Theta^1\} \\
(T_-)_3^1 &= -\overline{(T_-)_1^3} = \frac{1}{4}\{\mathscr{S}^1 \wedge \bar{\Theta}^3 - \bar{\mathscr{S}}^3 \wedge \Theta^1\} \\
(T_-)_4^1 &= -\overline{(T_-)_1^4} = \frac{1}{4}\{-\bar{\mathscr{S}}^2 \wedge \bar{\Theta}^3 + \bar{\mathscr{S}}^3 \wedge \bar{\Theta}^2\} \\
(T_-)_2^2 &= \frac{1}{4}\{\mathscr{S}^2 \wedge \bar{\Theta}^2 + \bar{\mathscr{S}}^1 \wedge \Theta^1 + \bar{\mathscr{S}}^3 \wedge \Theta^3\}
\end{aligned}
$$

$$
\begin{aligned}
(T_-)^2_3 &= -\overline{(T_-)^3_2} = \frac{1}{4}\{\mathscr{S}^2 \wedge \bar{\Theta}^3 - \bar{\mathscr{S}}^3 \wedge \Theta^2\} \\
(T_-)^2_4 &= -\overline{(T_-)^4_2} = \frac{1}{4}\{\bar{\mathscr{S}}^1 \wedge \bar{\Theta}^3 - \bar{\mathscr{S}}^3 \wedge \bar{\Theta}^1\} \\
(T_-)^3_3 &= \frac{1}{4}\{\mathscr{S}^3 \wedge \bar{\Theta}^3 + \bar{\mathscr{S}}^1 \wedge \Theta^1 + \bar{\mathscr{S}}^2 \wedge \Theta^2\} \\
(T_-)^3_4 &= -\overline{(T_-)^4_3} = \frac{1}{4}\{-\bar{\mathscr{S}}^1 \wedge \bar{\Theta}^2 + \bar{\mathscr{S}}^2 \wedge \bar{\Theta}^1\} \\
(T_-)^4_4 &= \frac{1}{4}\{\mathscr{S}^3 \wedge \bar{\Theta}^3 + \mathscr{S}^2 \wedge \bar{\Theta}^2 + \mathscr{S}^1 \wedge \bar{\Theta}^1\}.
\end{aligned}
$$

Thus we may express $(R_2)_-$ as follows:

$$
\begin{aligned}
(R_2)_- &= \frac{2}{n-2}(S \wedge I)_- - \frac{2\sigma}{n(n-2)}(I \wedge I)_- \\
&= \left[\frac{1}{2(n-2)} \left[\begin{array}{cc|cc} \mathscr{S}^1 & -\bar{\mathscr{S}}^2 & -\bar{\mathscr{S}}^3 & 0 \\ \mathscr{S}^2 & \bar{\mathscr{S}}^1 & 0 & -\bar{\mathscr{S}}^3 \\ \hline \mathscr{S}^3 & 0 & \bar{\mathscr{S}}^1 & \bar{\mathscr{S}}^2 \\ 0 & \mathscr{S}^3 & -\mathscr{S}^2 & \mathscr{S}^1 \end{array}\right] \right. \\
&\qquad \left. - \frac{\sigma}{2n(n-2)} \left[\begin{array}{cc|cc} \Theta^1 & -\bar{\Theta}^2 & -\bar{\Theta}^3 & 0 \\ \Theta^2 & \Theta^1 & 0 & -\bar{\Theta}^3 \\ \hline \Theta^3 & 0 & \bar{\Theta}^1 & \bar{\Theta}^2 \\ 0 & \Theta^3 & -\Theta^2 & \Theta^1 \end{array}\right] \right] \\
&\qquad \wedge \left[\begin{array}{cc|cc} \bar{\Theta}^1 & \bar{\Theta}^2 & \bar{\Theta}^3 & 0 \\ -\Theta^2 & \Theta^1 & 0 & \bar{\Theta}^3 \\ \hline -\Theta^3 & 0 & \Theta^1 & -\Theta^2 \\ 0 & -\Theta^3 & \Theta^2 & \bar{\Theta}^1 \end{array}\right].
\end{aligned}
$$

THEOREM 3. *The curvature of the negative spinor bundle admits the decomposition*

$$R_- = (R_1)_- + (R_2)_- + W_-$$

where

$$
\begin{aligned}
(R_1)_- &= \frac{\sigma}{4n(n-1)} {}^t\bar{\Theta} \wedge \Theta, \\
(R_2)_- &= \frac{1}{2(n-2)} \mathscr{S} \wedge \Theta - \frac{\sigma}{2n(n-2)} {}^t\bar{\Theta} \wedge \Theta, \\
W_- &= R_- - \frac{1}{2(n-2)} \mathscr{S} \wedge \Theta + \frac{\sigma}{4(n-1)(n-2)} {}^t\bar{\Theta} \wedge \Theta.
\end{aligned}
$$

Remark. Theorem 3 applies to the case $n = 2$ as well. The computations are indicated by the 2×2 submatrices (double-lined) in the upper left-hand corner.

Computing as above but using the positive spinor representation, we get the corresponding decomposition for the positive spinor bundle:

THEOREM 4. *The curvature of the positive spinor bundle admits the decomposition*

$$R_+ = (R_1)_+ + (R_2)_+ + W_+$$

where

$$(R_1)_+ = \frac{1}{4n(n-1)}\Theta \wedge {}^t\overline{\Theta},$$

$$(R_2)_+ = \frac{1}{2(n-2)}{}^t\overline{\mathcal{S}} \wedge {}^t\overline{\Theta} - \frac{\sigma}{2n(n-2)}\Theta \wedge {}^t\overline{\Theta},$$

$$W_+ = R_+ - \frac{1}{2(n-2)}{}^t\overline{\mathcal{S}} \wedge {}^t\overline{\Theta} + \frac{\sigma}{4(n-1)(n-2)}\Theta \wedge {}^t\overline{\Theta}.$$

In the above formulas, $\mathcal{S}$ and Θ represents the following matrice of 1-forms:

$$\mathcal{S} = \left[\begin{array}{cc||cc} \mathcal{S}^1 & -\overline{\mathcal{S}}^2 & -\overline{\mathcal{S}}^3 & 0 \\ \mathcal{S}^2 & \overline{\mathcal{S}}^1 & 0 & -\overline{\mathcal{S}}^3 \\ \hline\hline \mathcal{S}^3 & 0 & \overline{\mathcal{S}}^1 & \overline{\mathcal{S}}^2 \\ 0 & \mathcal{S}^3 & -\mathcal{S}^2 & \mathcal{S}^1 \end{array}\right],$$

$$\Theta = \left[\begin{array}{cc||cc} \overline{\Theta}^1 & \overline{\Theta}^2 & \overline{\Theta}^3 & 0 \\ -\Theta^2 & \Theta^1 & 0 & \overline{\Theta}^3 \\ \hline\hline -\Theta^3 & 0 & \Theta^1 & -\overline{\Theta}^2 \\ 0 & -\Theta^3 & \Theta^2 & \overline{\Theta}^1 \end{array}\right].$$

From Theorems 1 and 2 of §2, the integrability conditions for the canonical almost complex structures on the negative and positive spinor bundles are given respectively by

$$R_- = A \wedge \Theta, \quad R_+ = B \wedge {}^t\overline{\Theta}$$

Now from the decomposition of R_- and R_+, it is straightforward to show that the above conditions are equivalent to $W_- = 0$, $W_+ = 0$ respectively. We summarize these results in the following:

THEOREM 5. *The negative (resp. positive) spinor bundles $\mathcal{P}(V_-)$ (resp. $\mathcal{P}(V_+)$) over a $2n$-dimensional ($n = 2$ or 3) oriented Riemannian manifold M, with the canonical almost complex structure is a complex manifold if and only if $W_- = 0$ (resp. $W_+ = 0$) where W is the Weyl conformal tensor of M.*

Note that since **so**(6) is isomorphic to **su**(4) via the positive as well as the negative spinor representations. Thus for $n = 3$ the conditions $W_+ = 0$, $W_- = 0$ and $W = 0$ are equivalent. However for $n = 2$, $\mathbf{so}(4) \simeq \mathbf{su}(2) \oplus \mathbf{su}(2)$ and $W_\pm$ are respectively the self-dual and anti-self dual part of the Weyl tensor W. If M is a Kähler surface then $W_+ = 0$ is equivalent to the vanishing of the scalar curvature whereas $W_- = 0$ is equivalent to the vanishing of the Bochner tensor.

4. Integrability of submanifolds of the spinor bundle

The negative spinor bundle V_- contains natural subspaces F_- spanned by ξ_1, ξ_2 and ξ_3 and G_- spanned by $\xi_1\xi_2\xi_3$ whereas V_+ contains F_+ spanned by $\xi_1\xi_2$, $\xi_2\xi_3$ and $\xi_3\xi_1$ and G_+ spanned by 1. To study these subspaces we must restrict ourselves to the case where the base M is a complex 3-dimensional Kähler manifold with Kähler metric h and almost complex structure J. Let ν and Ω respectively be the hermitian connection and curvature of h. These are forms with values in **u**(3), the Lie algebra of the unitary group $U(3)$. Let e_α, $1 \le \alpha \le 6$, with $e_{2\alpha} = Je_{2\alpha-1}$ be an oriented orthonormal basis with respect to the Riemannian metric $g = \operatorname{Re} h$ for the real tangent bundle, then

$$f_\alpha = (e_{2\alpha-1} - ie_{2\alpha})/2, \quad 1 \le \alpha \le 3,$$

is a unitary frame for the holomorphic tangent bundle of M. With respect to these frames the hermitian connection ν of h and the Riemannian connection ω are related by the formulas

$$\omega_{2\alpha-1}^{2\beta-1} = \omega_{2\alpha}^{2\beta} = (\nu_I)_\alpha^\beta, \qquad \omega_{2\alpha-1}^{2\beta} = -\omega_{2\alpha}^{2\beta-1} = (\nu_{II})_\alpha^\beta \tag{4.1}$$

where ν_I and ν_{II} are respectively the real and imaginary parts of ν.
Similarly for the curvatures we also have the relations

$$R_{2\alpha-1}^{2\beta-1} = R_{2\alpha}^{\beta} = (\Omega_I)_\alpha^\beta, \qquad R_{2\alpha-1}^{2\beta} = -R_{2\alpha}^{2\beta-1} = (\Omega_{II})_\alpha^\beta \tag{4.2}$$

where Ω_I and Ω_{II} are the real and imaginary parts of Ω. We remark here that in the literature the basis are often chosen so that

$$Je_\alpha = e_{n+\alpha} \quad \text{and} \quad f_\alpha = e_\alpha - ie_{n+\alpha}, 1 \le \alpha \le n.$$

In that case the connections and curvatures are related by

$$\omega = \begin{bmatrix} \nu_I & \nu_{II} \\ -\nu_{II} & \nu_I \end{bmatrix}, \quad R = \begin{bmatrix} \Omega_I & \Omega_{II} \\ -\Omega_{II} & \Omega_I \end{bmatrix}$$

(cf. [27], p. 271).

With the help of formulas (4.1) and (4.2) we see that the spinor representations of the Kählerian connections and curvatures assume the following special forms:

$$\begin{aligned}
(\omega_-)_1^1 &= -\frac{i}{2}\left(\omega_2^1 - \omega_4^3 - \omega_6^5\right) \\
&= -\frac{i}{2}\left\{-(\nu_{II})_1^1 + (\nu_{II})_2^2 + (\nu_{II})_3^3\right\} \\
&= \frac{i}{2}\left\{(\nu_{II})_1^1 - (\nu_{II})_2^2 - (\nu_{II})_3^3\right\} = \frac{1}{2}\left(\nu_1^1 - \nu_2^2 - \nu_3^3\right), \\
(\omega_-)_2^1 &= \frac{1}{2}\left(\omega_3^1 + \omega_4^2\right) - i\left(\omega_4^1 - \omega_3^2\right)\}\big) \\
&= \frac{i}{2}\left\{(\nu_I)_2^1 + (\nu_I)_2^1 - i\left(-(\nu_{II})_2^1 - (\nu_{II})_2^1\right)\right\} \\
&= (\nu_I)_2^1 + i(\nu_{II})_2^1 = \nu_2^1, \\
(\omega_-)_3^1 &= \nu_3^1, \\
(\omega_-)_4^1 &= -\frac{1}{2}\left(\omega_5^3 - \omega_6^4\right) - i\left(\omega_6^3 + \omega_5^4\right)\} \\
&= -\frac{1}{2}\left\{(\nu_I)_3^2 - (\nu_I)_3^2 - i\left(-(\nu_{II})_3^2 + (\nu_{II})_3^2\right)\right\} = 0. \\
(\omega_-)_2^2 &= -\frac{i}{2}\left\{(\nu_{II})_1^1 - (\nu_{II})_2^2 + (\nu_{II})_3^3\right\} = -\left(\nu_1^1 - \nu_2^2 + \nu_3^3\right) \\
(\omega_-)_3^2 &= \nu_3^2 \\
(\omega_-)_4^2 &= 0 \\
(\omega_-)_3^3 &= -\frac{i}{2}\left\{-(\nu_{II})_1^1 + (\nu_{II})_2^2 - (\nu_{II})_3^3\right\} = -\left(\nu_1^1 + \nu_2^2 - \nu_3^3\right), \\
(\omega_-)_4^3 &= 0, \\
(\omega_-)_4^4 &= -\frac{i}{2}\left\{-(\nu_{II})_1^1 + (\nu_{II})_2^2 + (\nu_{II})_3^3\right\} = \left(\nu_1^1 + \nu_2^2 + \nu_3^3\right).
\end{aligned}$$

In other words,

(4.3)

$$\omega_- = \frac{1}{2}\left[\begin{array}{ccc|c} \nu_1^1 - \nu_2^2 - \nu_3^3 & 2\nu_1^1 & 2\nu_3^1 & 0 \\ 2\nu_1^2 & -\nu_1^1 + \nu_2^2 - \nu_3^3 & 2\nu_3^2 & 0 \\ 2\nu_1^3 & 2\nu_2^3 & -\nu_1^1 - \nu_2^2 + \nu_3^3 & 0 \\ \hline 0 & 0 & 0 & \nu_1^1 + \nu_2^2 + \nu_3^3 \end{array}\right].$$

Similarly,

(4.4)

$$R_- = \frac{1}{2}\left[\begin{array}{ccc|c} \Omega_1^1 - \Omega_2^2 - \Omega_3^3 & 2\Omega_1^1 & 2\Omega_3^1 & 0 \\ 2\Omega_1^2 & -\Omega_1^1 + \Omega_2^2 - \Omega_3^3 & 2\Omega_3^2 & 0 \\ 2\Omega_1^3 & 2\Omega_2^3 & -\Omega_1^1 - \Omega_2^2 + \Omega_3^3 & 0 \\ \hline 0 & 0 & 0 & \Omega_1^1 + \Omega_2^2 + \Omega_3^3 \end{array}\right].$$

Thus the connection and curvature of V_- splits into direct sums compatible with the decomposition $V_- = \mathbf{C}\langle \xi_1, \xi_2, \xi_3 \rangle \oplus \mathbf{C}\langle \xi_1\xi_2\xi_3 \rangle = F_- \oplus G_-$.

As for the integrability conditions, we choose the horizontal forms η^α, $1 \le \alpha \le 4$, as in (2.3) with $z^4 = 0$ since $F_- = \{z_4 = 0\}$. For vertical forms we now only need three: ∇z^1, ∇z^2 and ∇z^3. At any rate observe that from (2.2) and (4.3) we have $\nabla z^4 = dz^4 + \Sigma z^\beta (\omega_-)_\beta^4 \equiv 0$ on F_-. With this in mind, we see immediately from equations (2.4) and (2.5) that the integrability conditions are still satisfied by the horizontal forms. Similarly we can read off the integrability conditions for the vertical forms ∇z^1, ∇z^2 and ∇z^3 from (2.8) by setting $z^4 = 0$ and using (4.3). Summarizing, we have:

THEOREM 6. *Let M be a Kähler manifold of complex dimension 3. The submanifold $\mathscr{P}(F_-)$ with the induced almost complex structure is integrable iff*

$$(R_-)_\alpha^\beta = \sum_{\gamma=1}^{4} A_\gamma^\beta \wedge \Theta_\alpha^\gamma \quad \textit{for } 1 \le \alpha, \beta \le 3$$

where Θ_α^γ are the components of Θ as defined in (2.7).

Analogously, by setting $z^1 = z^2 = z^3 = 0$ we also obtain the integrability conditions for η^1, η^2, η^3, η^4 and ∇z^4 on G_-.

THEOREM 7. *Let M be a complex 3-dimensional Kähler manifold which is a spin manifold. Then the negative spinor bundle V_- (hence also G_-) is globally defined and the integrability of G_- is given by $(R_-)_4^4 = \Sigma_{\gamma=1}^4 A_\gamma^4 \wedge \Theta_4^\gamma$.*

The corresponding formulas for the connection and curvature of the positive spinor bundles are given by

$$\omega_+ = \frac{1}{2}\begin{bmatrix} -\nu_1^1 - \nu_2^2 - \nu_3^3 & 0 & 0 & 0 \\ 0 & \nu_1^1 + \nu_2^2 - \nu_3^3 & 2\nu_3^2 & -2\nu_3^1 \\ 0 & 2\nu_2^3 & \nu_1^1 - \nu_2^2 + \nu_3^3 & 2\nu_2^1 \\ 0 & -2\nu_1^3 & 2\nu_1^2 & -\nu_1^1 + \nu_2^2 + \nu_3^3 \end{bmatrix},$$

$$R_+ = \frac{1}{2}\begin{bmatrix} -\Omega_1^1 - \Omega_2^2 - \Omega_3^3 & 0 & 0 & 0 \\ 0 & \Omega_1^1 + \Omega_2^2 - \Omega_3^3 & 2\Omega_3^2 & -2\Omega_3^1 \\ 0 & 2\Omega_2^3 & \Omega_1^1 - \Omega_2^2 + \Omega_3^3 & 2\Omega_2^1 \\ 0 & -2\Omega_1^3 & 2\Omega_1^2 & -\Omega_1^1 + \Omega_2^2 + \Omega_3^3 \end{bmatrix}.$$

Let F_+ be the subspace of V_+ spanned by $\xi_1\xi_2$, $\xi_2\xi_3$ and $\xi_3\xi_1$ and G_+ spanned by 1, then we have the following integrability conditions:

THEOREM 8. *Let M be a Kähler manifold of complex dimension 3. Then the submanifold $\mathscr{P}(F_+) \subset \mathscr{P}(V_+)$ with the induced almost complex structure is integrable iff $(R_+)^\alpha_\beta = \Sigma^4_{\gamma=1} B^\alpha_\gamma \wedge \Psi^\gamma_\beta$ for $2 \le \alpha, \beta \le 4$, where Ψ^γ_β are the components of the matrix of 1-form $\Psi = {}^t\overline{\Theta}$. Furthermore, if M is spin then the integrability conditions for G_+ is given by $(R_+)^1_1 = \Sigma^4_{\gamma=1} B^1_\gamma \wedge \Psi^\gamma_1$.*

To understand the integrability conditions obtained above we begin by reviewing the decomposition of a Kählerian curvature tensor under the action of the unitary group. As before, let E be a real vector space of dimension $2n$ with an almost complex structure J and a hermitian inner product $\langle\ ,\ \rangle$. The complexification $E \otimes \mathbf{C}$ decomposes into E' and E'' where E' consists of vectors of type (1,0) and $E'' = \overline{E'}$. Denote by $E' \cdot E'$ the symmetric product of E' with E' and $\mathscr{U}(E' \cdot E')$ the unitary endomorphisms on $E' \cdot E'$. The elements of $\mathscr{U}(E' \cdot E')$ are referred to as Kählerian curvature tensors. In contrast to the Riemannian case, elements of $\mathscr{U}(E' \cdot E')$ satisfies automatically the Bianchi identity. The space $\mathscr{U}(E' \cdot E')$ decomposes under the action of the unitary group into three invariant subspaces $F_1 \oplus F_2 \oplus \mathscr{B}$. The subspace F_1 is spanned (over $\mathscr{R}$) by $I \cdot I$ where I is the identity on E'. Denote by $(\ ,\ )$ the hermitian inner product on E', i.e., $(Z, W) = 2\langle Z, \overline{W}\rangle$ for, $Z, W \in E'$. Then we have

$$(4.5)\qquad \begin{aligned} (I \cdot I(Z \cdot W), U \cdot V) &= \tfrac{1}{2}(Z \cdot W + W \cdot Z, U \cdot V) \\ &= (Z \cdot W, U \cdot V) \\ &= (Z, U)(W, V) + (Z, V)(W, U). \end{aligned}$$

Hence $I \cdot I$ is identified with the tensor

$$L(Z,\bar{V})W = (Z,V)W + (W,V)Z$$

which is characterized by the property that the holomorphic sectional curvature

$$k(Z \wedge \bar{Z}) = \langle R(Z,\bar{Z})Z, \bar{Z}\rangle / |Z|^4 \equiv 1.$$

The subspace F_2 consists of endomorphisms of the form $\Omega_0 \cdot I$ where $\Omega_0 \in \mathscr{U}_0 E'$, the set of unitary endomorphisms on E' with zero trace. The subspace $\mathscr{B}$ is the orthogonal complements of $F_1 \oplus F_2$ in $\mathscr{U}(E' \cdot E')$. Elements of $\mathscr{B}$ are called Bochner tensors. The dimensions of the various subspaces are

$$\begin{aligned}
\dim_{\mathbf{R}} \mathscr{U}(E' \cdot E') &= \frac{1}{4} n^2 (n+1)^2, \\
\dim_{\mathbf{R}} F_1 &= 1, \\
\dim_{\mathbf{R}} F_2 &= n^2 - 1, \\
\dim_{\mathbf{R}} \mathscr{B} &= \frac{1}{4} n^2 (n-1)(n+3)
\end{aligned}$$

where $\dim_{\mathbf{C}} E' = n \geq 2$.

Let M be a complex n-dimensional Kähler manifold with Kähler metric and Ω the associated curvature, Ω_1, Ω_2 and B the projections into the respective invariant subspace, then $\Omega = \Omega_1 + \Omega_2 + B$ with

$$\begin{aligned}
\Omega_1 &= \frac{\sigma}{n(n+1)} I \cdot I, \\
\Omega_2 &= \frac{2}{n+2} K \cdot I - \frac{2\sigma}{n(n+2)} I \cdot I, \qquad (4.6)\\
B &= \Omega - \frac{2}{n+2} K \cdot I + \frac{\sigma}{(n+1)(n+2)} I \cdot I
\end{aligned}$$

where K is the Ricci tensor and σ is the scalar curvature.

Components of the curvature matrix of $I \cdot I$ w.r.t. unitary frames $\Theta^\alpha = \theta^{2\alpha-1} + i\theta^{2\alpha}$ are given by

$$L_\alpha^\beta = \Theta^\beta \wedge \bar{\Theta}^\alpha + \delta_\alpha^\beta \sum_\gamma \Theta^\gamma \wedge \bar{\Theta}^\gamma.$$

By a straightforward calculation we obtain the negative spinor representation of $I \cdot I$:

$$(I \cdot I) = \begin{bmatrix} -\Theta^2 \wedge \overline{\Theta}^2 - \Theta^3 \wedge \overline{\Theta}^3 & \Theta^1 \wedge \overline{\Theta}^2 & \Theta^1 \wedge \overline{\Theta}^3 & 0 \\ \Theta^2 \wedge \overline{\Theta}^1 & -\Theta^1 \wedge \overline{\Theta}^1 - \Theta^3 \wedge \overline{\Theta}^3 & \Theta^2 \wedge \overline{\Theta}^3 & 0 \\ \Theta^3 \wedge \overline{\Theta}^1 & \Theta^3 \wedge \overline{\Theta}^2 & -\Theta^1 \wedge \overline{\Theta}^1 - \Theta^2 \wedge \overline{\Theta}^2 & 0 \\ 0 & 0 & 0 & A \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\overline{\Theta}^2 & -\overline{\Theta}^3 & 0 \\ 0 & \overline{\Theta}^1 & 0 & -\overline{\Theta}^3 \\ 0 & 0 & \overline{\Theta}^1 & \overline{\Theta}^2 \\ 0 & 2\Theta^3 & -2\Theta^2 & 2\Theta^1 \end{bmatrix} \wedge \begin{bmatrix} \overline{\Theta}^1 & \overline{\Theta}^2 & \overline{\Theta}^3 & 0 \\ -\Theta^2 & \Theta^1 & 0 & \overline{\Theta}^3 \\ -\Theta^3 & 0 & \Theta^1 & -\Theta^2 \\ 0 & -\Theta^3 & \Theta^2 & \overline{\Theta}^1 \end{bmatrix}$$

$$+ \left[\begin{array}{ccc|c} & & & -2\overline{\Theta}^2 \wedge \overline{\Theta}^3 \\ & 0 & & 2\overline{\Theta}^1 \wedge \overline{\Theta}^3 \\ & & & -2\overline{\Theta}^1 \wedge \overline{\Theta}^2 \\ \hline 2\Theta^2 \wedge \Theta^3 & -2\Theta^1 \wedge \Theta^3 & 2\Theta^1 \wedge \Theta^2 & 0 \end{array}\right]$$

where $A = 2\sum_{\alpha=1}^{3} \Theta^\alpha \wedge \overline{\Theta}^\alpha$.

Denoting respectively by $(I \cdot I)'_-$ and $(I \cdot I)''_-$ the F_- and G_- components of $(I \cdot I)_-$, we have

$$(I \cdot I)'_- = \mathscr{L} \wedge \Theta', \tag{4.7}$$

$$(I \cdot I)''_- = \nu \wedge \Theta'' \tag{4.8}$$

where

$$\mathscr{L} = \begin{bmatrix} 0 & -\overline{\Theta}^2 & -\overline{\Theta}^3 & 0 \\ 0 & \overline{\Theta}^1 & 0 & -\overline{\Theta}^3 \\ 0 & 0 & \overline{\Theta}^1 & \overline{\Theta}^2 \end{bmatrix},$$

$$\nu = (\nu_1, \nu_2, \nu_3, \nu_4)$$

with $\nu_1 = 0$, $\nu_2 = 2\Theta^3$, $\nu_3 = -2\Theta^2$, $\nu_4 = 2\Theta^1$. We also denote by Θ' the (4×3)-matrix consisting of the first three columns of the matrix Θ and Θ'' the 4-th column of Θ.

For the curvature tensor $K \cdot I$ we have

$$\begin{aligned} (K \cdot I(z \cdot w), u \cdot v) &= \frac{1}{2}(Kz \cdot w + Kw \cdot z, u \cdot v) \\ &= \frac{1}{2}\{(Kz, u)(w, v) + (Kz, v)(w, u) \\ &\qquad + (Kw, u)(z, v) + (Kw, v)(z, u)\}; \end{aligned} \tag{4.8}$$

hence it is identified with the (1, 3) tensor

$$T(z,\bar{v})w = \frac{1}{2}\{(w,v)Kz + (Kz,v)w + (z,v)Kw + (Kw,v)z\}.$$

Relative to the unitary frame $\xi_\alpha = \frac{1}{2}(e_{2\alpha-1} - ie_{2\alpha})$, the components of T are given by

$$\begin{aligned}(4.9)\quad T_\alpha^\beta &= (T\xi_\alpha, \xi_\beta)\\ &= \frac{1}{4}\Big\{\delta_\alpha^\beta\rho + K_{\alpha\bar{\beta}}\sum_\gamma \Theta^\gamma\wedge\overline{\Theta}^\gamma + \sum_\gamma K_{\alpha\bar{\gamma}}\Theta^\beta\wedge\overline{\Theta}^\gamma + \sum_\gamma K_{\gamma\bar{\beta}}\Theta^\gamma\wedge\overline{\Theta}^\alpha\Big\}\end{aligned}$$

where $K_{\alpha\bar{\beta}} = (K\xi_\alpha, \xi_\beta) = \sum_\mu R^\mu_{\mu\alpha\bar{\beta}}$ and $\rho = \sum_{\alpha,\beta} K_{\alpha\bar{\beta}}\Theta^\alpha\wedge\overline{\Theta}^\beta$.

Applying the negative spinor representation we obtain

$$\begin{aligned}
(T_-)_1^1 &= \frac{1}{2}(T_1^1 - T_2^2 - T_3^3)\\
&= \frac{1}{4}\Big\{K^1\wedge\overline{\Theta}^1 - K^2\wedge\overline{\Theta}^2 - K^3\wedge\overline{\Theta}^3\\
&\qquad + \Theta^1\wedge\overline{K^1} - \Theta^2\wedge\overline{K^2} - \Theta^3\wedge\overline{K^3}\\
&\qquad + (K_{1\bar{1}} - K_{2\bar{2}} - K_{3\bar{3}})\sum_\gamma\Theta^\gamma\wedge\overline{\Theta}^\gamma - \rho\Big\},\\
(T_-)_2^1 &= -\overline{(T_-)_1^2} = \frac{1}{2}\Big\{K^1\wedge\overline{\Theta}^2 + \Theta^1\wedge\overline{K^2} + K_{2\bar{1}}\sum_\gamma\Theta^\gamma\wedge\overline{\Theta}^\gamma\Big\},\\
(T_-)_3^1 &= -\overline{(T_-)_1^3} = \frac{1}{2}\Big\{K^1\wedge\overline{\Theta}^3 + \Theta^1\wedge\overline{K^3} + K_{3\bar{1}}\sum_\gamma\Theta^\gamma\wedge\overline{\Theta}^\gamma\Big\},\\
(T_-)_4^1 &= -\overline{(T_-)_1^4} = 0,\\
(T_-)_2^2 &= -\frac{1}{4}\Big\{K^1\wedge\overline{\Theta}^1 - K^2\wedge\overline{\Theta}^2\\
&\qquad + K^3\wedge\overline{\Theta}^3 + \Theta^1\wedge\overline{K^1} - \Theta^2\wedge\overline{K^2} - \Theta^3\wedge\overline{K^3}\\
&\qquad + (K_{1\bar{1}} - K_{2\bar{2}} + K_{3\bar{3}})\sum_\gamma\Theta^\gamma\wedge\overline{\Theta}^\gamma + \rho\Big\},\\
(T_-)_3^2 &= -\overline{(T_-)_2^3} = \frac{1}{2}\Big\{K^2\wedge\overline{\Theta}^3 + \Theta^2\wedge\overline{K^3} + K_{3\bar{2}}\sum_\gamma\Theta^\gamma\wedge\overline{\Theta}^\gamma\Big\},\\
(T_-)_4^2 &= -\overline{(T_-)_2^4} = 0,
\end{aligned}$$

$$\begin{aligned}(T_-)_3^3 = -\frac{1}{4}\Big\{&K^1 \wedge \overline{\Theta}^1 + K^2 \wedge \overline{\Theta}^2 - K^3 \wedge \overline{\Theta}^3 \\ &+ \Theta^1 \wedge \overline{K^1} - \Theta^3 \wedge \overline{K^3} \\ &+ (K_{1\bar{1}} + K_{2\bar{2}} - K_{3\bar{3}})\sum_\gamma \Theta^\gamma \wedge \overline{\Theta}^\gamma + \rho\Big\},\end{aligned}$$

$$(T_-)_4^3 = -\overline{(T_-)_3^4} = 0,$$

$$(T_-)_4^4 = \frac{1}{4}\Big\{\sum_\alpha K^\alpha \wedge \overline{\Theta}^\alpha + \sum_\alpha \wedge \overline{K^\alpha} + \Big(\sum_\alpha K_{\alpha\bar{\alpha}}\Big)\Big(\sum_\gamma \Theta^\gamma \wedge \overline{\Theta}^\gamma\Big) + 3\rho\Big\}.$$

Since the Ricci tensor is hermitian, we may choose unitary frames so that K is in diagonal form with real eigenvalues λ_1, λ_2 and λ_3. The formulas above reduce to

$$\begin{aligned}(T_-)_1^1 &= \frac{1}{4}\Big\{2\lambda_1\Theta^1 \wedge \overline{\Theta}^1 - 2\lambda_2\Theta^2 \wedge \overline{\Theta}^2 - 2\lambda_3\Theta^3 \wedge \overline{\Theta}^3 \\ &\qquad + (\lambda_1 - \lambda_2 - \lambda_3)\sum_\gamma \Theta^\gamma \wedge \overline{\Theta}^\gamma - \sum_\gamma \lambda_\gamma \Theta^\gamma \wedge \overline{\Theta}^\gamma\Big\} \\ &= \frac{1}{4}\{2\lambda_1 - \lambda_2 - \lambda_3)\Theta^1 \wedge \overline{\Theta}^1 + (\lambda_1 - 4\lambda_2 - \lambda_3)\Theta^2 \wedge \overline{\Theta}^2 \\ &\qquad + (\lambda_1 - \lambda_2 - 4\lambda_3\}\Theta^3 \wedge \overline{\Theta}^3\},\end{aligned}$$

$$(T_-)_2^1 = -\overline{(T_-)_1^2} = \tfrac{1}{2}(\lambda_1 + \lambda_2)\Theta^1 \wedge \overline{\Theta}^2,$$

$$(T_-)_3^1 = -\overline{(T_-)_1^3} = \tfrac{1}{2}(\lambda_1 + \lambda_3)\Theta^1 \wedge \overline{\Theta}^3,$$

$$(T_-)_4^1 = -\overline{(T_-)_1^4} = (T_-)_4^2 = -\overline{(T_-)_2^4} = (T_-)_4^3 = -\overline{(T_-)_3^4} = 0,$$

$$\begin{aligned}(T_-)_2^2 = \frac{1}{4}\{&(-4\lambda_1 + \lambda_2 - \lambda_3)\Theta^1 \wedge \overline{\Theta}^1 + (-\lambda_1 + 2\lambda_2 - \lambda_3)\Theta^2 \wedge \overline{\Theta}^2 \\ &+ (-\lambda_1 + \lambda_2 - 4\lambda_3)\Theta^3 \wedge \overline{\Theta}^3\},\end{aligned}$$

$$(T_-)_3^2 = \frac{1}{2}(\lambda_2 + \lambda_3)\Theta^2 \wedge \overline{\Theta}^3,$$

$$\begin{aligned}(T_-)_3^3 = \frac{1}{4}\{&(-4\lambda_1 - \lambda_2 + \lambda_3)\Theta^1 \wedge \overline{\Theta}^1 + (-\lambda_1 - 4\lambda_2 + \lambda_3)\Theta^2 \wedge \overline{\Theta}^2 \\ &+ (-\lambda_1 - \lambda_2 + 2\lambda_3)\Theta^3 \wedge \overline{\Theta}^3\},\end{aligned}$$

$$\begin{aligned}(T_-)_4^4 = \frac{1}{4}\{&(6\lambda_1 + \lambda_2 + \lambda_3)\Theta^1 \wedge \overline{\Theta}^1 + (\lambda_1 + 6\lambda_2 + \lambda_3)\Theta^2 \wedge \overline{\Theta}^2 \\ &+ (\lambda_1 + \lambda_2 + 6\lambda_3)\Theta^3 \wedge \overline{\Theta}^3\}.\end{aligned}$$

From these expressions we observe that T_- can be expressed as

$$(4.10)\qquad T_- = \frac{1}{4}\left[\begin{array}{cccc} & \mathcal{M} & & \\ \hline \mu_1 & \mu_2 & \mu_3 & \mu_4 \end{array}\right] \wedge \Theta + \frac{1}{4}\left[\begin{array}{ccc|c} & & & \psi_1 \\ & H & & \psi_2 \\ & & & \psi_3 \\ \hline \nu_1 & \nu_2 & \nu_3 & 0 \end{array}\right]$$

where

$$\begin{aligned}
\mu_1 &= 0,\ \mu_2 = (\lambda_1 + \lambda_2 + 6\lambda_3)\Theta^3, \\
\mu_3 &= -(\lambda_1 + 6\lambda_2 + \lambda_3)\Theta^2, \\
\mu_4 &= (6\lambda_1 + \lambda_2 + \lambda_3)\Theta^1, \\
\nu_1 &= (2\lambda_1 + 7\lambda_2 + 7\lambda_3)\Theta^2 \wedge \Theta^3, \\
\nu_2 &= -(7\lambda_1 + 2\lambda_2 + 7\lambda_3)\Theta^1 \wedge \Theta^3, \\
\nu_3 &= (7\lambda_1 + 7\lambda_2 + 2\lambda_3)\Theta^1 \wedge \Theta^2, \\
\psi_1 &= (-2\lambda_1 + 5\lambda_2 + 5\lambda_3)\overline{\Theta}^2 \wedge \overline{\Theta}^3, \\
\psi_2 &= -(5\lambda_1 - 2\lambda_2 + 5\lambda_3)\overline{\Theta}^1 \wedge \overline{\Theta}^3
\end{aligned}$$

and

$$\psi_3 = (5\lambda_1 + 5\lambda_2 - 2\lambda_3)\overline{\Theta}^1 \wedge \overline{\Theta}^2.$$

The two matrices $\mathcal{M}$ and H are given by

(4.11)

$$\mathcal{M} = \begin{bmatrix} (2\lambda_1 - \lambda_2 - \lambda_3)\Theta^1 & (\lambda_1 - 4\lambda_2 - \lambda_3)\overline{\Theta}^2 & (\lambda_1 - \lambda_2 - 4\lambda_3)\overline{\Theta}^3 & 0 \\ (-\lambda_1 + 2\lambda_2 - \lambda_3)\Theta^2 & (4\lambda_1 - \lambda_2 + \lambda_3)\overline{\Theta}^1 & 0 & (-\lambda_1 + \lambda_2 - 4\lambda_3)\overline{\Theta} \\ (-\lambda_1 - \lambda_2 + 2\lambda_3)\Theta^3 & 0 & (4\lambda_1 + \lambda_2 - \lambda_3)\overline{\Theta}^1 & (\lambda_1 + 4\lambda_2 - \lambda_3)\overline{\Theta}^2 \end{bmatrix},$$

(4.12)

$$H = \begin{bmatrix} 0 & (\lambda_1 - \lambda_2)\Theta^1 \wedge \overline{\Theta}^2 & (\lambda_1 - \lambda_3)\Theta^1 \wedge \overline{\Theta}^3 \\ (\lambda_2 - \lambda_1)\Theta^2 \wedge \overline{\Theta}^1 & 0 & (\lambda_2 - \lambda_3)\Theta^2 \wedge \overline{\Theta}^3 \\ (\lambda_3 - \lambda_1)\Theta^3 \wedge \overline{\Theta}^1 & (\lambda_3 - \lambda_2)\Theta^3 \wedge \overline{\Theta}^2 & 0 \end{bmatrix}.$$

Denoting by T'_- and T''_- the F_- and G_- components of T_-, we have

$$(4.13)\qquad T'_- = \frac{1}{4}(\mathcal{M} \wedge \Theta' + H),$$

$$(4.14)\qquad T'' = \frac{1}{4}(\mu_1, \mu_2, \mu_3, \mu_4) \wedge \Theta'' = \frac{1}{4}\sum_{\beta=1}^{4} \mu_\beta \wedge \Theta_4^\beta$$

where Θ' is the first three columns of Θ and Θ'' is the 4-th column of Θ. Notice that the curvature T'_- satisfies the integrability condition of Theorem 6 iff $H = 0$ iff $\lambda_1 = \lambda_2 = \lambda_3 = 0$ iff the Ricci tensor is a multiple of the metric tensor, i.e., the metric is Einstein. On the other hand we see that T''_- satisfies automatically the integrability condition of Theorem 7.

From the decomposition (4.6) of the curvature tensor Ω we obtain the following decomposition of the F_- and G_- components of the curvature Ω_-:

$$\Omega'_- = (\Omega_1)'_- + (\Omega_2)'_- + B'_-, \quad \Omega''_- = (\Omega_1)''_- + (\Omega_2)''_- + B''_- \tag{4.15}$$

with

$$(\Omega_1)'_- = \frac{\sigma}{n(n+1)}(I \cdot I)'_- = \frac{\sigma}{n(n+1)}\mathscr{L} \wedge \Theta', \tag{4.16}$$

$$\begin{aligned}(\Omega_2)'_- &= \frac{2}{n+2}(K \cdot I)'_- - \frac{2\sigma}{n(n+1)}(I \cdot I)'_- \\ &= \left[\frac{1}{2(n+2)}\mathscr{M} - \frac{\sigma}{n(n+1)}L\right] \wedge \Theta' + \frac{1}{2(n+2)}H,\end{aligned} \tag{4.17}$$

$$\begin{aligned}B'_- = \Omega'_- &- \left[\frac{1}{2(n+2)}\mathscr{M} - \frac{\sigma}{(n+1)(n+2)}\mathscr{L}\right] \\ &\wedge \Theta' - \frac{1}{2(n+2)}H,\end{aligned} \tag{4.18}$$

$$(\Omega_1)''_- = \frac{\sigma}{n(n+1)}\mu \wedge \Theta'', \tag{4.19}$$

$$(\Omega_2)''_- = \left[\frac{1}{2(n+2)}\nu - \frac{2\sigma}{n(n+2)}\mu\right] \wedge \Theta'', \tag{4.20}$$

$$B''_- = \text{trace of } B = 0. \tag{4.21}$$

From the decompositions above and Theorems 6 and 7, it is straightforward to show:

THEOREM 9. *Let M be a complex 3-dimensional Kähler manifold then the submanifold $\mathscr{P}(F_-) \subset \mathscr{P}(V_-)$ with the induced almost complex structure is a complex manifold iff $B'_- = 0$ and that the metric is Einstein.*

Remark. It is easily seen from (4.4) that $B'_- = 0$ iff $B = 0$ so that the conditions in the above theorem means that M is Kähler-Einstein with vanishing Bochner tensor. The fiber in $\mathscr{P}(F_-)$ over each point of M is $\mathbf{C}P^2$.

THEOREM 10. *Let M be a complex 3-dimensional Kähler manifold such that M is spin. Then $G_- \subset V_-$ with the induced almost complex structure is a complex manifold.*

Remark. The fiber of G_- over each point $x \in M$ is $\mathbf{C}$. Note that the projections of $\mathscr{P}(F_-)$ and of G_- onto M are not holomorphic.

The corresponding decomposition and integrability for F_+ and G_+ are entirely analogous and will be omitted here.

5. Examples

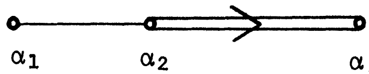
Example I. The standard spheres S^n, the Euclidean spaces $\mathbf{R}^n$ and the hyperbolic space $H^n = SO_0(1, n)/SO(n)$ are conformally flat; i.e., the Weyl tensor W is 0 and so are their quotients. For non-Einstein examples, the spaces $\mathbf{R}^p \times S^q$, $S^p \times S^q$, $\mathbf{R}^p \times H^q$, $H^p \times H^q$ with $p, q \geq 1$ and with the standard product metric are conformally flat iff $p = 1$ or $q = 1$; the space $H^p \times S^q$, $p, q \geq 1$ with the product metric is conformally flat iff $p = q$. It is also known that every orientable, locally irreducible, locally symmetric 4-manifold is half conformally flat, i.e. $W_- = 0$ (cf. Derdziński [6]).

Example II. The complex projective spaces $\mathbf{C}P^n$ with Fubini-Study metric, the complex Euclidean spaces $\mathbf{C}^n$ and the complex hyperbolic spaces D^n (i.e. unit ball in $\mathbf{C}^n$ with Bergmann metric) are Kähler-Einstein with vanishing Bochner tensor and, so are their quotients. For non-Einstein examples, the space $D^p \times \mathbf{C}P^q$, $p, q \geq 1$ with product metric has vanishing Bochner tensor iff $p = q$. For Kähler surfaces, $\mathbf{C}P^2$, $\mathbf{C}^2$, D^2 and $D^1 \times \mathbf{C}P^1$ exhaust the list of simply connected surfaces with vanishing Bochner tensor (cf. Chen [5]). For Kähler surfaces, $W_- = 0 \Leftrightarrow$ vanishing Bochner tensor; whereas $W_+ = 0 \Leftrightarrow$ scalar curvature vanishes (cf. Tricerri-Vanhecke [19], Derdziński [6]). Kähler surfaces with vanishing scalar curvature are classified by Hitchen [12]; they are either glat or a K-3 surface, an Enriques-surface or the orbit space of an Enriques-surface by an anti-holomorphic involution.

Example III. The twistor space $\mathscr{P}(V_-)$ over S^4 is $\mathscr{C}P^3$ and that of S^6 is the complex hyperquadric Q^6. In fact the twistor space of S^{2n} is $SO(2n+1)/U(n)$ with the unique (up to conjugation) invariant almost complex structure. We remark here that the uniqueness of the invariant almost complex structure corresponds to the fact that the center of $U(n)$ is one dimension. Also the 2-nd Betti number of such manifold is 1. The oriented orthonormal frame bundle of S^{2n} is $SO(2n+1)$ and the twistor space is the associate fiber bundle

$$SO(2n+1) \times_{SO(2n)} SO(2n)/U(n) \approx SO(2n+1)/U(n).$$

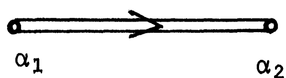
For S^4 and S^6, it is straightforward to show that canonical almost complex structure defined on $\mathscr{P}(V_-)$ is invariant by (and commutes with) the action of SO(7). Thus $\mathscr{P}(V_-)$ is identified with $SO(5)/U(2)$ (resp. $SO(7)/U(3)$) as complex homogeneous Kähler manifold. It is known the $SO(2n+1)/U(n) \approx SO(2n+2)/U(n+1)$ which are the hermitian symmetric spaces of type D_{n+1}, which for $n = 2$ and 3 are respectively $\mathbf{C}P^3$ and Q^6. Actually for these two cases we can check directly by computating the first chern number c_1 (we follow the notations of Bourbaki [4]). For $SO(5)/U(2)$ the Dynkin diagram is B_2:



The roots of $SO(5)$ are $\pm\xi_1$, $\pm\xi_2$, $\pm(\xi_1+\xi_2)$, $\pm(\xi_1-\xi_2)$ and the roots for $U(2)$ are $\pm(\xi_1-\xi_2)$. The positive roots are respectively $\xi_1=\alpha_1+\alpha_2$, $\xi_2=\alpha_2$, $\xi_1-\xi_2=\alpha_1$, $\xi_1+\xi_2=\alpha_1+2\alpha_2$ (and for $U(2)$, $\xi_1-\xi_2$). Thus the complementary positive roots are ξ_1, ξ_2 and $\xi_1+\xi_2$. Then c_1 is given by

$$2(\xi_1+\xi_2+\xi_1+\xi_2,\xi_2)=4(\xi_2,\xi_2)=4=1+\dim_{\mathbf{C}} SO(5)/U(2).$$

Analogously for $SO(7)/U(3)$, the Dynkin diagram is B_3:



The positive roots for $SO(7)$ are $\xi_1=\alpha_1+\alpha_2+\alpha_3$, $\xi_2=\alpha_2+\alpha_3$, $\xi_3=\alpha_3$, $\xi_1-\xi_2=\alpha_1$, $\xi_1-\xi_3=\alpha_1+\alpha_2$, $\xi_2-\xi_3=\alpha_2$, $\xi_1+\xi_2=\alpha_1+2\alpha_2+2\alpha_3$, $\xi_1+\xi_3=\alpha_1+\alpha_2+\alpha_3$, $\xi_2+\xi_3=\alpha_2+2\alpha_3$ (and for $U(3)$, $\xi_1-\xi_2$, $\xi_1-\xi_3$, $\xi_2-\xi_3$). The complementary positive roots are ξ_1, ξ_2, ξ_3, $\xi_1+\xi_2$, $\xi_1+\xi_3$, $\xi_2+\xi_3$ and so c_1 is given by

$$\begin{aligned}&2(\xi_1+\xi_2+\xi_3+\xi_1+\xi_2+\xi_1+\xi_3+\xi_2+\xi_3,\xi_3)\\&\quad=2(3\xi_3,\xi_3)=6=\dim_{\mathbf{C}} SO(7)/U(3).\end{aligned}$$

It is a classical result that a projective manifold M with $c_1=1+\dim_{\mathbf{C}}M$ is the complex projective space and if $c_1=\dim_{\mathbf{C}}M$ then M is the hyperquadric.

Example IV. The twistor space of $\mathbf{C}P^2$ is $U(3)/T^3$ ($T^3=U(1)\times U(1)\times U(1)$), i.e., the flag manifold $F(1,2)$ of lines in $\mathbf{C}P^2$. It can be shown that $\mathscr{P}(F_-)$ over $\mathbf{C}P^3$ is $U(4)/U(2)\times U(1)\times U(1)$. It is a simply connected homogeneous Kähler manifold. The canonical almost complex structure on $\mathscr{P}(F_-)$ corresponds to the invariant almost complex structure described below. In this case, the dimension of the center is larger than 1; thus there are inequivalent invariant almost complex structures (cf. Borel-Hirzebruch [3]). The roots of $U(4)$ are given by $\pm(\xi_i-\xi_j)$, $1\le i<j\le 4$; those of $U(2)\times U(1)\times U(1)$ are $\pm(\xi_1-\xi_2)$ with $\xi_1-\xi_2$ the positive root. The complementary positive roots are $\xi_1-\xi_3$, $\xi_1-\xi_4$, $\xi_2-\xi_3$, $\xi_2-\xi_4$, $\xi_3-\xi_4$. This is the root system of an invariant almost complex structure. The image in $H^2(U(4)/T^4,Z)$ of the first Chern class of the complex structure defined above is given by the sum of the positive roots $2\xi_1+2\xi_2-\xi_3-3\xi_4$.

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