ILLINOIS JOURNAL OF MATHEMATICS
Volume 31, Number 2, Summer 1987

SHARP INEQUALITIES FOR HOLOMORPHIC FUNCTIONS

BY

JACOB BURBEA

1. Introduction

In a recent paper, Mateljević and Pavlović [6] gave new proofs for the isoperimetric inequality by using the boundary behaviors of holomorphic functions belonging to certain Hardy classes on the unit disk Δ. These proofs are based on sharp norm inequalities for holomorphic functions which are of interest on their own right. For example, the following sharp inequality is proved in [6]. Let $f \in H^1(\Delta)$, then

$$4\pi \int_\Delta |f(z)|^2\, dA(z) \le \left\{ \int_{\partial\Delta} |f(z)|\, |dz| \right\}^2,$$

where dA denotes the area Lebesgue measure, and $H^p(\Delta)$ $(0 < p < \infty)$ denotes Hardy class. Equality holds if and only if f is of the form $f(z) = C(1 - z\bar{\zeta})^{-2}$, $z \in \Delta$, for some constant C and some point $\zeta \in \Delta$. Other sharp inequalities, similar to the one above, were proved by Aronszajn [1], Saitoh [9] and Burbea [3], [4]. The main purpose of this paper is to give an extension of these results to various situations which were not covered in [1], [3], [4], [6], [9]. The method of proof will be based on ingredients taken from a rather general theory expounded in [4] (see also [2], [3]).

2. Preliminaries and notation

For $z = (z_1, \dots, z_n) \in \mathbf{C}^n$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{Z}_+^n$ we use the standard multinomial notation

$$\alpha! = \alpha_1! \cdots \alpha_n!, \quad |\alpha| = \alpha_1 + \cdots + \alpha_n, \quad z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n},$$

$$\|z\|_\infty = \max_{1 \le j \le n} |z_j| \quad \text{and} \quad \|z\| = \left(|z_1|^2 + \cdots + |z_n|^2\right)^{1/2}.$$

Received May 20, 1985.

Moreover, if also $\zeta = (\zeta_1, \dots, \zeta_n) \in \mathbf{C}^n$, then we let

$$z \cdot \zeta = (z_1\zeta_1, \dots, z_n\zeta_n) \in \mathbf{C}^n \quad \text{and} \quad \langle z, \zeta \rangle = z_1\bar{\zeta}_1 + \cdots + z_n\bar{\zeta}_n.$$

We also let

$$\Delta = \{\lambda \in \mathbf{C} : |\lambda| < 1\},$$
$$T = \partial\Delta = \{\lambda \in \mathbf{C} : |\lambda| = 1\}, \quad \Delta^n = \{z \in \mathbf{C}^n : \|z\|_\infty < 1\}$$

and

$$B = \{z \in \mathbf{C}^n : \|z\| < 1\}, \quad S = \partial B = \{z \in \mathbf{C}^n : \|z\| = 1\}.$$

For a complex manifold D, $H(D)$ denotes the class of all holomorphic functions on D. An open set Ω in $\mathbf{C}^n$ is said to be a *complete Reinhardt domain* if $z \in \Omega$ implies $z \cdot \zeta \in \Omega$ for every $\zeta \in \bar{\Delta}^n$. In this case Ω is a star shaped domain containing the origin. Moreover, for any $f \in H(\Omega)$ there exists a unique power series

$$f(z) = \sum_\alpha a_\alpha z^\alpha \quad (z \in \Omega)$$

with normal convergence in Ω, i.e., the power series converges absolutely and uniformly on compacta of Ω to f, and with

$$a_\alpha = a_\alpha(f) = \{\partial^\alpha f\}(0)/\alpha! \quad (\alpha \in \mathbf{Z}_+^n).$$

Here, for $z = (z_1, \dots, z_n) \in \mathbf{C}^n$ and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{Z}_+^n$,

$$\partial^\alpha = \partial_n^{\alpha_1} \cdots \partial_n^{\alpha_n} \quad \text{where} \quad \partial_j = \partial/\partial z_j, 1 \le j \le n.$$

For a subset Λ of $\mathbf{Z}_+^n$, we let

$$H(\Omega : \Lambda) = \{f \in H(\Omega) : \{\partial^\alpha f\}(0) = 0, \alpha \in \Lambda\}.$$

We fix a complete Reinhardt domain Ω in $\mathbf{C}^n$. A function ϕ, holomorphic on a neighborhood of $0 \in \mathbf{C}^n$ with $c_\alpha = a_\alpha(\phi)$, $\alpha \in \mathbf{Z}_+^n$, i.e.,

$$\phi(z) = \sum_\alpha c_\alpha z^\alpha,$$

is said to belong to $\mathscr{P}(\Omega)$ if $c_\alpha \ge 0$ for every $\alpha \in \mathbf{Z}_+^n$ and if $\phi(z \cdot \bar{z}) < \infty$ for every $z \in \Omega$. It is said to belong to $\mathscr{P}_\infty(\Omega)$ if $\phi \in \mathscr{P}(\Omega)$ and also $\phi(z \cdot \bar{z}) = \infty$ for every z on the boundary $\partial\Omega$ of Ω. For $\phi \in \mathscr{P}(\Omega)$ we let $\Lambda_\phi = \{\alpha \in \mathbf{Z}_+^n : c_\alpha = 0\}$ and $\Gamma_\phi = \mathbf{Z}_+^n \setminus \Lambda_\phi$, and define

$$k_\phi(z, \zeta) = \phi(z \cdot \bar{\zeta}) \quad (z, \zeta \in \Omega).$$

Evidently, for any $\zeta \in \Omega$, $k_\phi(\cdot, \zeta) \in H(\Omega : \Lambda_\phi)$, and

$$\sum_{k,m=1}^{N} a_k \bar{a}_m k_\phi(z_k, z_m) \geq 0$$

for every $z_1, \ldots, z_N \in \Omega$ and every $a_1, \ldots, a_N \in \mathbf{C}$ $(N = 1, 2, \ldots)$. It follows that k_ϕ is a sesqui-holomorphic positive-definite kernel on $\Omega \times \Omega$. In particular,

$$\overline{k_\phi(z, \zeta)} = k_\phi(\zeta, z) \quad \text{and} \quad |k_\phi(z, \zeta)|^2 \leq k_\phi(z, z) k_\phi(\zeta, \zeta)$$

for every $z, \zeta \in \Omega$. From the general theory of reproducing kernels (see Aronszajn [1]) follows that there exists a unique functional Hilbert space $\mathscr{H}_\phi$ of functions f in $H(\Omega : \Lambda_\phi)$ with k_ϕ as its reproducing kernel. To identify this Hilbert space we introduce the quadratic norm (see [4])

$$\|f\|_\phi^2 = \sum_{\alpha \in \Gamma_\phi} c_\alpha^{-1} |a_\alpha|^2$$

for any $f \in H(\Omega : \Lambda_\phi)$ with $a_\alpha = a_\alpha(f)$, $\alpha \in \mathbf{Z}_+^n$, and denote by $\langle \ , \ \rangle_\phi$ the induced inner product. This gives

$$\mathscr{H}_\phi = \left\{ f \in H(\Omega : \Lambda_\phi) : \|f\|_\phi < \infty \right\}$$

and

$$f(\zeta) = \langle f, k_\phi(\cdot, \zeta) \rangle_\phi \quad \left(f \in \mathscr{H}_\phi, \zeta \in \Omega \right).$$

We shall need the following theorem. Its proof is found in [4] (see also [3]).

THEOREM 2.1. *Let ϕ and ψ be in $\mathscr{P}(\Omega)$. Then $\phi\psi \in \mathscr{P}(\Omega)$ with*

$$\Gamma_{\phi\psi} = \left\{ \gamma \in \mathbf{Z}_+^n : \gamma = \alpha + \beta, \alpha \in \Gamma_\phi, \beta \in \Gamma_\psi \right\}.$$

Moreover, if $f \in \mathscr{H}_\phi$ and $g \in \mathscr{H}_\psi$ then $fg \in \mathscr{H}_{\phi\psi}$ with

$$\|fg\|_{\phi\psi} \leq \|f\|_\phi \|g\|_\psi.$$

Equality holds if and only if either $fg = 0$ or f and g are of the form

$$f = C_1 k_\phi(\cdot, \zeta), \quad g = C_2 k_\psi(\cdot, \zeta)$$

for some nonzero constants C_1 and C_2 and for some point $\zeta \in \mathbf{C}^n$ with $\phi(\zeta \cdot \bar{\zeta}) < \infty$ and $\psi(\zeta \cdot \bar{\zeta}) < \infty$. In particular, if also either ϕ or ψ is in $\mathscr{P}_\infty(\Omega)$ then the point ζ must lie in Ω.

We also note that for $\phi \in \mathscr{P}(\Omega)$ with $c_\alpha = a_\alpha(\phi)$, $\alpha \in \mathbf{Z}^n_+$, the monomials $\sqrt{c_\alpha}\, z^\alpha$, $\alpha \in \Gamma_\phi$, form an orthonormal basis for $\mathscr{H}_\phi$.

For $q > 0$ and $m \in \mathbf{Z}_+$, $(q)_m$ stands for 1 if $m = 0$ and

$$(q)_m = \Gamma(q+m)/\Gamma(q) = q(q+1)\cdots(q+m-1) \quad (m \geq 1).$$

3. Inequalities in the plane

In the one dimensional case ($n = 1$) we take the unit disk Δ as our fixed Reinhardt domain Ω. On Δ we consider the function $\phi_q(z) = (1-z)^{-q}$ where $q > 0$. Evidently, $\phi_q \in \mathscr{P}_\infty(\Delta)$ with $a_m(\phi_q) = (q)_m/m!$ for $m \in \mathbf{Z}_+$ and with $\Gamma_{\phi_q} = \mathbf{Z}_+$. The corresponding Hilbert space $\mathscr{H}_{\phi_q}$, norm $\|\cdot\|_{\phi_q}$ and reproducing kernel k_{ϕ_q} are denoted by $\mathscr{H}_q(\Delta)$, $\|\cdot\|_q$ and k_q, respectively. Thus

$$k_q(z,\zeta) = \phi_q(z\bar{\zeta}) = (1 - z\bar{\zeta})^{-q} \quad (z, \zeta \in \Delta)$$

and

$$\|f\|_q^2 = \sum_{m=0}^{\infty} \frac{m!}{(q)_m} |a_m|^2,$$

where $f \in H(\Delta)$ with $a_m = a_m(f)$, $m \in \mathbf{Z}_+$, and therefore $\mathscr{H}_q(\Delta) = \{f \in H(\Delta) : \|f\|_q < \infty\}$.As an immediate consequence of Theorem 2.1, we have:

THEOREM 3.1. *Let $f_j \in \mathscr{H}_{q_j}(\Delta)$ where $q_j > 0$ for $j = 1, \ldots, m$, $m \geq 2$. Then*

$$\prod_{j=1}^{m} f_j \in \mathscr{H}_{q_1 + \cdots + q_1}(\Delta)$$

with

$$\left\| \prod_{j=1}^{m} f_j \right\|_{q_1 + \cdots + q_1} \leq \prod_{j=1}^{m} \|f_j\|_{q_j}.$$

Equality holds if and only if either $\prod_{j=1}^m f_j = 0$ or each f_j is of the form $f_j = C_j k_{q_j}(\cdot, \zeta)$ for some point $\zeta \in \Delta$ and some nonzero constants $C_j (1 \leq j \leq m)$.

We let $dA_0(z) = |dz|/2\pi$ be the normalized boundary measure on $\partial\Delta$, and we consider the family $\{dA_q\}_{q>0}$ of probability measures on $\bar{\Delta}$ given by

$$da_q(z) = q\pi^{-1}(1 - |z|^2)^{q-1}\, dA(z) \quad (z \in \Delta).$$

As a measure on $\bar{\Delta}$, $dA_q \to dA_0$ as $q \to 0^+$. In particular, if f is a continuous function on $\bar{\Delta}$, then

$$\int f\,dA_0 = \int_T f\,dA_0 = \lim_{q\to 0^+} \int_\Delta f\,dA_q.$$

On the other hand

$$\int f\,dA_q = \int_\Delta f\,dA_q \quad (q > 0)$$

if f is integrable with respect to dA_q.

For $q \geq 0$ and $0 < p < \infty$, we let $A_q^p(\Delta)$ stand for the space of all functions $f \in H(\Delta)$ such that $\|f\|_{p,q} < \infty$, where

$$\|f\|_{p,q} = \left\{\int |f|^p\,dA_q\right\}^{1/p},$$

and where for $q = 0$ the integration is carried over the nontangential boundary values of $f \in A_0^p(\Delta)$. It follows that $A_0^p(\Delta)$ is the *Hardy space* $H^p(\Delta)$, that $A_q^p(\Delta)$, $q > 0$, is a *weighted Bergman space* and that $A_1^p(\Delta)$ is the ordinary *Bergman space* $A^p(\Delta)$. Moreover, it also follows that the space $A_q^2(\Delta)$ is identical with the space $\mathscr{H}_{1+q}(\Delta)$ and that $\|\cdot\|_{2,q} = \|\cdot\|_{1+q}$ for $q \geq 0$. Note also, that for $0 < p < \infty$, the Hardy space $H^p(\Delta) = A_0^p(\Delta)$ is a projective limit, as $q \to 0^+$, of the weighted Bergman spaces $A_q^p(\Delta)$, $q > 0$.

Another functional Hilbert space of interest is the *Dirichlet space*

$$\mathscr{D}(\Delta) = \{f \in H(\Delta : \{0\}) : \|f'\|_{2,1} < \infty\}.$$

This space can be generated by $\phi_0(z) = -\log(1 - z)$, and thus its reproducing kernel k_0 is given by

$$k_0(z,\zeta) = \phi_0(z\bar{\zeta}) = -\log(1 - z\bar{\zeta}) \quad (z,\zeta \in \Delta).$$

Moreover, for any $f \in H(\Delta : \{0\})$ with $a_m = a_m(f)$, $m \in \mathbf{Z}_+$, the quadratic form $\|f\|_0$ of $\mathscr{D}(\Delta)$ is given by

$$\|f\|_0^2 = \|f'\|_1^2 = \sum_{m=1}^{\infty} m|a_m|^2.$$

Note also that $(k_q - 1)/q \to k_0$ and that $q\|f\|_q^2 \to \|f\|_0^2 (f \in H(\Delta : \{0\}))$ as $q \to 0^+$, and thus $\mathscr{D}(\Delta)$ may be viewed as a projective limit of the space $\sqrt{q} \cdot \{f \in \mathscr{H}_q(\Delta) : f(0) = 0\}$ when $q \to 0^+$.

THEOREM 3.2. *Let f and g be in $\mathcal{D}(\Delta)$. Then $fg \in A_0^2(\Delta) = H^2(\Delta)$ with*

$$\|fg\|_{2,0} \leq \|f\|_0 \|g\|_0.$$

Equivalently,

$$\pi \int_{\partial\Delta} |fg|^2 \, ds \leq 2\left\{\int_\Delta |f'|^2 \, dA\right\} \cdot \left\{\int_\Delta |g'|^2 \, dA\right\},$$

where $ds(z) = |dz|$, $z \in \partial\Delta$. Equality holds if and only if either $fg = 0$ or f and g are of the form $f(z) = C_1 z$, $g(z) = C_2 z$ for some nonzero constants C_1 and C_2.

Proof. Let $a_m = a_m(f)$, $b_m = a_m(g)$ and $c_m = a_m(fg)$, $m \in \mathbf{Z}_+$. It follows that $a_0 = b_0 = c_0 = 0$, $c_1 = 0$ and

$$c_m = \sum_{k=1}^{m-1} a_k b_{m-k} \quad (m = 2, 3, \ldots).$$

This and the Cauchy-Schwarz inequality give

$$\|fg\|_{2,0}^2 = \sum_{m=1}^{\infty} \left| \sum_{k=1}^{m} a_k b_{m+1-k} \right|^2 \leq \sum_{m=1}^{\infty} m \sum_{k=1}^{m} |a_k|^2 |b_{m+1-k}|^2.$$

On the other hand

$$\begin{aligned} \|f\|_0^2 \|g\|_0^2 &= \left\{ \sum_{m=1}^{\infty} m|a_m|^2 \right\} \cdot \left\{ \sum_{m=1}^{\infty} m|b_m|^2 \right\} \\ &= \sum_{m=1}^{\infty} \sum_{k=1}^{m} k|a_k|^2 (m+1-k)|b_{m+1-k}|^2. \end{aligned}$$

Since $k(m+1-k) - m = (m-k)(k-1)$ is non-negative for every $1 \leq k \leq m$, $m = 1, 2, \ldots$, the desired inequality follows. If equality holds then for every $m = 1, 2, \ldots$, there exists a scalar $\lambda_m \in \mathbf{C}$ so that $\lambda_m = a_k b_{m+1-k}$ and $(m-k)(k-1)|a_k|^2 |b_{m+1-k}|^2 = 0$ for every $1 \leq k \leq m$. It follows that $\lambda_m = 0$ for every $m \geq 2$ and $\{fg\}(z) = \lambda_1 z^2$. This gives the equality statement of the theorem, and the proof is complete.

THEOREM 3.3. *Let $f_j \in H^{p_j}(\Delta)$ with $0 < p_j < \infty$ for $j = 1, 2, \ldots, m$, $m \geq 2$. Then*

$$\int_\Delta |f_1|^{p_1} \cdots |f_m|^{p_m} \, dA_{m-1} \leq \prod_{j=1}^{m} \int_{\partial\Delta} |f_j|^{p_j} \, dA_0.$$

Equality holds if and only if either $\prod_{j=1}^m f_j = 0$ *or each* f_j *is of the form*

$$f_j = C_j k_{2/p_j}(\cdot, \zeta),$$

i.e.,

$$f_j(z) = C_j(1 - z\bar{\zeta})^{-2/p_j},$$

for some point $\zeta \in \Delta$ *and some nonzero constants* C_j $(1 \le j \le m)$.

Proof. For $1 \le j \le m$ we let $\mathscr{B}_j$ be a Blaschke product formed from the zeros in Δ, if any, of f_j, and define $g_j = (f_j/\mathscr{B}_j)^{p_j/2}$. Then $g_j \in \mathscr{H}_1(\Delta)$ $(= H^2(\Delta))$ for $j = 1, \ldots, m$, and, by Theorem 3.1, $\prod_{j=1}^m g_j \in \mathscr{H}_m$ with

$$\left\| \prod_{j=1}^m g_j \right\|_m \le \prod_{j=1}^m \|g_j\|_1. \tag{3.1}$$

Equality holds if and only if each g_j is of the form $g_j = \tilde{C}_j k_1(\cdot, \zeta)$ for some $\zeta \in \Delta$ and some nonzero constants $\tilde{C}_j$ $(1 \le j \le m)$. We are assuming without loss, of course, that $\prod_{j=1}^m f_j \ne 0$ and hence also $\prod_{j=1}^m g_j \ne 0$. Now, inequality (3.1) is, by definition, equivalent to the inequality

$$\int_\Delta |\mathscr{B}_1|^{-p_1} \cdots |\mathscr{B}_m|^{-p_m} |f_1|^{p_1} \cdots |f_m|^{p_m}\, dA_{m-1} \le \prod_{j=1}^m \int_{\partial\Delta} |f_j|^{p_j}\, dA_0,$$

and hence, since $|\mathscr{B}_j| \le 1$ $(1 \le j \le m)$ on Δ, the desired inequality follows. If equality holds then each $\mathscr{B}_j$ must be a constant λ_j with $|\lambda_j| = 1$ $(1 \le j \le m)$ and each g_j is of the above mentioned form. It follows that each f_j is of the form

$$f_j = \lambda_j \tilde{C}_j^{2/p_j}[k_1(\cdot, \zeta)]^{2/p_j} \quad \text{or} \quad f_j = C_j k_{2/p_j}(\cdot, \zeta)$$

where $C_j = \lambda_j \tilde{C}_j^{2/p_j}$ $(1 \le j \le m)$. This concludes the proof

A special case of this theorem, namely when $m = 2$, was also obtained by Mateljević and Pavlović [6], by using different methods.

COROLLARY 3.4. *Let* $f \in H^p(\Delta)$ *with* $0 < p < \infty$. *Then for any integer* $m \ge 2$, $f \in A_{m-1}^{mp}(\Delta)$ *with*

$$\|f\|_{mp, m-1} \le \|f\|_{p,0}.$$

Equivalently

$$\int_\Delta |f|^{mp}\, dA_{m-1} \le \left\{ \int_{\partial\Delta} |f|^p\, dA_0 \right\}^m.$$

Equality holds if and only if f *is of the form* $f(z) = C(1 - z\bar{\zeta})^{-2/p}$, $z \in \Delta$, *for some constant* C *and some point* $\zeta \in \Delta$.

Putting $m = 2$ and $p = 1$ in this corollary we obtain the result, mentioned in the introduction, of Mateljević and Pavlović [6].

Let D be a hyperbolic simply connected plane domain and let λ_D be its Poincaré metric. The latter is defined by

$$\lambda_D(z) = k_1(\phi(z), \phi(z))|\phi'(z)| \quad (z \in D),$$

where ϕ is a Riemann mapping of D onto Δ, and is independent of the particular choice of ϕ. According to a theorem of Warschawski [10], if ∂D is of class C^1 with a Dini-continuous normal, in particular if $\partial D \in C^{1,\varepsilon}$ $(0 < \varepsilon < 1)$, then the conformal mapping $\phi: D \to \Delta$ extends to a C^1-diffeomorphism of $\bar{D}$ onto $\bar{\Delta}$ and there exist positive constants a and b such that

$$0 < a \le |\phi'(z)| \le b < \infty \quad (z \in \bar{D}).$$

It follows that for any $0 < p < \infty$ the Hardy space $H^p(D)$ coincides with the Smirnov class $E^p(D)$ (see [5, p. 169]), and that the "norm" in $H^p(D)$ may be given by

$$\|f\|_{p,D} = \left\{\frac{1}{2\pi}\int_{\partial D} |f(z)|^p |dz|\right\}^{1/p} < \infty,$$

where the integration is carried over the nontangential boundary values of $f \in H^p(D)$. In particular, $\{\phi^m \cdot (\phi')^{1/2}\}_{m \ge 0}$ forms an orthonormal basis for $H^2(D)$ and

$$K_{0,D}(z,\zeta) = k_1(\phi(z), \phi(\zeta))\left\{\phi'(z)\overline{\phi'(\zeta)}\right\}^{1/2} \quad (z, \zeta \in D)$$

is the Szegö reproducing kernel of $H^2(D)$.

Let $0 < p < \infty$. For $q > 0$, we let $L_q^p(D)$ be the L^p-space with respect to the measure $(q/\pi)\lambda_D^{1-q}\, dA$, and we let $A_q^p(D) = H(D) \cap L_q^p(D)$. It follows that $A_q^p(D)$ is a closed subspace of $L_q^p(D)$. It is natural to extend these definitions to $q = 0$ by letting $L_0^p(D)$ stand for the L^p-space with respect to the boundary measure $|dz|/2\pi$ on ∂D, and by defining $A_0^p(D)$ to be $H^p(D)$ as above. In this case we adopt the usual convention of identifying Hardy classes $A_0^p(D) = H^p(D)$ with closed subspaces of $L_0^p(D)$. We now observe that if ψ is any biholomorphic mapping of D onto another domain D^* such that ∂D^* is of class C^1 with a Dini-continuous normal, then the mapping

$$f \mapsto (f \circ \psi) \cdot (\psi')^{(q+1)/p}$$

constitutes a linear isometry of $L_q^p(D^*)$ and $A_q^p(D^*)$ onto $L_q^p(D)$ and

$A_q^p(D)$, respectively, for any $q \geq 0$ and $0 < p < \infty$. In particular,

$$\left\{\sqrt{(q+1)_m/m!}\,\phi^m \cdot (\phi')^{(q+1)/2}\right\}_{m \geq 0}$$

forms an orthonormal basis for $A_q^2(D)$ and $K_{q,D}(z, \zeta) = \{K_{0,D}(z, \zeta)\}^{q+1}$ $(z, \zeta \in D)$ is the reproducing kernel of $A_q^2(D)$ $(q \geq 0)$. Note also that $A_1^p(D)$ is the ordinary Bergman space and that $A_0^p(D)$ is the projective limit of $A_q^p(D)$ as $q \to 0^+$ $(0 < p < \infty)$.

In view of the above discussion, the following theorem may be regarded as a corollary of Theorem 3.3. Once again, a special case of this theorem, namely when $m = 2$ and D is a simply connected plane domain whose boundary is analytic is due to Mateljević and Pavlović [6]. (Note, however, that the corresponding equality statement in [6] contains a trivial error.)

THEOREM 3.5. *Let D be a simply connected plane domain whose boundary ∂D is of class C^1 with a Dini-continuous normal. Let $f_j \in H^{p_j}(D)$ with $0 < p_j < \infty$ for $j = 1, 2, \ldots, m$, $m \geq 2$. Then $\prod_{j=1}^m |f_j|^{p_j} \in L^1_{m-1}(D)$ with*

$$\frac{m-1}{\pi} \int_D \left(\prod_{j=1}^m |f_j|^{p_j} \right) \cdot \lambda_D^{2-m} \, dA \leq \prod_{j=1}^m \|f_j\|_{p_j, D}^{p_j}.$$

Equality holds if and only if either $\prod_{j=1}^m f_j = 0$ or each f_j is of the form

$$f_j = C_j \cdot (\phi')^{1/p_j}$$

where ϕ is a Riemann mapping of D onto Δ and C_j are nonzero constants $(1 \leq j \leq m)$.

Proof. Let ψ be any biholomorphic mapping of Δ onto D, and define

$$g_j = (f_j \circ \phi) \cdot (\phi')^{1/p_j} \quad (1 \leq j \leq m).$$

Since $g_j \in H^{p_j}(\Delta)$ we may apply Theorem 3.3 with g_j in place of f_j. This gives the present inequality statement. Equality holds if and only if either $\prod_{j=1}^m g_j = 0$ or each g_j is of the form $g_j = C_j' k_{2/p_j}(\cdot, \tau)$ for some point $\tau \in \Delta$ and some nonzero constants C_j' $(1 \leq j \leq m)$. Equivalently, either $\prod_{j=1}^m f_j = 0$ or each f_j is of the form

$$f_j = C_j' \left[\overline{\psi'(\tau)}\right]^{1/p_j} \left[K_{1,D}(\cdot, \psi(\tau))\right]^{1/p_j}.$$

Letting ϕ be a Riemann mapping of D onto Δ with $\phi[\psi(\tau)] = 0$, and then letting $C_j = C_j' \overline{\left\{ \psi'(\tau)\phi'(\psi(\tau)) \right\}}^{1/p_j}$, we obtain the desired result.

COROLLARY 3.6. *Let D be a simply connected plane domain whose boundary ∂D is of class C^1 with a Dini-continuous normal, and let $f \in H^p(D)$ with $0 < p < \infty$. Then for any integer $m \geq 2$, $f \in A^{mp}_{m-1}(D)$ with*

$$\frac{m-1}{\pi} \int_D |f|^{mp} \lambda_D^{2-m}\, dA \leq \|f\|^{mp}_{p,D}.$$

Equality holds if and only if f is of the form $f = C(\phi')^{1/p}$ for some Riemann mapping ϕ of D onto Δ and some constant C.

4. Inequalities on the polydisk

We take the unit polydisk Δ^n as our fixed Rienhardt domain Ω. On Δ^n we consider the function

$$\phi_{\mathbf{q}}(z) = \prod_{j=1}^{n} (1 - z_j)^{-q_j} \quad (z = (z_1, \ldots, z_n) \in \Delta^n)$$

where $\mathbf{q} = (q_1, \ldots, q_n) \in \mathbf{R}^n_+ \setminus \{0\}$. Obviously, $\phi_{\mathbf{q}} \in \mathscr{P}_\infty(\Delta^n)$ with $a_\alpha(\phi_{\mathbf{q}}) = (\mathbf{q})_\alpha / \alpha!$ for $\alpha \in \mathbf{Z}^n_+$ and with $\Gamma_{\phi_{\mathbf{q}}} = \mathbf{Z}^n_+$, where

$$(\mathbf{q})_\alpha = (q_1)_{\alpha_1} \cdots (q_n)_{\alpha_n} \quad (\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{Z}^n_+).$$

The corresponding Hilbert space, norm and reproducing kernel are denoted by $\mathscr{H}_{\mathbf{q}}(\Delta^n)$, $\|\cdot\|_{\mathbf{q}}$ and $k_{\mathbf{q}}$, respectively. Thus

$$k_{\mathbf{q}}(z, \zeta) = \phi_{\mathbf{q}}(z \cdot \bar{\zeta}) = \prod_{j=1}^{n} \left(1 - z_j \bar{\zeta}_j\right)^{-q_j} \quad (z, \zeta \in \Delta^n)$$

and

$$\|f\|^2_{\mathbf{q}} = \sum_\alpha \frac{\alpha!}{(\mathbf{q})_\alpha} |a_\alpha|^2,$$

where $f \in H(\Delta^n)$ with $a_\alpha = a_\alpha(f)$, $\alpha \in \mathbf{Z}^n_+$, and therefore

$$\mathscr{H}_{\mathbf{q}}(\Delta^n) = \left\{ f \in H(\Delta^n) : \|f\|_{\mathbf{q}} < \infty \right\}.$$

Theorem 2.1 has now the following form:

THEOREM 4.1. *Let $f_j \in \mathscr{H}_{\mathbf{q}_j}(\Delta^n)$ where $\mathbf{q}_j \in \mathbf{R}^n_+ \setminus \{0\}$ for $j = 1, \ldots, m$, $m \geq 2$. Then*

$$\prod_{j=1}^{m} f_j \in \mathscr{H}_{\mathbf{q}_1 + \cdots \mathbf{q}_m}(\Delta^n)$$

with

$$\left\| \prod_{j=1}^{m} f_j \right\|_{\mathbf{q}_1 + \cdots + \mathbf{q}_m} \le \prod_{j=1}^{m} \|f_j\|_{\mathbf{q}_j}.$$

Equality holds if and only if either $\prod_{j=1}^{m} f_j = 0$ *or each* f_j *is of the form*

$$f_j = C_j k_{\mathbf{q}}(\cdot, \zeta)$$

for some $\zeta \in \Delta^n$ *and some nonzero constants* C_j $(1 \le j \le m)$.

When $\mathbf{q} \ge \mathbf{1} = (1, \dots, 1)$ the quadratic norm $\|\cdot\|_{\mathbf{q}}$ admits an integral representation. To see this we consider the probability measure

$$d\mu_{\mathbf{q}}(z) = dA_{q_1 - 1}(z_1) \cdots dA_{q_n - 1}(z_n) \quad \text{for } z = (z_1, \dots, z_n) \in \bar{\Delta}^n.$$

As in the unit disk Δ, $d\mu_{\mathbf{q}} \to d\mu_{\mathbf{1}}$ as $\mathbf{q} \to \mathbf{1}^+$, and

$$\|f\|_{\mathbf{q}}^2 = \int |f|^2 d\mu_{\mathbf{q}} \quad \left(f \in \mathscr{H}_{\mathbf{q}}(\Delta^n), \mathbf{q} \ge \mathbf{1}\right).$$

Here, the integration is carried over Δ^n if $\mathbf{q} > \mathbf{1}$ and over the distinguished boundary T^n if $\mathbf{q} = \mathbf{1}$. In the latter case, f in the integral represents the nontangential (distinguished) boundary values of f. In a similar and obvious manner one may describe the intermediate situation where some, but not all, of the components q_j of $\mathbf{q} = (q_1, \dots, q_n) \ge \mathbf{1}$ are equal to $\mathbf{1}$. It follows that $\mathscr{H}_{\mathbf{1}}(\Delta^n)$ is the *Hardy space* $H^2(\Delta^n)$ and that $\mathscr{H}_{\mathbf{q}}(\Delta^n)$ for $\mathbf{q} > \mathbf{1}$ is the *weighted Bergman space* $A^2_{\mathbf{q}-\mathbf{1}}(\Delta^n)$ with $\mathscr{H}_{\mathbf{2}}(\Delta^n) = A^2_{\mathbf{1}}(\Delta^n)$ as the ordinary *Bergman space*. Moreover, any space $\mathscr{H}_{\mathbf{q}_0}(\Delta)$ with $\mathbf{q}_0 \ge \mathbf{1}$ may be viewed as a projective limit of weighted Bergman spaces $A^2_{\mathbf{q}-\mathbf{1}}(\Delta^n)$, $\mathbf{q} > \mathbf{1}$, as $\mathbf{q} \to \mathbf{q}_0^+$.

Let $R: H(\Delta^n) \to H(\Delta)$ be the diagonal restriction mapping defined by

$$\{Rf\}(\omega) = f(\omega, \dots, \omega).$$

Since the diagonal restriction of $k_{\mathbf{q}}$ ($\mathbf{q} \in \mathbf{R}^n_+ \setminus \{0\}$), the reproducing kernel of $\mathscr{H}_{\mathbf{q}}(\Delta^n)$, is the reproducing kernel $k_{|\mathbf{q}|}$ of $\mathscr{H}_{|\mathbf{q}|}(\Delta)$, where $|\mathbf{q}| = q_1 + \cdots + q_n$, we deduce from the general theory of reproducing kernels [1] (see also [2]) that R is a contractive linear transformation of $\mathscr{H}_{\mathbf{q}}(\Delta^n)$ onto $\mathscr{H}_{|\mathbf{q}|}(\Delta)$. Moreover, it also follows that R^*, the adjoint of R, is a linear isometry of $\mathscr{H}_{|\mathbf{q}|}(\Delta)$ onto $N(R)^\perp$, the orthogonal complement of the null-space $N(R) = \{f \in \mathscr{H}_{\mathbf{q}}(\Delta^n) : Rf = 0\}$ in $\mathscr{H}_{\mathbf{q}}(\Delta^n)$, with RR^* as the identity operator on $\mathscr{H}_{|\mathbf{q}|}(\Delta)$ and R^*R as the orthogonal projector of $\mathscr{H}_{\mathbf{q}}(\Delta^n)$ onto $N(R)^\perp$, and thus $\|R\| = \|R^*\| = 1$. In particular, R maps the Hardy space $H^2(\Delta^n)$ onto the weighted Bergman space $A^2_{n-1}(\Delta)$. For these and related results we refer the reader to Beatrous and Burbea [2].

A somewhat more precise formulation may be given by considering certain power expansions. For $\mathbf{q} \in \mathbf{R}^n$ and $m \in \mathbf{Z}_+$, we consider the homogeneous polynomial of degree m,

$$\phi_{\mathbf{q},m}(z) = \sum_{|\alpha|=m} \frac{(\mathbf{q})_\alpha}{\alpha!} z^\alpha \quad (z \in \mathbf{C}^n). \tag{4.1}$$

Since $\phi_{\mathbf{q},m}$ is the m-th coefficient in the expansion of $\phi_{\mathbf{q}}(\boldsymbol{\omega} \cdot z) = \prod_{j=1}^n (1 - \omega z_j)^{-q_j}$ in powers of ω, where $\boldsymbol{\omega} = (\omega, \dots, \omega)$, we deduce that

$$\phi_{\mathbf{q},m}(\mathbf{1}) = \sum_{|\alpha|=m} \frac{(\mathbf{q})_\alpha}{\alpha!} = \frac{1}{m!}(|\mathbf{q}|)_m, \tag{4.2}$$

and hence

$$\phi_{\mathbf{1},m}(\mathbf{1}) = \sum_{|\alpha|=m} 1 = \binom{m+n-1}{m}.$$

Let $f \in H(\Delta^n)$ with $a_\alpha = a_\alpha(f)$, $\alpha \in \mathbf{z}_+^n$. Then

$$\{Rf\}(\omega) = \sum_{m=0}^{\infty} \left(\sum_{|\alpha|=m} a_\alpha \right) \omega^m \quad (\omega \in \Delta), \tag{4.3}$$

and so

$$N(R) = \left\{ f \in H(\Delta^n) : \sum_{|\alpha|=m} a_\alpha(f) = 0,\ m = 0, 1, \dots \right\}.$$

THEOREM 4.2. *Let* $\mathbf{q} \in \mathbf{R}_+^n \setminus \{0\}$. *Then*:

(i) *R maps $\mathscr{H}_{\mathbf{q}}(\Delta^n)$ into $\mathscr{H}_{|\mathbf{q}|}(\Delta)$ and $\|Rf\|_{|\mathbf{q}|} \le \|f\|_{\mathbf{q}}$ for every $f \in \mathscr{H}_{\mathbf{q}}(\Delta^n)$, with equality holding if and only if there is a sequence $\{\lambda_m\}$ of complex numbers such that*

$$\sum_{m=0}^{\infty} \frac{1}{m!}(|\mathbf{q}|)_m |\lambda_m|^2 < \infty$$

and such that $a_\alpha(f) = \lambda_{|\alpha|}(\mathbf{q})_\alpha/\alpha!$ for every $\alpha \in \mathbf{Z}_+^n$ or, equivalently

$$f = \sum_{m=0}^{\infty} \lambda_m \phi_{\mathbf{q},m};$$

(ii) For $g \in \mathscr{H}_{|\mathbf{q}|}(\Delta)$ with $b_m = a_m(g)$, $m \in \mathbf{Z}_+$ we have

$$\{R^* g\}(z) = \sum_{m=0}^{\infty} b_m \frac{m!}{(|\mathbf{q}|)_m} \phi_{\mathbf{q},m}(z) \quad (z \in \Delta^n);$$

(iii) RR^* is the identity operator on $\mathscr{H}_{|\mathbf{q}|}(\Delta)$ and R^* is a linear isometry of $\mathscr{H}_{|\mathbf{q}|}(\Delta)$ onto $N(R)^{\perp}$. Moreover, $N(R)^{\perp}$ is the closure in $\mathscr{H}_{\mathbf{q}}(\Delta^n)$ of the linear span of $\{\phi_{\mathbf{q},m}\}_{m\geq 0}$;

(iv) R is a linear transformation of $\mathscr{H}_{\mathbf{q}}(\Delta^n)$ onto $\mathscr{H}_{|\mathbf{q}|}(\Delta)$ with $\|R\| = 1$, and R^*R is the orthogonal projector of $\mathscr{H}_{\mathbf{q}}(\Delta^n)$ onto $N(R)^{\perp}$.

Proof. To prove (i) we assume that $f \in \mathscr{H}_{\mathbf{q}}(\Delta^n)$ with $a_\alpha = a_\alpha(f)$, $\alpha \in \mathbf{Z}_+^n$ and use (4.3). Then, by the Cauchy-Schwarz inequality and (4.2),

$$\begin{aligned}
\|Rf\|_{|\mathbf{q}|}^2 &= \sum_{m=0}^{\infty} \frac{m!}{(|\mathbf{q}|)_m} \left| \sum_{|\alpha|=m} a_\alpha \right|^2 \\
&\leq \sum_{m=0}^{\infty} \left(\sum_{|\alpha|=m} \frac{\alpha!}{(\mathbf{q})_\alpha} |a_\alpha|^2 \right) \\
&= \|f\|_{\mathbf{q}}^2,
\end{aligned}$$

and the desired inequality follows. Equality holds if and only if for every $m \in \mathbf{Z}_+$ there exists a number $\lambda_m \in \mathbf{C}$ so that $a_\alpha(\alpha!/(\mathbf{q})_\alpha)^{1/2} = \lambda_m((\mathbf{q})_\alpha/\alpha!)^{1/2}$ for all $\alpha \in \mathbf{Z}_+^n$ with $|\alpha| = m$. This, together with (4.1) and (4.2), concludes the proof of (i). Item (ii) follows from (i) by a direct calculation based on (4.1) and (4.3). We now prove (iii). That RR^* is the identity operator on $\mathscr{H}_{|\mathbf{q}|}(\Delta)$ is a straightforward consequence of (ii), (4.1) and (4.3). From this it follows easily that R^* is a linear isometry of $\mathscr{H}_{|\mathbf{q}|}(\Delta)$ onto $R^*(\mathscr{H}_{|\mathbf{q}|}(\Delta))$, and the latter is a closed subspace of $\mathscr{H}_{\mathbf{q}}(\Delta^n)$. In particular, $R^*(\mathscr{H}_{|\mathbf{q}|}(\Delta)) = N(R)^{\perp}$ and the first part of (iii) follows. The second part follows from this and (ii), and (iii) is proved. To prove (iv), we first observe that by (i), R is a linear transformation of $\mathscr{H}_{\mathbf{q}}(\Delta^n)$ into $\mathscr{H}_{|\mathbf{q}|}(\Delta)$ with $\|R\| \leq 1$. We then use (iii) to conclude that R is onto $\mathscr{H}_{|\mathbf{q}|}(\Delta)$ and that $\|R\| = \|R^*\| = 1$. Finally, we let $P = R^*R$, and note that, by the last observation and (iii), P is a linear transformation of $\mathscr{H}_{\mathbf{q}}(\Delta^n)$ onto $N(R)^{\perp}$. Since $P^* = P$ and, by (iii), $P^2 = R^*RR^*R = R^*R = P$, we conclude that P is the orthogonal projector of $\mathscr{H}_{\mathbf{q}}(\Delta^n)$ onto $N(R)^{\perp}$. The proof is now complete.

A special case of part (i) of this theorem, namely when $n = 2$ and $\mathbf{q} = \mathbf{1} = (1,1)$ was also observed in Mateljević and Pavlović [6]. In this case, by (4.1),

$$\phi_{\mathbf{1},m}(z_1, z_2) = \sum_{\alpha_1+\alpha_2=m} z_1^{\alpha_1} z_2^{\alpha_2} = \frac{z_1^{m+1} - z_2^{m+1}}{z_1 - z_2} \quad (n = 2),$$

and thus, as a corollary, we obtain:

COROLLARY 4.3. *Let $f \in H^2(\Delta^2) = \mathscr{H}_1(\Delta^2)$. Then $Rf \in A^2(\Delta)$ and*

$$\int_{\Delta} |f(\omega, \omega)|^2 \, dA_1(\omega) \leq \int_{T^2} |f(z_1, z_2)|^2 \, dA_0(z_1) \, dA_0(z_2).$$

Equality holds if and only if f is of the form

$$f(z_1, z_2) = \sum_{m=0}^{\infty} \lambda_m (z_1 - z_2)^{-1} \left(z_1^{m+1} - z_2^{m+1}\right),$$

where

$$\sum_{m=0}^{\infty} (m+1) |\lambda_m|^2 < \infty.$$

The last condition on $\{\lambda_m\}$ is implicit, but not mentioned explicitly, in [6].

For other approaches to the problem of diagonal restrictions on polydisks we refer to Rudin [7, p. 53] (see also the references in [2]).

5. Inequalities on the ball

We now take the unit ball B as our fixed Reinhardt domain Ω and consider the function

$$\psi_q(z) = (1 - \langle z, \mathbf{1} \rangle)^{-q} \quad (z \in \mathbf{C}^n)$$

where $q > 0$. Clearly, $\psi_q \in \mathscr{P}_\infty(B)$ with $a_\alpha(\psi_q) = (q)_{|\alpha|}/\alpha!$ for $\alpha \in \mathbf{Z}_+^n$ and with $\Gamma_{\psi_q} = \mathbf{Z}_+^n$. The corresponding Hilbert space, norm and reproducing kernel are denoted by $\mathscr{H}_q(B)$, $|||\cdot|||_q$ and K_q, respectively. Thus

$$K_q(z, \zeta) = \psi_q(z \cdot \bar{\zeta}) = (1 - \langle z, \zeta \rangle)^{-q} \quad (z, \zeta B)$$

and

$$|||f|||_q^2 = \sum_{\alpha} \frac{\alpha!}{(q)_{|\alpha|}} |a_\alpha|^2$$

where $f \in H(B)$ with $a_\alpha = a_\alpha(f)$, $\alpha \in \mathbf{Z}_+^n$, and hence $\mathscr{H}_q(B) = \{f \in H(B) : |||f|||_q < \infty\}$. Theorem 2.1 has now the following form:

THEOREM 5.1. *Let $f_j \in \mathscr{H}_{q_j}(B)$ where $q_j > 0$ for $j = 1, \ldots, m$, $m \geq 2$. Then*

$$\prod_{j=1}^{m} f_j \in \mathscr{H}_{q_1 + \cdots + q_m}(B)$$

with

$$|||\prod_{j=1}^{m} f_j|||_{q_1+\cdots+q_m} \le \prod_{j=1}^{m} |||f_j|||_{q_j}.$$

Equality holds if and only if either $\prod_{j=1}^{m} f_j = 0$ *or each* f_j *is of the form* $f_j = C_j K_{q_j}(\cdot, \zeta)$ *for some* $\zeta \in B$ *and some nonzero constants* C_j $(1 \le j \le m)$.

When $q \ge n$ the quadratic norm $|||\cdot|||_q$ admits an integral representation. To see this we let dv stand for the Lebesgue measure on $\mathbf{C}^n$ and $d\sigma$ for the surface measure on $S = \partial B$, normalized so that $\sigma(B) = 1$. For $s \ge 0$ we consider the probability measure dv_s on $\overline{B}$, defined by $dv_0 = d\sigma$ when $s = 0$ and by

$$dv_s(z) = \pi^{-n}(s)_n(1 - \|z\|^2)^{s-1}\,dv(z)$$

when $s > 0$. As a measure on $\overline{B}$, $dv_s \to dv_0$ as $s \to 0^+$, and

$$|||f|||_{n+q}^2 = \int |f|^2 dv_q \quad \left(f \in \mathcal{H}_{n+q}(B),\ q \ge 0\right).$$

Here, the integration is carried over B when $q > 0$ and over $S = \partial B$ when $q = 0$. In the latter case, f in the integral represents the nontangential boundary values of f. It follows that $\mathcal{H}_n(B)$ is the Hardy space $H^2(B)$ and that $\mathcal{H}_{n+q}(B)$ for $q > 0$ is the weighted Bergman space $A_q^2(B)$ with $\mathcal{H}_{n+1}(B) = A_1^2(B)$ as the ordinary Bergman space. It also follows that $H^2(B)$ is a projective limit of $A_q^2(B)$ as $q \to 0^+$.

Let $R_n: H(B) \to H(\Delta)$ be the *n-diagonal restriction mapping* defined by

$$\{R_n f\}(\omega) = f(n^{-1/2}\omega, \ldots, n^{-1/2}\omega).$$

As in the case of the polydisk, the n-diagonal restriction of K_q, the reproducing kernel of $\mathcal{H}_q(B)$, $q > 0$, is the reproducing kernel k_q of $\mathcal{H}_q(\Delta)$. This observation leads to the following theorem. Its proof follows either from the general theory of reproducing kernels [1], [2] or from arguments similar to those given in the proof of Theorem 4.2.

THEOREM 5.2. *Let* $q > 0$. *Then*:

(i) R_n *maps* $\mathcal{H}_q(B)$ *into* $\mathcal{H}_q(\Delta)$ *with* $\|R_n f\|_q \le |||f|||_q$ *for every* $f \in \mathcal{H}_q(B)$. *Equality holds if and only if* f *is of the form*

$$f = \sum_{m=0}^{\infty} \lambda_m P_m$$

where

$$P_m(z) = (z_1 + \cdots + z_n)^m \quad (z = (z_1, \ldots, z_n) \in \mathbf{C}^n).$$

and $\lambda_m \in \mathbf{C}$ *with*

$$\sum_{m=0}^{\infty} \frac{m!}{(q)_m} n^m |\lambda_m|^2 < \infty;$$

(ii) *For* $g \in \mathscr{H}_q(\Delta)$ *with* $b_m = a_m(g)$, $m \in \mathbf{Z}_+$, *we have*

$$R_n^* g = \sum_{m=0}^{\infty} b_m n^{-m/2} P_m;$$

(iii) $R_n R_n^*$ *is the identity operator on* $\mathscr{H}_q(\Delta)$ *and* R_n^* *is a linear isometry of* $\mathscr{H}_q(\Delta)$ *onto* $N(R_n)^{\perp}$. *Here* $N(R_n)$ *is the null-space in* $\mathscr{H}_q(B)$ *of* R_n *and* $N(R_n)^{\perp}$ *is its orthogonal complement in* $\mathscr{H}_q(B)$. *Moreover,* $N(R_n)^{\perp}$ *is the closure in* $\mathscr{H}_q(B)$ *of the linear span of* $\{P_m\}_{m \geq 0}$;

(iv) R_n *is a linear transformation of* $\mathscr{H}_q(B)$ *onto* $\mathscr{H}_q(\Delta)$ *with* $\|R_n\| = 1$, *and* $R_n^* R_n$ *is the orthogonal projector of* $\mathscr{H}_q(B)$ *onto* $N(R_n)^{\perp}$.

The following corollary is a special case of part (i) of Theorem 5.2 (compare Rudin [8, p. 127]).

COROLLARY 5.3. *Let* $f \in H^2(B)$. *Then* $R_n f \in A^2_{n-1}(\Delta)$ *and*

$$\int_{\Delta} |R_n f|^2 \, dA_{n-1} \leq \int_S |f|^2 \, d\sigma.$$

Equality holds if and only if f *is of the form*

$$f = \sum_{m=0}^{\infty} \lambda_m P_m \quad \text{with} \quad \sum_{m=0}^{\infty} m! n^m |\lambda_m|^2 / (n)_m < \infty.$$

REFERENCES

1. N. ARONSZAJN, *Theory of reproducing kernels*, Trans. Amer. Math. Soc., vol. 68 (1950), pp. 337–404.
2. F. BEATROUS and J. BURBEA, *Positive-definiteness and its applications to interpolation problems for holomorphic functions*, Trans. Amer. Math. Soc., vol. 284 (1984), pp. 247–270.
3. J. BURBEA, *Inequalities for reproducing kernel spaces*, Illinois J. Math., vol. 27 (1983), pp. 130–137.
4. ______, *Inequalities for holomorphic functions of several complex variables*, Trans. Amer. Math. Soc., vol. 276 (1983), pp. 247–266.
5. P.L. DUREN, *Theory of H^p-Spaces*, Academic Press, New York, 1970.

6. M. Mateljević and M. Pavlović, *New proofs of the isoperimetric inequality and some generalizations*, J. Math. Anal. Appl., vol. 98 (1984), pp. 25–30.
7. W. Rudin, *Function theory in polydisks*, Benjamin, New York, 1969.
8. ______, *Function theory in the unit ball of* $\mathbf{C}^n$, Springer-Verlag, New York, 1980.
9. S. Saitoh, *The Bergman norm and Szegö norm*, Trans. Amer. Math. Soc., vol. 249 (1974), pp. 79–82.
10. S.E. Warschawski, *On the differentiability at the boundary in conformal mapping*, Proc. Amer. Math. Soc., vol. 12 (1961), pp. 614–620.

University of Pittsburgh
Pittsburgh, Pennsylvania