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GENERATORS AND RELATIONS FOR FINITELY GENERATED GRADED NORMAL RINGS OF DIMENSION TWO

BY

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Chapter 1. Introduction

Assume that R is a finitely generated graded normal ring of dimension 2 over $\mathbf{C}$ such that $R = \oplus_k R_k$ where $R_k = 0$ if $k < 0$ and $R_0 = \mathbf{C}$. This implies that R is the coordinate ring of a normal affine surface which admits a $\mathbf{C}^*$-action with a unique fixed point P, corresponding to the maximal ideal $\oplus_{k>0}^{\infty} R_k$ (see [5]). Henry Pinkham has shown that R is isomorphic to $\mathscr{L}(D) = \oplus_{n=0}^{\infty} L(nD)$ where D is a divisor on a Riemann surface X of genus g of the form

$$D = \sum_{p \in X} n_p P + \sum_{\substack{i=1 \\ p_i \in X}}^{k} \left(\frac{\beta_i}{\alpha_i}\right) P_i \qquad (*)$$

where $n_p \in Z$, all but finitely many $n_p = 0$, $0 < \beta_i/\alpha_i < 1$, and $L(nD)$ denotes the set of meromorphic functions f, such that $\operatorname{div}(f) + nD \geq 0$. It is easily seen that for each n, $L(nD)$ is a vector space over C.

It is always possible to choose a minimal set $S = \{y_1, \ldots, y_k\}$ of generators for $\mathscr{L}(D)$ such that the elements of S are homogeneous i.e. $y_j \in L(q_j D)$ for some q_j. In the polynomial ring $C[Y_1, \ldots, Y_k]$ give the variable Y_i degree q_i; then there exists a graded surjective homomorphism

$$\varphi\colon C[Y_1, \ldots, Y_k] \to \mathscr{L}(D), \quad \varphi(Y_i) = y_i.$$

Let I be the kernel of φ. We call I the ideal of relations for $\mathscr{L}(D)$ corresponding to S.

In the following paper it is shown that in many cases a minimal set of homogeneous generators S and generators for the corresponding ideal of relations I for $\mathscr{L}(D)$ can be determined if homogeneous generators and relations are known for $\mathscr{L}(D_1)$ where $D_1 < D$ and $\mathscr{L}(D_1)$ has a much simpler

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structure than $\mathscr{L}(D)$. In particular, the degrees of homogeneous generators and relations can be deduced for all divisors $D = D_0 + \Sigma(\beta_i/\alpha_i)P_i$ where $\deg D_0 \geq 2g + 1$ or D_0 is the canonical divisor on a nonhyperelliptic curve of genus $g > 3$. Here and throughout the paper D_0 refers to an integral divisor. These results generalize Mumford's and Saint-Donat's work on $\mathscr{L}(D_0)$ where D_0 is of degree $> 2g + 1$ (see [4] and [8]). They also generalize Saint-Donat's results in his paper on Petri's analysis of the linear system of quadrics through a canonical curve (see [7]).

Given D as in $(*)$ above, to find the degrees of the elements in a minimal set of homogeneous generators S for $\mathscr{L}(D)$, we show that it is necessary to consider the convergents p_{ij}/q_{ij} of the decomposition of the continued fraction

$$\frac{\alpha_i}{\beta_i} = a_{i1} - \cfrac{1}{a_{i2} - \cfrac{}{\ddots - \cfrac{1}{a_{ik_1}}}}$$

Here

$$\frac{p_{ij}}{q_{ij}} = a_{i1} - \cfrac{1}{a_{i2} - \cfrac{1}{a_{i3} - \cfrac{}{\ddots - \cfrac{1}{a_{ij}}}}} = [a_{i1}, \ldots, a_{ij}].$$

We find a divisor D_1 such that $\mathscr{L}(D_1)$ is understood and

$$D_1 = \sum_{P \in X} n_P P + \sum_{i=1}^{n} \frac{q_{ij_i}}{p_{ij_i}} P_i$$

where $q_{ij_i} = 0$ or p_{ij_i}/q_{ij_i} is one of the convergents of α_i/β_i. The precise conditions D_1 must satisfy are given in Theorem 2.10. We then start with a set of generators for $\mathscr{L}(D_1)$ and build on to it. The additional generators correspond to the convergents which appear in fractions in D but not in D_1. By Riemann-Roch it is clear that whenever $\deg p_{ij}D \geq 2g$ where $j > j_i$, there exists a rational function $y \in L(p_{ij}D) - L(p_{ij}D - P_i)$. In Lemma 2.9 it will be shown that y is a primitive element and therefore if S is a minimal set of homogeneous generators for $\mathscr{L}(D)$ and $\deg p_{ij}D \geq 2g$, S must contain an element (which will be called $x_{i,j}$) of degree p_{ij}. We say that such an $x_{i,j}$ is a generator corresponding to the convergent p_{ij}/q_{ij} at P_i.

Once the necessity of having elements $x_{i,j}$ in any minimal set of homogeneous generators for $\mathscr{L}(D)$ has been shown, the sufficiency is demonstrated by showing that a basis for the vector space $L(tD_1)$ for any t can be completed to a basis B_t for $L(tD)$ using certain powers and multiples of the newly acquired generators $x_{i,j}$. In each example to be considered the completed basis for $L(tD)$ is

$$B_t = \text{basis } L(tD_1) \cup \{ x_{i,j}^m x_{i,j+1}^n \}$$

where $mp_{ij} + np_{ij+1} = t$. Linear independence is shown by seeing that each function $x_{i,j}^m x_{i,j+1}^n$ has a pole at P_i of a different order.

Having established that a minimal set of homogeneous generators for $\mathscr{L}(D)$ can be obtained using a minimal set $S_1 = \{y_1, \ldots, y_s\}$ for $\mathscr{L}(D_1)$ and a set of new generators $\{x_{i,j}\}$ we now have a surjective graded homomorphism

$$\varphi\colon C[Y_1, \ldots, X_{n,k_n}] \to \mathscr{L}(D)$$

where the variables Y_l and $X_{i,j}$ have been given the degrees of the generators y_l and $x_{i,j}$. Let I be the kernel of this map and let I_1 be the kernel of the map

$$\varphi\colon C[Y_1, \ldots, Y_s] \to \mathscr{L}(D_1).$$

It will be shown by induction on the number of new generators that

$$I = \langle I_1, M - \sum_l c_l B_l \rangle$$

where M is a quadratic monomial such that $X_{i,j} | M$, $c_i \in C$ and $\varphi(B_l)$ is a basis element of the same degree as M. By a quadratic monomial we will always mean that $M = Z_i Z_j$ which of course does not imply that M is of degree two relative to the grading.

The conditions on D_1 stated in Theorem 2.10 ensure that the elements $x_{i,j}$ exist in $\mathscr{L}(D)$, suffice to generate $\mathscr{L}(D)$ but do not make any of the generators $y_1, \ldots, y_s$ of $\mathscr{L}(D_1)$ unnecessary as generators for $\mathscr{L}(D)$. First of all it is required that $\deg nD_1 \geq 2g - 1$ for all $n \geq \min p_{ij_i+1}$. This requirement ensures the existence of new generators

$$x_{i,j} \in L(p_{ij}D) - L(p_{ij}D - P_i), \quad j > j_i$$

as well as the fact that for all $m > \min p_{ij_i+1}$,

$$l(mD) - l(mD_1) = \sum_{i=1}^{n} \left[m \frac{\beta_i}{\alpha_i} \right] - t_i$$

where $t_i = [mp_{ij_i}/q_{ij_i}]$ if $j_i \neq 0$ and 0 otherwise. To prevent the possibility that some $y_k \in \langle S_1 - \{y_k\}, \{x_{i,j}\} \rangle$ it is required that each $y_k \in S_1$ be of degree

$\leq \min p_{ij_i+1}$ or that y_k be a primitive element,

$$y_k \in L(p_{is}D) - L(p_{is}D - P_i).$$

Finally, to form the basis for $L(tD)$ it will be necessary to have a generator x_{i,j_i} whenever $j_i \neq k_i$ and this is assured if degree $p_{ij_i}D_1 \geq 2g$.

As an example we consider two divisors on a Riemann surface of genus 0.

Example 1.1. Let D and D_1 be divisors on a Riemann surface X of genus 0 such that

$$D_1 = -P + \frac{1}{2}P_1 + \frac{1}{2}P_2 \quad \text{and} \quad D = -P + \frac{\beta_1}{\alpha_1}P_1 + \frac{\beta_2}{\alpha_2}P_2$$

where

$$\frac{\alpha_i}{\beta_i} = a_{i1} - \cfrac{1}{a_{i2} - \cfrac{}{\ddots \cfrac{1}{-a_{ik_i}}}}$$

has convergents p_{ij}/q_{ij} and $1/2 \leq \beta_i/\alpha_i \leq 1$. It is not difficult to see $\mathscr{L}(D_1) \simeq C[Y_1]$ where Y_1 is of degree two, $\varphi(Y_1) = y_1$ and we can take

$$y_1 = \frac{(z-P)^2}{(z-P_1)(z-P_2)}.$$

For $k \in Z^+$ a basis for $L(2kD_1)$ is $\{y_1^k\}$ and $L((2k+1)D_1) = \{0\}$. We have the necessary conditions of Theorem 2.10 since $\deg nD_1 \geq 2g - 1$ for all n and y_1 is a generator corresponding to the first convergent for α_1/β_1 and α_2/β_2. To form a minimal set S of homogeneous generators for $\mathscr{L}(D)$ one can take y_1 as above and then elements

$$x_{i,j} = \frac{(z-P)^{p_{ij}}}{(z-P_i)^{q_{ij}}(z-P_s)^{p_{ij}-q_{ij}}}, \quad s \in \{1,2\} - \{i\}, j > 1$$

The elements $x_{i,j}$ are functions of degree p_{ij} with poles at P_i of degree q_{ij}. They are necessary as generators as in $L(p_{ij}D)$ no product of functions of lower degree will have poles at P_i of as high an order as q_{ij}. For $k \in Z^+$ a basis for $\mathscr{L}(2kD)$ is

$$\{y_1^k, x_{i,j}^m x_{i,j+1}^n\} \quad \text{where} \quad mp_{ij} + np_{ij+1} = 2k$$

and a basis for $\mathscr{L}((2k+1)D)$ is

$$\{x_{i,j}^{m}x_{i,j+1}^{n}\} \quad \text{where} \quad mp_{ij} + np_{ij+1} = 2k+1.$$

Using properties of convergents one shows the sets are linearly independent and of the right order. The degrees of the generators and relations as well as the Poincaré power series for the ring can be found in Table 2.

In Chapter 2 the assertions of the foregoing paragraphs are proved in detail. Applications of the theorems in Chapter 2 are given in Chapter 3. Here it is shown that if D is a divisor on a smooth projective curve of genus g where $\deg D_0 \geq 2g+1$, $D_0 = \Sigma n_p P$, and

$$D = D_0 + \sum\left(\frac{\beta_i}{\alpha_i}\right)P_i, \quad 0 < \frac{\beta_i}{\alpha_i} < 1,$$

the degrees of the elements in a minimal homogeneous set of generators for $\mathscr{L}(D)$ and the degrees of the generators for the corresponding ideal of relations are obtained from the numerators of the convergents of the fractions α_i/β_i. For a second application, rings of automorphic forms are considered. Let G be a finitely generated Fuchsian group of the first kind and X the Riemann surface which is the compactification of H_+/G. Suppose $Q_1,\ldots,Q_s$ are the parabolic points of X and $P_1,\ldots,P_r$ the elliptic points with branching numbers $e_1,\ldots,e_r$. Let $A(k)$ be the vector space of automorphic forms of weight k relative to G, i.e.,

$$f \in A(k) \Leftrightarrow f(g(z)) = \frac{dg}{dz}^{-k} f(z).$$

We consider the ring $A(G) = \Sigma_{k=0}^{\infty} A(K)$ which we say has signature

$$(g; s; e_1,\ldots,e_r).$$

Gunning has shown that

$$A(G) = \sum_{k=0}^{\infty} A(k) \simeq \mathscr{L}(D)$$

where

$$D = K + Q_1 + \cdots + Q_s + \sum_{i=1}^{r} \frac{e_i - 1}{e_i} P_i$$

and K is the canonical divisor on X. Given any such divisor D, the degrees of the elements in a minimal set of homogenous generators for $\mathscr{L}(D)$ and the degrees of the homogeneous generators for the corresponding ideal of relations can be deduced using the theorems of Chapter 2. This work is started here and will be completed in a subsequent paper. In many cases to find $G_D(t)$ and $R_D(t)$ one can apply the work of Chapter 2 to Wagreich's results on rings of automorphic forms with few generators. I am grateful to Wagreich for these results and his good suggestions on the final copy of this paper.

Chapter 2

As stated in the introduction Henry Pinkham has shown that every finitely generated graded normal ring of dimension two over $\mathbf{C}$ is isomorphic to $\oplus_{n=0}^{\infty} L(nD)$ where D is a "fractional" divisor on a Riemann surface of genus g. More specifically D is of the form

$$D = D_0 + \sum_{i=1}^{k} \frac{\beta_i}{\alpha_i} P_i, \tag{1}$$

where $\beta_i/\alpha_i \in Q$ and $D_0 = \Sigma_{p \in X} n_p P$, n_p an integer such that $n_p = 0$ for all but finitely many P. The fractions can be added so it is assumed without loss of generality that the P_i's are distinct. We consider the vector space $L(tD) = \{f | (f) \geq -tD\}$, where $(f) = \Sigma v_p(f) P$ denotes the divisor of a meromorphic function f, and then study the ring $\oplus_{n=0}^{\infty} L(nD)$ which will be denoted by $\mathscr{L}(D)$. We will use the notation $l(nD)$ to denote the dimension of $L(nD)$. These rings are precisely the coordinate rings of normal affine surfaces with good C^*-action.

For each t, it is clear that $L(tD) = L(D^t)$ where

$$D^t = tD_0 + \sum_{i=1}^{k} \left[t \frac{\beta_i}{\alpha_i} \right] P_i$$

and $[t\beta_i/\alpha_i]$ is the greatest integer $\leq (t\beta_i/\alpha_i)$. Therefore, $\deg tD$ is defined in the following way.

DEFINITION 2.1. If

$$D = D_0 + \sum_{i=1}^{k} \frac{\beta_i}{\alpha_i} P_i,$$

for $t \in Z$ we have

$$\deg tD = \deg tD_0 + \sum_{i=1}^{k} \left[t \frac{\beta_i}{\alpha_i} \right].$$

Note. We have $\deg tD \geq 0 \nRightarrow \deg(t+1)D \geq 0$. Consider the divisor of Example 1.1,

$$D = -P + \frac{1}{2}P_1 + \frac{1}{2}P_2.$$

We have $\deg 2kD = 0$ while $\deg(2k+1)D = -1$. If $l_i < \beta_i/\alpha_i < l_i + 1$, $l_i \in Z$, then for all $n \in Z$,

$$\left[n\frac{\beta_i}{\alpha_i}\right] = nl_i + \left[n\left(\frac{\beta_i}{\alpha_i} - l_i\right)\right].$$

We therefore assume the fractions β_i/α_i in (1) are such that $0 \leq \beta_i/\alpha_i < 1$. Finally, it is clear if $D_0 \sim D_0'$ (i.e., $D_0 - D_0' = (f)$) then for

$$D = D_0 + \sum_{i=1}^{k}\left(\frac{\beta_i}{\alpha_i}\right)P_i \quad \text{and} \quad D' = D_0' + \sum_{i=1}^{k}\left(\frac{\beta_i}{\alpha_i}\right)P_i,$$

we have

$$\mathscr{L}(D') \overset{\varphi}{\cong} \mathscr{L}(D) \quad (\varphi(g) = gf^n \quad \text{for all} \quad g \in L(nD')).$$

It should be noted that in this paper the term "D a divisor" will refer to a divisor of the form given in (1) where $0 < \beta_i/\alpha_i < 1$ if $k \geq 1$ (the possibility $D = D_0$ is not excluded.)

All the results of the paper are written down in Table 2 of Chapter 3 where the Poincaré generating polynomial, the Poincaré relational polynomial and the Poincaré power series are given for $\mathscr{L}(D)$ for each divisor D which is considered in the paper. These polynomials are defined below. For

$$R = \bigoplus_{i=0}^{\infty} R_i \cong K[X_1, \dots, X_d]/I,$$

let m be the maximal ideal $m = \oplus_{i=1}^{\infty} R_i$. The elements $x_j \in m$, $1 \leq j \leq d$ are a set of algebra generators for R if and only if the residue classes $\bar{x}_1, \dots, \bar{x}_d \in m/m^2$ are a basis of m/m^2 as a K-vector space. m/m^2 is a graded vector space $m/m^2 = \oplus_{i=1}^{\infty}(m/m^2)_i$.

DEFINITION 2.2. The Poincaré generating polynomial of

$$R = \bigoplus_{i=0}^{\infty} R_i \cong K[X_1, \dots, X_d]/I$$

is defined to be

$$G_R(t) = \sum_{i=0}^{\infty} a_i t^i \quad \text{where} \quad a_i = \dim(m/m^2)_i.$$

Similarly the elements x_j, $j = 1, \dots, n$ are a minimal set of generators for I if and only if the residue classes $\bar{x}_j$, $j = 1, \dots, m$ form a basis for the graded vector space I/mI where $m = \langle X_1, \dots, X_d \rangle$.

DEFINITION 2.3. The Poincaré relational polynomial of

$$R = \bigoplus_{i=0}^{\infty} R_i \cong K[X_1, \dots, X_d]/I$$

is defined to be

$$R_R(t) = \bigoplus_{i=0}^{\infty} a_i t^i \quad \text{where} \quad a_i = \dim(I/mI)_i.$$

DEFINITION 2.4. The Poincaré power series of R is defined to be

$$P_R(t) = \sum_{i=0}^{\infty} a_i t^i \quad \text{where} \quad a_i = \dim R_i.$$

For $\mathscr{L}(D)$ we use the notation $P_D(t)$, $R_D(t)$ and $G_D(t)$.

The key results of the paper are proved in Theorem 2.10 and Theorem 2.12. Five short lemmas precede Theorem 2.10. The first three give very elementary facts about graded rings that are used throughout the rest of the paper. The fourth lemma is an equally elementary fact about certain sets of elements in the vector space $L(D)$ which will often be used to determine the independence of a chosen set of rational functions. The proofs of these lemmas are all straightforward and will be omitted. Lemma 2.9 is of utmost importance for the proof of Theorem 2.10. Its proof depends on properties of the convergents of continued fractions which are given in the appendix.

LEMMA 2.5. *Suppose R is a finitely generated graded ring over F where*

$$\varphi\colon F[X_1, \dots, X_k]/I \to R = \bigoplus_{n=0}^{\infty} V_n$$

is a graded F-isomorphism.

(1) *Given a set of elements $b_i \in F[X_1, \dots, X_k]$ such that $\phi(b_i + I)$ are a basis for V_n, for an arbitrary monomial m of degree n in $F[X_1, \dots, X_k]$ there exists a ! expression $m - \sum c_i b_i \in I$ where $c_i \in F$.*

(2) *$I = \langle m - c_i b_i \rangle$ where m, c_i, b_i are as in* (1).

DEFINITION 2.5A. Suppose $M \in F[X_1, X_2, \dots, X_k]$ is a monomial of degree n and we are given a set of elements $b_i \in F[X_1, \dots, X_k]$ such that

$\varphi(b_i + I)$ are a basis for V_n. By Lemma 2.5 there exists a unique expression $\Sigma c_i b_i$ such that $M - \Sigma c_i b_i \in I$. The notation $f^M = \Sigma c_i b_i$ will be used.

Part (2) in Lemma 2.5 implies that $I = \langle M - f^M | M \text{ a monomial}\rangle$.

Suppose $F[X_1, \ldots, X_k, Y_1, \ldots, Y_l]$ and $F[X_1, \ldots, X_k]$ are finitely generated graded polynomial rings over a field F such that X_i is of the same degree in both rings. Assume $D < D'$ where D and D' are divisors on a smooth projective curve of genus g. If φ_1 and φ_2 are graded F-isomorphisms where $\varphi_1(X_j + I_1) = \varphi_2(X_j + I_2)$ for all J then Lemma 2.6 and 2.7 show that

$$I_2 \cap F[X_1, \ldots, X_k] = I_1$$

where

$$\begin{tikzcd} F[X_1, \ldots, X_k]/I_1 \arrow[r, "\varphi_1"] \arrow[d, dashed] & \mathscr{L}(D) \arrow[d, dashed] \\ F[X_1, \ldots, X_k, Y_1, \ldots, Y_l]/I_2 \arrow[r, "\varphi_2"] & \mathscr{L}(D'). \end{tikzcd}$$

LEMMA 2.6. *Assume we are as above. A basis $\varphi_2(b_i + I_2)$ for $L(tD)$ can be completed to a basis for $L(tD')$.*

LEMMA 2.7. *Assume the hypotheses of Lemma 2.6 where*

$$m = f(X_1, \ldots, X_k) \in F[X_1, \ldots, X_k, Y_1, \ldots, Y_l]$$

is of degree t and a basis for $L(tD')$, $\{\varphi_2(b_i + I_2)\}$ $i = 1, \ldots, r$, is such that $b_i = g_i(X_1, \ldots, X_k)$ $i = 1, \ldots, s$ and $\varphi_1(b_i + I_1)$ $i = 1, \ldots, s$, $s \le r$ is a basis for $L(tD)$. It then follows that in the relation

$$m - \sum_{i=1}^{r} c_i b_i \in I_2,\ c_i = 0 \quad \text{for all} \quad i > s \quad \text{and} \quad m - \sum_{i=1}^{s} c_i b_i \in I_1.$$

LEMMA 2.8. *Suppose $D = \Sigma_{i=1}^r m_i P_i$, $m_i \in Z$, is a divisor on a smooth projective curve X of genus g with function field $K(X)$. Given a set of rational functions $\{x_1, \ldots, x_k\}$, $x_i \in \mathscr{L}(D)$, choose any s, $1 \le s \le r$. Suppose in $\{x_1, \ldots, x_k\}$ there exists at most one rational function x_j such that $v_{p_s}(x_j) = \mathscr{O}$ for each integer $\mathscr{O}$, $-m_s \le \mathscr{O} \le t$, then if $\Sigma_i k_i x_i = 0$, $k_i = 0$ for all x_i such that $v_{p_s}(x_i) \le t$.*

The above will be used in the following situation.

LEMMA 2.8A. *Suppose*

$$D = D_0 + \sum_{i=1}^{k} \frac{\beta_i}{\alpha_i} P_i,\ D_0 = \sum_{p \in X} n_p P, n_p \in Z,$$

is a divisor on a smooth projective curve of genus g. Suppose

$$x_{i,j} \in L(p_{ij}D) - L(p_{ij}D - P_i)$$

where

$$\frac{p_{ij}}{q_{ij}} = [a_{i1}, \dots, a_{ij}](p_{i0} = 1, q_{i0} = 0).$$

Then in $L(nD)$ the set $\{x_{i,j}^s x_{i,j+1}^t\}$ where i is fixed and $sp_{ij} + tp_{ij+1} = n$ is linearly independent. Here s and t range over all nonnegative solutions to the diophantine equation $sp_{ij} + tp_{ij+1} = n$.

Proof. By the theorem on convergents in the appendix each element in the set is a rational function r with $v_{p_i}(r)$ of a different order $\mathcal{O}$,

$$-nn_{p_i} \geq \mathcal{O} \geq -nn_{p_i} - \left[n\frac{\beta_i}{\alpha_i}\right].$$

Now Lemma 2.8 applies. □

DEFINITION 2.9A. Let x be a homogeneous element of $\mathcal{L}(D)$. Thus $x \in L(nD)$ for some n and suppose x is not in the image of

$$\varphi_i\colon L(iD) \oplus L((n-i)D) \to L(nD)$$

for any $i = 1, 2, \cdots (n-1)$. Then x is called a primitive element of $\mathcal{L}(D)$.

It is clear if there exists a primitive element in $L(nD)$, any set of homogeneous generators for $\mathcal{L}(D)$ must have an element of degree n.

LEMMA 2.9. *Suppose $D = D_1 + (\beta/\alpha)P$ is a divisor on a smooth projective curve of genus g with function field $K(X)$. Suppose $0 < \beta/\alpha < 1$ and $s \in Z$ is the order of P in D_1. Let $\alpha/\beta = [a_1, \dots, a_k]$ have convergents $p_j/q_j = [a_1, \dots, a_j]$ (by convention $p_0 = 1, q_0 = 0$).*

(1) *If* $\deg p_j D \geq 2g$ *then in $\mathcal{L}(D)$ any x_j of degree p_j such that*

$$x_j \in L(p_jD) - L(p_jD - P)$$

is primitive.

(2) *Suppose* $\deg p_jD \geq 2g$ *for $j = 0, \dots, k$ and $t \in Z^+$. The set*

$$S = \{x_j^m x_{j+1}^n\colon mp_j + np_{j+1} = t\}$$

is a set of $[t\beta/\alpha] + 1$ linearly independent elements of $L(tD)$.

(3) *For a fixed positive integer* $v < k$, *the set*

$$S_1 = \{x_j^m x_{j+1}^n \colon mp_j + np_{j+1} = t,\ j \geq v,\ n \neq 0 \text{ if } j = v\}$$

is a linearly independent set with $[t\beta/\alpha] - [tq_v/p_v]$ *elements. In* (2) *and* (3), m *and* n *are defined as* s *and* t *were in Lemma* 2.8A.

Proof. If $\deg p_1 D \geq 2g$ then there exists $f \in L(p_j D) - L(p_j D - P)$ by Riemann Roch. We claim that f is primitive. It is sufficient to see that if m is an arbitrary monomial $m = \Pi y_l^{m_l} \in L(p_j D)$, then $v_p(m) > v_p(f)$ where each y_l is an element of degree n_l, $n_l < p_j$. Now $v_p(f) = -p_j s - q_j$. Given y_l of $\deg n_l$, $n_l < p_j$, y_l is such that $v_p(y_l) \geq -[n_l\beta/\alpha] - n_l s$.

Now we know by Fact 7 in the appendix that

$$\left[n_l \frac{\beta}{\alpha}\right] = \left[n_l \frac{q_{j-1}}{p_{j-1}}\right] \quad \text{if} \quad n_l < p_j.$$

We have $\sum n_l m_l = p_j$ and

$$v_p(m) \geq -\sum n_l m_l s - \sum m_l \left[n_l \frac{q_{j-1}}{p_{j-1}}\right] = -p_j s - \sum m_l \left[n_l \frac{q_{j-1}}{p_{j-1}}\right].$$

Suppose

$$\sum m_l \left[n_l \frac{q_{j-1}}{p_{j-1}}\right] \geq q_j.$$

Then

$$\sum m_l \left(n_l \frac{q_{j-1}}{p_{j-1}}\right) \geq q_j$$

which implies

$$\sum m_l n_l q_{j-1} \geq p_{j-1} q_j.$$

Using Fact 2 of the appendix ($p_{j-1}q_j - q_{j-1}p_j = 1$), we get

$$\sum m_l n_l q_{j-1} \geq 1 + q_{j-1} p_j$$

which implies

$$p_j q_{j-1} \geq 1 + q_{j-1} p_j.$$

This is impossible so $v_p(m) > -p_j s - q_j = v_p(f)$ and $\mathscr{L}(D)$ has a primitive x_j of $\deg p_j$ where $x_j \in L(p_j D) - L(p_j D - P)$.

Proof of (2). By the theorem on convergents in the appendix each element in S is a rational function with a pole at P of a different order $\mathcal{O}$ such that $-ts \geq \mathcal{O} \geq -ts - [\beta_t/\alpha]$. Furthermore for each $\mathcal{O}$ there exists a rational function $f \in S$ such that $v_p(f) = \mathcal{O}$. The set S therefore has $[t\beta/\alpha] + 1$ elements; it is linearly independent by Lemma 2.8.

Proof of (3). Using the argument of (2), the subset S' of S given by

$$S' = \left\{ x_j^m x_{j+1}^n \colon j = 0, \dots, v - 1,\ mp_j + np_{j+1} = t \right\}$$

is a linearly independent set of $[tq_v/p_v] + 1$ elements. In $L(tD)$, $S_1 = S - S'$ and so is a linearly independent set of $[t\beta/\alpha] - [tq_v/p_v]$ elements. ■

In the theorems that follow it is always assumed that

$$D = D_0 + \sum_{i=1}^{n} \frac{\beta_i}{\alpha_i} P_i$$

is a divisor on a smooth projective curve X such that $D_0 = \Sigma_{p \in X} n_p P$, $n_p \in Z$, $0 < \beta_i/\alpha_i < 1$, and the P_i's are distinct.

The jth convergent of α_i/β_i refers to the fraction

$$\frac{p_{ij}}{q_{ij}} = a_{i1} - \cfrac{1}{a_{i2} - \cfrac{1}{a_{i3} - \ddots - \cfrac{1}{a_{ij}}}} = [a_{i1}, \dots, a_{ij}]$$

where $\alpha_i/\beta_i = [a_{i1}, \dots, a_{ik_i}]$. $x_{i,j} \in \mathcal{L}(D)$ is always a rational function of degree p_{ij} such that $x_{i,j} \in L(p_{ij}D) - L(p_{ij}D - P_i)$.

THEOREM 2.10. *Suppose $a_{i_1}, \dots, a_{ik_1} \geq 2$ and $[a_{i1}, \dots, a_{ij}] = p_{ij}/q_{ij}$ for all i. Let*

$$D_1 = D_0 + \sum_{i=1}^{n} \frac{q_{ij_i}}{p_{ij_i}} P_i \quad \text{and} \quad D = D_0 + \sum_{i=1}^{n} \frac{\beta_i}{\alpha_i} P_i.$$

Here if $j_i > 0$, p_{ij_i}/q_{ij_i} is a convergent of $\alpha_i/\beta_i = p_{ik_i}/q_{ik_i}$ and D_0 as always is an integral divisor. If $D_0 \geq 0$ we may have $j_i = 0$ using the convention $q_{i0} = 0$, $p_{i0} = 1$. Otherwise $1 \leq j_i \leq k_i$. Assume further:

(1) *For $1 \leq i \leq n$ if $j_i \neq k_i$ there exists*

$$x_{i,j_i} \in L\left(p_{ij_i} D_1\right) - L\left(p_{ij_i} D_1 - P_i\right).$$

In the case $j_i = 0$ *this is the constant function which we denote by* $x_{i,0}$.

(2) *We have* $\deg mD_1 \geq 2g - 1$ *whenever* $m \geq \min p_{ij_i+1}$, $1 \leq i \leq n$.

(3) *There exists a minimal set of generators* $\{y_1, \ldots, y_l\}$ *for* $L(D_1)$ *such that for all r either* (a) $\deg y_r \leq \min p_{ij_i+1}$, $1 \leq i \leq n$, *or* (b) y_r *is the sole generator corresponding to a convergent, i.e.*,

$$y_r \in L(p_{ts}D_1) - L(p_{ts}D_1 - P_t)$$

for some $s \leq j_t$ *and for* $i \neq r$, $y_i \notin L(p_{ts}D_1) - L(p_{ts}D_1 - P_t)$.

Choose elements $x_{i,j}$ *of degree* $p_{ij} \in L(p_{ij}D) - L(p_{ij}D - P_i)$ *where* $j_i < j \leq k_i$, $1 \leq i \leq n$. *It then follows that* $\{y_1, \ldots, y_l, x_{i,j}, \ldots, x_{n,k_n}\}$ *is a minimal set of homogeneous generators for* $\mathscr{L}(D)$.

Proof. For each m let B_m be a basis of $L(mD_1)$ and let

$$c_m = \Big\{ x_{i,j}^s x_{i,j+1}^t \colon j_i \leq j \leq k_i - 1,\ s \neq 0 \text{ if } j = j_i,\ 1 \leq i \leq n, \text{ and}$$
$$s, t \text{ range over all nonnegative integers such that } sp_{ij} + tp_{ij+1} = m \Big\}.$$

We will show that $B_m \cup \mathbf{C}_m$ is a basis for $L(mD)$.

Step 1. We show that

$$l(mD) = l(mD_1) + \sum_{i=1}^{n} \left[\frac{q_{ik_i}}{p_{ik_i}} m \right] - \left[\frac{q_{ij_i}}{p_{ij_i}} m \right] \quad \text{for all} \quad m.$$

If $m < \min p_{ij_i+1}$ then $L(mD) = L(mD_1)$ since $[q_{ik_i}/p_{ik_i}m] = [q_{ij_i}/p_{ij_i}m]$ for all i by Facts 7 and 11 in the appendix.

For $m \geq \min p_{ij_i+1}$, by Riemann Roch, condition 2 and the fact that $\deg mD \geq \deg mD_1$ we have

$$\begin{aligned} l(mD) &= \deg mD + 1 - g \\ &= \deg mD_1 + \sum_{i=1}^{n} \left[\frac{q_{ik_i}}{p_{ik_i}} m \right] - \left[\frac{q_{ij_i}}{p_{ij_i}} m \right] + 1 - g \\ &= l(mD_1) + \sum_{i=1}^{n} \left[\frac{q_{ik_i}}{p_{ik_i}} m \right] - \left[\frac{q_{ij_i}}{p_{ij_i}} m \right]. \end{aligned}$$

Step 2. By repeated application of Lemma 2.9,

$$\begin{aligned} |B_m \cup C_m| &= l(mD_1) + \sum_{i=1}^{n} \left[\frac{q_{ik_i}}{p_{ik_i}} M \right] - \left[\frac{q_{ij_i}}{p_{ij_i}} m \right] \\ &= l(mD). \end{aligned}$$

By Lemma 2.9 and Lemma 2.8, $B_m \cup C_m$ is a linearly independent set. Thus

$$\{ y_1, \ldots, y_2, x_{i,j}, \ldots, x_{n,k_n} \}$$

generates the ring $\mathscr{L}(D)$. It must now be seen the set is minimal. The element $x_{i,j}$ cannot be generated by other elements since $x_{i,j}$ is primitive in $\mathscr{L}(D)$ by Lemma 2.9. It is clear that

$$y_i \notin \langle y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_l, x_{i,j}, \ldots, x_{k,k_n} \rangle$$

as $\deg y_i \leq \min p_{ij_i+1}$ or y_i is a primitive element in $\mathscr{L}(D)$.

COROLLARY 2.10A. *Let D_1 and D be as above. If $P_R(t)$ is the Poincaré power series for $\mathscr{L}(D_1)$ then*

$$P_R(t) + \sum_{i=1}^{n} \sum_{l=j_i}^{k_i-1} \frac{1}{(1 - t^{p_{il}})(1 - t^{p_{il+1}})} - \frac{1}{(1 - t^{p_{il}})}$$

is the Poincaré power series for $\mathscr{L}(D)$.

Proof. This follows by comparing $\bigcup B_m$ and $\bigcup (B_m \cup C_m)$ where B_m is a basis for $L(mD_1)$ and $B_m \cup C_m$ is the basis chosen in Theorem 2.10 for $L(mD)$.

Given $D_1 < D$ as in Theorem 2.10, it has been seen that the degrees of the elements in a minimal set of homogeneous generators for $\mathscr{L}(D)$ can be easily obtained when those for $\mathscr{L}(D_1)$ are known. Theorem 2.12 shows that if the degrees of the generators for the corresponding ideal of relations for $\mathscr{L}(D_1)$ are given, those for $\mathscr{L}(D)$ can be deduced.

Proposition 2.11 is an elementary fact which is used extensively in the proof of Theorem 2.12. We can assume that if $\{ y_1, \ldots, y_s \}$ is a set of homogeneous generators for $\mathscr{L}(D_1)$ the basis for each vector space $L(nD_1)$ consists of elements of the form $\prod y_i^{n_i}$, i.e., the basis elements are monomials. The proof is straightforward and will be omitted.

PROPOSITION 2.11. *Let D be a divisor on a smooth projective curve of genus g and suppose $\mathscr{L}(D)$ is isomorphic to $F[X_1, \ldots, X_s]/I$. Suppose M is a monomial of* $\deg t$ *in $F[X_1, \ldots, X_s]$. Given $M - m_i \in J \subset I$, m_i monomials of degree t, to show $M - f^M \in J$ one need only see that $m_i - f^{m_i} \in J$ for each i.*

THEOREM 2.12. *Suppose D_1 and D are as in Theorem 2.10. Recall that*

$$D_1 = D_0 + \sum_{i=1}^{n} \frac{q_{ij_i}}{p_{ij_i}} P_i, \quad D = D_0 + \sum_{i=1}^{n} \frac{\beta_i}{\alpha_i} P_i$$

where if $j_i \neq 0$, p_{ij_i}/q_{ij_i} *is a convergent of* $\alpha_i/\beta_i = p_{ik_i}/q_{ik_i}$. *By Theorem* 2.10 *we know that if*

$$\mathscr{L}(D_1) \cong \mathbf{C}[Y_1,\dots,Y_s]/I_1$$

then

$$\mathscr{L}(D) \cong \mathbf{C}\left[Y_1,\dots,Y_s,X_{i,j},\dots,X_{n,k_n}\right]/I.$$

We now show I is generated by I_1 together with $\{M - f^M | M$ is a quadratic monomial and $X_{i,j}|M$ for some i, j $1 \le i \le n$, $j_i + 1 \le j \le k_i\}$. The proof is by induction on $\sum_{i=1}^n(k_i - j_i)$. Suppose $\sum_{i=1}^n(k_i - j_i) = 1$. By the hypotheses and Theorem 2.10 we have a set of homogeneous generators $\{y_1,\dots,y_s,x_{i,j_i+1}\}$ for $L(D)$, for some particular i, and we have some

$$y_k = x_{i,j_i} \in L(p_{ij}D_1) - L(p_{ij}D_1 - P_i).$$

Now $x_{i,j_i+1} \in L(p_{ij_i+1}D) - L(p_{ij_i+1}D - P_i)$ is the only "new" element. Let

$$J = \langle I_1, Y_l X_{i,j_i+1} - f^{Y_l X_{i,j_i+1}}\rangle,\quad 1 \le l \le s.$$

If $J \neq I$ there exists a relation $R \in I$ of smallest degree r such that $R \notin J$. To prove R cannot exist we look at the set

$$S = \left\{M_k - f^{M_k}\colon M_k \textit{ is a monomial of degree } r \textit{ and } M_k - f^{M_k} \notin J\right\}$$

and show that S is empty. For each $M_k - f^{M_k} \in S$ we must have $X_{i,j_i+1}|M_k$ as otherwise $M_k - f^{M_k} \in I_1 \subset J$. Find $M_s - f^{M_s} \in S$ so that t_s is the smallest positive integer where $X_{i,j_i+1}^{t_s}|M_s$ but $X_{i,j_i+1}^{t_s+1} \nmid M_s$. We know there exists a Y_t such that $Y_t \neq X_{i,j_i}$ and $Y_t|M_s$ since $X_{i,j_i}^{m_1}X_{i,j_i+1}^{m_2}$ is a basis element. We have

$$\frac{M_s}{Y_t} - f^{M_s/Y_t} \in J$$

by choice of r. By Proposition 2.11, $M_s - f^{M_s} \in J$ if for each summand m_k of $Y_t f^{M_s/Y_t}$ we have $m_k - f^{m_k} \in J$. By Lemma 2.6, 2.7 and the fact that $I_1 \subset J$ we need only see that $m_l - f^{m_l} \in J$ for $m_l = c_l X_{ij_i+1}^{m_1}X_{ij_i}^{m_2}Y_t$. By Lemma 2.13 which follows the proof, we get $m_1 \le t_s$. We know that

$$X_{i,j_1+1}^{m_1-1}X_{i,j_i}^{m_2}\left(X_{i,j_i+1}Y_t - f^{X_{i,j_i+1}Y_t}\right) \in J.$$

Each summand of $X_{i,j_i+1}^{m_1-1}X_{i,j_i}^{m_2}f^{X_{i,j_i+1}Y_t}$ is a basis element or a monomial m for which $m - f^m$ must be in J by choice of t_s, so in either case Proposition 2.11 applies to show $m_l - f^{m_l} \in J$. Thus $M_s - f^{M_s} \notin S$. The supposition that S is nonempty leads to a contradiction and the proof of the case $\Sigma(k_i - j_i) = 1$ is finished.

For $\Sigma_{i=1}^{n}(k_i - j_i) > 1$ consider an appropriate D'' such that $D_1 < D'' < D$. Use the induction hypothesis on $\mathscr{L}(D_1)$ and $\mathscr{L}(D'')$ and then again on $L(D'')$ and $\mathscr{L}(D)$. □

In Lemma 2.13 we simplify notation by writing P for the particular P_i in Theorem 2.11; X_{i,j_i} will be simplified to X_j, X_{i,j_i+1} to X_{j+1}, and the labels for convergents will be simplified accordingly.

LEMMA 2.13. *If $X_{j+1}^t | M$ but $X_{j+1}^{t+1} \nmid M$ and $f^M = \Sigma_l c_l B_l$, then $X_{j+1}^{t+1} \nmid B_l$ for any l.*

Proof. Suppose Y_i is of degree s_i then $M = X_{j+1}^t \Pi Y_i^{n_i}$ is of degree $d = tp_{j+1} + \Sigma n_i s_i$ and

$$v_p\left(X_{j+1}^t \prod Y_i^{n_i}\right) \geq -tq_{j+1} - \sum n_i \left[s_i \frac{q_j}{p_j}\right] - dl_p$$

where l_p is the order of P in D_1.

Suppose there exists $B_l = X_{j+1}^{m_1} X_j^{m_2}$ where $m_1 \geq t+1$, in f^M. We have

$$\left.\begin{aligned} m_1 p_{j+1} + m_2 p_j &= d \\ tp_{j+1} + \sum n_i s_i &= d \end{aligned}\right\} \Rightarrow \sum n_i s_i = (m_1 - t) p_{j+1} + m_2 p_j.$$

$$v_p\left(x_{j+1}^{m_1} x_j^{m_2}\right) = -dl_p - m_1 q_{j+1} - m_2 q_j.$$

Lemma 2.9 and Lemma 2.8 imply

$$tq_{j+1} + \sum n_i \left[s_i \frac{q_j}{p_j}\right] \geq m_1 q_{j+1} + m_2 q_j. \tag{1}$$

As

$$\left[\sum n_i s_i \frac{q_j}{p_j}\right] \geq \sum n_i \left[s_i \frac{q_j}{p_j}\right],$$

(1) implies

$$tq_{j+1} + \left[\left((m_1 - t) p_{j+1} + m_2 p_j\right) \frac{q_j}{p_j}\right] \geq m_1 q_{j+1} + m_2 q_j.$$

Using Fact 2 of the appendix we get

$$tq_{j+1} + \left[(m_1 - t) \frac{(q_{j+1} p_j - 1)}{p_j} + m_2 q_j\right] \geq m_1 q_{j+1} + m_2 q_j$$

which implies

$$m_1 q_{j+1} + m_2 q_j + \left[\frac{t - m_1}{p_j}\right] \geq m_1 q_{j+1} + m_2 q_j.$$

But this is impossible if $m_1 > t$ as then

$$\left[\frac{t - m_1}{p_j}\right] < 0.$$

Therefore $m_1 \leq t$. □

It should be noted that if $\{r_1, \ldots, r_l\}$ is a minimal set of homogeneous generators for I_1, $\{r_1, \ldots, r_l, M - f^M\colon M$ is quadratic and $x_{i,j}|M\}$ may not be a minimal set for I. Each element $M - f^M$ is necessary but one can have some $r_k \in \langle M - f^M \rangle$. A case in which this occurs is given in Example 3.5. If, however, each r_i is of the form $r_i = M' - f^{M'}$, M' quadratic, then $\{r_1, \ldots, r_l, M - f^M\}$ is a minimal set for I.

As this is an important fact and will be used in Chapter 3 it is proved as a lemma.

LEMMA 2.14. *Assume $\mathscr{L}(D)$ is minimally generated by the rational functions $y_1, \ldots, y_s$ and, for each n, B_n is a chosen basis of $L(nD)$. Then $\mathscr{L}(D)$ is isomorphic to $K[Y_1, \ldots, Y_s]/I$, $\varphi(y_i) = Y_i$. If the elements in the set $\{M - f^M\colon$ M is quadratic$\}$ are a sufficient set of generators for I, they are a minimal set.*

Proof. We only prove the last statement. From Lemma 2.5 and the fact that $K[Y_1, \ldots, Y_s]$ is noetherian we know I can be generated by a finite set of elements r_i, $i = 1, \ldots, k$, such that $r_i = M - f^M$ where M is a monomial and f^M is the expression for M in terms of the basis. If $y_j y_l$ is not a basis element, $Y_j Y_l - f^{Y_j Y_l} \in I$ is a nontrivial element in I. As $\mathscr{L}(D)$ is minimally generated by the y_i's, the terms cY_k and cY_l do not appear as summands in any r_i for any $c \in K$.

If $Y_j Y_l - f^{Y_j Y_l} = \sum_{i=1}^k g_i r_i$ where $g_i \in K[Y_1, \ldots, Y_s]$ the quadratic term $Y_j Y_l$ implies that for some i, $g_i \in K$ and $Y_j Y_l$ appears as a summand in r_i. But now since $Y_j Y_l$ is not a basis element, we must have $r_i = Y_j Y_l - f^{Y_j Y_l}$. □

COROLLARY 2.14A. *If $S = \{M_i - f^{M_i}\}$ is a set of generators for I, and M' is quadratic but not a basis element then $M' - f^{M'} \neq 0$ and is in S.*

THEOREM 2.15. *Assume D is a divisor on a smooth projective curve of genus g such that*

$$D = D_0 + \sum_{i=1}^{n} \frac{\beta_i}{\alpha_i} P_i$$

where β_i/α_i *has convergents* p_{ij}/q_{ij}. *Suppose*

(1) $D_0 \geq 0$, $\mathscr{L}(D_0) \cong C[Y_1,\ldots,Y_s]/I$ *and* $L(nD_0)$ *has a basis* B_n,

(2) N *is the largest positive integer such that* $\beta_i/\alpha_i < 1/N$ *for all* i;

(3) $\{y_1,\ldots,y_s\}$ *is a minimal set of generators for* $\mathscr{L}(D_0)$ *such that each* y_s *is of degree* $\leq N+1$,

(4) $\deg D_0 \geq (2g-1)/(N+1)$.

Then

$$\mathscr{L}(D) \cong C\left[Y_1,\ldots,Y_s, X_{1,1},\ldots,X_{n,k_n}\right]/I'$$

where $I' = \langle I, M - f^M$: *M is a quadratic monomial with* $X_{i,j}|M$ *for some* $i, j\rangle$. *A basis for* $L(nD)$ *is*

$$\left\{B_n, x_{i,j}^{m_1}x_{i,j+1}^{m_2}: m_1p_{ij} + m_2p_{ij+1} = n.\right\}$$

Proof. This follows from Theorems 2.10 and 2.12 where in the notation of those theorems, min $p_{ij_i+1} = N+1$. □

Chapter 3

In Chapter 3 several applications of the theorems of Chapter 2 are given. It is also shown that if D_1 and D fulfill the hypotheses of Theorem 2.10 where $D_1 < D$ and $G_D(1) > G_{D_1}(1) + 1 \geq 3$ then $\mathscr{L}(D)$ is not isomorphic to the coordinate ring of a complete intersection. The results obtained in this chapter are collected together at the end in Table 2. The single most general result follows in Theorem 3.1.

THEOREM 3.1. *Let* D *be a divisor on a smooth projective curve of genus* g *such that*

$$D = D_0 + \sum_{i=1}^{k}\frac{\beta_i}{\alpha_i}P_i, \quad \frac{\beta_i}{\alpha_i} \in Q.$$

We assume without loss of generality that D_0 *has integer coefficients*, $0 < \beta_i/\alpha_i < 1$, *and the* P_i*'s are distinct. Suppose* $\deg D_0 \geq 2g+2$. *For each* β_i/α_i *find the convergents* p_{ij}/q_{ij}, $j = 1,\ldots,k_i$, *of the decomposition of*

$$\frac{\alpha_i}{\beta_i} = [a_{i1},\ldots,a_{ik_i}].$$

Let $p_{i0} = 1$ *and let* $\varphi_{\alpha_i,\beta_i} = \sum_{j=1}^{k_i}t^{p_{ij}}$. *Suppose* $S = \{(i, j, i', j'): 1 \leq i \leq i' \leq k,$

$1 \leq j \leq k_i$, $1 \leq j' \leq k_{i'}$, *and finally* $j - j' \geq 2$ *if* $i = i'\}$. *Then*

$$G_D(t) = G_{D_0}(t) + \sum_{i=1}^{k} \varphi_{\alpha_i, \beta_i},$$

$$R_D(t) = R_{D_0}(t) + \sum_{i=1}^{k} \left(G_{D_0}(t)\varphi_{\alpha_i, \beta_i} - t^{p_{i1}+1}\right) + \sum_{(i, j, i'j') \in S} t^{p_{ij}+p_{i'j'}},$$

$$P_D(t) = P_{D_0}(t) + \sum_{i=1}^{k} \sum_{j=0}^{k_i - 1} \left(\frac{1}{(1 - t^{p_{ij}})(1 - t^{p_{ij+1}})} - \frac{1}{(1 - t^{p_{ij}})} \right),$$

$$G_D(1) = G_{D_0}(1) + \sum_{i=1}^{k} k_i,$$

$$R_D(1) = R_{D_0}(1) + \frac{1}{2}(\Sigma k_i)\left(2G_{D_0}(1) + \Sigma k_i - 3\right).$$

Proof. In [8], Saint-Donat proves that

$$G_{D_0}(t) = (\deg D_0 + 1 - g)t$$

and

$$R_{D_0}(t) = \frac{\left(G_{D_0}(t)\right)^2 - tG_{D_0}(t) - 2t^2 \deg D_0}{2}$$

Now if $\deg D_0 > 2g - 1$ then D_0 is base point free so we can assume D_0 is effective. It is not difficult to see that D_0 and D satisfy the hypotheses of Theorem 2.10. As an immediate consequence of Theorem 2.10,

$$G_D(t) = (\deg D_0 + 1 - g)t + \sum_{i=1}^{k} \varphi_{\alpha_i, \beta_i}.$$

The elements represented in $G_D(t) - G_{D_0}(t)$ are rational functions $x_{i,j}$, $j > 0$ of degree p_{ij} such that $x_{i,j} \in L(p_{ij}D) - L(p_{ij}D - P_i)$. Let $x_{i,0}$ be the constant function and one of the generators for $\mathscr{L}(D_0)$. For each n, a basis for $L(nD)$ is $B_n = \{$basis for $L(nD_0)$, $x_{i,j}^{n_1} x_{i,j+1}^{n_2}\}$, $n_1 p_{ij} + n_2 p_{ij+1} = n$, $n_2 \neq 0$ if $j = 0$. It follows that

$$P_D(t) = P_{D_0}(t) + \sum_{i=1}^{k} \sum_{j=0}^{k_i} \left(\frac{1}{(1 - t^{p_{ij}})(1 - t^{p_{ij+1}})} - \frac{1}{(1 - t^{p_{ij}})} \right)$$

by considering $\bigcup_n B_n$.

By Theorem 2.12, Lemma 2.14, and Saint-Donat's result, the ideal of relations is minimally generated by $\langle M - f^M \rangle$ where M is quadratic (i.e., has precisely 2 factors). The terms in $R_D(t) - R_{D_0}(t)$ are as follows. The summands in

$$\sum_{i=1}^{k} G_{D_0}(t)\varphi_{\alpha_i,\beta_i}$$

represent the elements $X_{i,j}Y_s - f^{X_{i,j}Y_s}$ where Y_s corresponds to a generator for $\mathscr{L}(D_0)$. As there exists $Y_s = X_{i,0}$ and $x_{i,0}x_{i,1}$ is a basis element, $\sum_{i=1}^{k} t^{p_{i1}+1}$ must be subtracted out. The summands in

$$\sum_{(i,j,i'j')\in S} t^{p_{ij}+p_{i'j'}}$$

represent elements $X_{i,j}X_{i',j'} - f^{X_{i,j}X_{i',j'}}$. $G_D(1)$ gives the number of elements in a minimal set of homogeneous generators for $\mathscr{L}(D)$. $R_D(1)$ is as stated as each time a new generator $x_{i,j}$ is added to a set of l generators one gets $l-1$ new nontrivial relations $M - f^M$ where M is quadratic and $x_{i,j}|M$. Therefore if $G_D(1) - G_{D_0}(1) = s$ then

$$R_D(1) - R_{D_0}(1) = G_{D_0}(1) - 1 + G_{D_0}(1) + \cdots + G_{D_0}(1) + s - 2.$$

With one additional proposition one can extend the results of Theorem 3.1 to the case in which $\deg D_0 = 2g+1$ or $D_0 = K$ where K is the canonical divisor on a nonhyperelliptic curve of genus $g > 3$. Saint Donat has shown that in both these cases $G_{D_0}(t)$ is a polynomial of degree one and $R_{D_0}(t)$ has at most degree 3. Theorem 3.3 can then be proved as a direct consequence of the following.

PROPOSITION 3.2. *Let D be as in Theorem* 3.1 *only without the supposition that* $\deg D_0 \geq 2g+2$. *Suppose instead we are given* $D_0 > 0$, $\deg 2D_0 \geq 2g-1$ *and*

$$G_{D_0}(t) = a_1t + a_2t^2, \qquad R_{D_0}(t) = b_2t^2 + b_3t^3.$$

Then the conclusions of Theorem 3.1 *hold for* $\mathscr{L}(D)$. *Furthermore the coefficient of t^3 in $R_D(t)$ is $b_3 + (a_1 - 1)N$ where N is the number of fractions in D which are* $> 1/2$.

Proof. All follows as in Theorem 3.1 except for the statement about $R_D(t)$. Assume S is a minimal set of relations for the ideal of relations of $\mathscr{L}(D_0)$. We claim that if $f \in S$, then $f \notin \langle S - \{f\}, M - f^M\rangle$, M a quadratic monomial such that $X_{i,j}|M$ where $x_{i,j}$ is one of the new generators for $\mathscr{L}(D)$. Let N be the number of fractions in D which are $\geq 1/2$. The degrees of the new generators are all ≥ 2 and $x_{i,j}$ is of degree 2 if and only if $j = 1$ and $\beta_i/\alpha_i \geq 1/2$. The degrees of the quadratics M are therefore ≥ 3 and there are precisely $N(a_1 - 1)$ of degree 3 which are not basis elements. (Recall $L(D)$ has a generator x_0 which is the constant function and $x_0 x_{i1}$ is a basis element). Suppose

$$f = r + \sum_{i=1}^{k} g_i\left(M_i - f^{M_i}\right), \quad r \in \langle S \rangle, \text{ some } g_i \neq 0.$$

Now it must be that $\deg f = 3$ and $g_i = c$ but this implies that the right hand side has a quadratic term which is divisible by a new generator while the left hand side does not. Therefore

$$f \notin \langle S - \{f\}, M - f^M \rangle.$$

THEOREM 3.3. *Let D be a divisor on a smooth projective curve such that*

$$D = D_0 + \sum_{i=1}^{k} \frac{\beta_i}{\alpha_i} P_i \quad \text{where } 0 < \frac{\beta_i}{\alpha_i} < 1 \text{ and } D_0 = \sum_{p \in X} n_p P,\ n_p \in Z.$$

Suppose $\deg D_0 = 2g + 1$ *or* $D_0 = K$ *where K is the canonical divisor on a non-hyperelliptic curve of genus $g > 3$. Then the conclusions of Theorem* 3.1 *hold for* $\mathcal{L}(D)$.

Proof. One can assume without loss of generality that $D_0 > 0$. Now apply Saint Donat's results from [7] and [8] along with Proposition 3.2 to get the result. □

Thus far we have looked at examples in which $D_0 > 0$. The divisor of Example 1.1,

$$D = -P + \frac{\beta_1}{\alpha_1} P_1 + \frac{\beta_2}{\alpha_2} P_2,\ g = 0,\ \frac{\beta_i}{\alpha_i} \geq \frac{1}{2},$$

provides a clear illustration of Theorem 2.10 where $D_0 < 0$.

LEMMA 3.4. *Let D be a divisor on a Riemann surface of genus 0 such that*

$$D = -P + \frac{\beta_1}{\alpha_1} P_1 + \frac{\beta_2}{\alpha_2} P_2,\ \frac{\beta_i}{\alpha_i} \geq \frac{1}{2} \quad \textit{for all } i.$$

Let $\varphi_{\alpha_i, \beta_i}$ *and* S *be as in Theorem* 3.1. *Then*

$$G_D(t) = \sum_{i=1}^{2} \varphi_{\alpha_i, \beta_i} - t^2,$$

$$R_D(t) = \sum_{(i,j,i',j') \in S} t^{p_{ij}+p_{i',j'}} - t^2 \sum_{i=1}^{2} \varphi_{\alpha_i, \beta_i} + t^4,$$

$$P_D(t) = \sum_{i=1}^{2} \sum_{j=1}^{k_i - 1} \left(\frac{1}{(1 - t^{p_{ij}})(1 - t^{p_{ij+1}})} - \frac{1}{(1 - t^{p_{ij}})} - \frac{t^2}{(1 - t)^2} \right) + 1.$$

Proof. For $D_1 = -P + 1/2P_1 + 1/2P_2$ one gets $\mathscr{L}(D_1) \simeq C[Y_1]$ where

$$y_1 = x_{1,1} = x_{2,1} = \frac{(z - P)^2}{(z - P_1)(z - P_2)}$$

and is of degree 2. Choose any other D satisfying the conditions of the lemma. The three conditions of Theorem 2.10 are shown to hold for D_1 and D.

Condition 1 holds since we have $\deg 2D = 0 \geq 2g$.

For condition 2, $\deg nD \geq -1$ for all $n \geq 1$, and for condition 3 we see that the generator corresponds to a convergent.

The lemma now follows using Theorem 2.10 and 2.12 while keeping in mind that there is only one generator of degree 2, no relations of degree 4 or of degree $p_{i,2} + 2$, $i = 1, 2$. □

As mentioned in the introduction, the theorems of Chapter 2 can be used to find $G_D(t)$ and $R_D(t)$ for all rings of automorphic forms and in many cases the theorems are applied to Wagreich's results on automorphic forms with three and four generators. The following example uses one of Wagreich's results and illustrates the process of finding D_1 given D.

LEMMA 3.5. *Suppose* D *is a divisor on a Riemann surface of genus* 1, $D = (\beta/\alpha)P$, *where* $\beta/\alpha \geq 4/5$. *The entries for* $G_D(t)$ *and* $R_D(t)$ *are as in the table.*

Let α/β have convergents p_j/q_j, $j = 1, \ldots, k_i$. By Fact 10 of the appendix for $1 \leq j \leq 4$ we have $p_j/q_j = (j+1)/j$.

We would like to find a divisor D_l, $D_l < D$, such that $\mathscr{L}(D_l)$ is understood and D_l together with D fulfills the hypotheses of Theorem 2.10.

As $A(G)$ with signature $(1; 0; e_i)$ is isomorphic to $\mathscr{L}(D_i)$ where D_i is a divisor on a Riemann surface of genus 1 such that

$$D_i = \frac{e_i - 1}{e_i}$$

we consider the signatures $(1; 0; e_i)$, $2 \leq e_i \leq 5$ (see Table 1). The ring associated with $(1; 0; 2)$ is $\mathscr{L}(D_2)$ where $D_2 = (q_1/p_1)P$ but $\mathscr{L}(D_2)$ does not fulfill condition 3 of Theorem 2.10 as the generators do not correspond to convergents and are of degree higher than p_2. Likewise for $(1; 0; 3)$ we have the ring $\mathscr{L}(D_3)$ where $D_3 = (q_2/p_2)P$ but again $\mathscr{L}(D_3)$ does not fulfill condition 3 of Theorem 2.10. In the case of $(1; 0; 4)$ however, one has $D_4 = (q_3/p_3)P$ where $\mathscr{L}(D_4)$ has generators of degrees p_0, p_2, p_3. By Lemma 2.9 one knows these generators are rational functions

$$x_j \in L(q_j P) - L((q_j D - P).$$

The hypotheses of Theorem 2.10 are easily seen to be satisfied and therefore

$$G_D(t) = t + \sum_{i=2}^{k} t^{p_i}.$$

Using only Theorem 2.12 one can not completely determine $R_D(t)$ however. We know $R_{D_4}(t) = t^9$ and $R_D(t)$ will necessarily have terms $\sum_{i=2}^{k-2}\sum_{j=2}^{k-i} t^{p_i + p_{i+j}} + \sum_{i=4}^{k} t^{1+p_i}$ (these correspond to elements $(X_i X_{i+s} - f^{X_i X_{i+s}})$ $i \geq 2$, $s \geq 2$ or $i = 0$, $s \geq 4$), but one cannot tell whether or not $R_D(t)$ will contain the t^9 term. Consider the monomials in $L(9D_4)$:

$$(*) \qquad \begin{aligned} &x_0 x_3^2 \in L(6P) - L(5P), \quad x_2^3 \in L(6P) - L(5P), \quad x_0^9, \\ &x_0^6 x_2 \in L(2P) - L(P), \\ &x_0^5 x_3 \in L(3P) - L(2P), \quad x_0^3 x_2^2 \in L(4P) - L(3P), \\ &x_0^2 x_2 x_3 \in L(5P) - L(4P). \end{aligned}$$

By Lemma 2.8, the relation for $\mathscr{L}(D_4)$ is necessarily of the form

$$r = x_0 x_3^2 - c_0 x_2^3 - c_1 x_0^9 - c_2 x_0^6 x_2 - c_3 x_0^5 x_3 - c_4 x_0^3 x_2^2 - c_5 x_0^2 x_2 x_3, \quad c_0 \neq 0.$$

Next, consider $\mathscr{L}(D_5)$ where $D_5 = 4/5P = (q_4/p_4)P$. As could be predicted using Theorem 2.10, $G_{D_5}(t) = t + t^3 + t^4 + t^5$. But now $R_{D_5}(t) = t^6 + t^8 = t^{p_4 + p_0} + t^{p_4 + p_2}$ and the t^9 term does not appear. This implies

$$r \in \langle X_0 X_4 - f^{X_0 X_4}, X_2 X_4 - f^{X_2 X_4} \rangle.$$

It can be seen that

$$X_2\left(X_0 X_4 - f^{X_0 X_4}\right) - X_0\left(X_2 X_4 - f^{X_2 X_4}\right) = cr$$

as

$$X_0X_4 - f^{X_0X_4} = X_0X_4 - b_1X_2^2 - b_2X_3X_0^2 - b_3X_2X_0^3 - b_4X_0^6,$$
$$X_2X_4 - f^{X_2X_4} = X_2X_4 - d_1X_3^2 - d_2X_3X_2X_0 - d_3X_2^2X_0^2$$
$$-d_4X_3X_0^4 - d_5X_2X_0^5 - d_6X_0^8,$$

and Lemma 2.8 implies $b_1 \neq 0$, $d_1 \neq 0$. Therefore

$$\frac{1}{d_1}\left[X_0\left(X_2X_4 - f^{X_2X_4}\right) - X_2\left(X_0X_4 - f^{X_0X_4}\right)\right] - r$$
$$= a_1X_2^3 - a_2X_0^9 - a_3X_0^6X_2 - a_4X_0^5X_3 - a_5X_0^3X_2^2 - a_6X_0^2X_2X_3.$$

Using (*) and Lemma 2.8 we get $a_i = 0$ for all i. By Lemma 2.14,

$$R_D(t) = \sum_{i=2}^{k-2}\sum_{j=2}^{k-i} t^{p_i+p_{i+j}} + \sum_{i=4}^{k} t^{1+p_i}.$$

If $3/4 < \beta/\alpha < 4/5$ then the degree 9 relation will be necessary as a generator in the ideal of relations as one can see, from Facts 10, 11 and Lemma A.2 of the appendix, that all new additional generators necessary for $\mathscr{L}(D)$ will be of degree > 8. □

In finding the degrees of generators and relations for all rings of automorphic forms the case of $g = 0$ is the most complicated. Here if $A(G)$ has signature $(0; s; e_1, e_2, \dots, e_r)$, $A(G) \simeq \mathscr{L}(D)$ where D is a divisor on a Riemann surface of genus 0 and

$$D = -2P + sP + \sum_{i=1}^{r} \frac{e_i - 1}{e_i} P_i.$$

If $s \geq 2$, Theorem 2.10 and 2.12 can be applied and the results are listed in Table 2. The case $s = 1$, $r = 2$ is taken care of by Lemma 3.4. Rings with signatures $s = 0$, $r \leq 5$ can be understood by applying Theorem 2.10 and 2.12 to Wagreich's result on rings of automorphic forms with few generators. The rest of the cases, i.e., $s = 1$, $r > 2$ and $s = 0$, $r \geq 6$, will be taken care of by Lemma 3.8. For the purpose of understanding $A(G)$ one is only interested in

$$D = -nP + \sum_{i=1}^{k} \frac{e_i - 1}{e_i} P_i$$

where $n = 1$ or 2 and $k \geq 3n$ but the more general case of

$$D = -rP + \sum_{i=1}^{k} \frac{\beta_i}{\alpha_i} P_i, \quad k \geq 3r, \ \frac{\beta_i}{\alpha_i} \geq \frac{1}{2},$$

is just as easily studied.

We first consider the divisor $D_1 = -rP + \sum_{i=1}^{k} 1/2 P_i$, $k \geq 3r$, and then apply Theorems 2.10 and 2.12 to understand the general case.

LEMMA 3.6. *Let D_1 be a divisor on a Riemann surface of genus* 0 *such that* $D_1 = -rP + \Sigma_{i=1}^{k} 1/2P_i$, $k \geq 3r$, $P \neq P_i$. *One can make a specific choice of generators to show that for* $\mathscr{L}(D_1)$:

(1) $G_{D_1}(t) = (k-2r+1)t^2 + (k-3r+1)t^3$.

(2) $R_{D_1}(t) = \binom{k-2r}{2}t^4 + (k-2r)(k+3r)t^5 + \binom{k-3r+2}{2}t^6$.

(3) $P_{D_1}(t) = \dfrac{(k-2r)(1+t^3)}{(1-t^2)^2} - \dfrac{(k-2r-1)(1+t^3)}{(1-t^2)}$.

(4) *A basis for* $L(tD_1)$, *t even, is*

$$B_{te} = \left\{x_{2r}^{t/2-k_1}x_{2r+l}^{k_1}, x_{2r}^{t/2}\right\},\ k_1 = 1,\dots,t/2,\ l = 1,\dots,k-2r$$

Here x_{2r+m} *is a generator of degree* 2 *and*

$$x_{2r+m} = \frac{(z-P)^{2r}}{\left[\prod_{i=1}^{2r-1}(z-P_i)\right](z-P_{2r+m})},\quad m = 0,\dots,k-2r.$$

A basis for $L(tD_1)$, *t odd, is*

$$B_{t0} = \left\{x_{2r}^{(t-3)/2-k_1}x_{2r+l}^{k_1}y_{3r}, x_{2r}^{(t-3)/2-k_2}x_{3r+s}^{k_2}, y_{3r+s}, x_{2r}^{(t-3)/2}y_{3r}\right\}$$

where $1 \leq k_1 \leq (t-3)/2, 1 \leq l \leq r, 0 \leq k_2 \leq (t-3)/2$ *and* $1 \leq s \leq k-3r$. *Here*

$$y_{3r+m} = \frac{(z-P)^{3r}}{\left[\prod_{i=1}^{3r-1}(z-P_i)\right](z-P_{3r+m})},\quad m = 0,1,\dots,k-3r,$$

and y_{3r+m} *is a generator of degree* 3.

(5) *The relations can be chosen as follows.*

(a) *For degree* 4,

$$X_{2r+k_1}X_{2r+s} - c_1X_{2r}X_{2r+k_1} - c_2X_{2r}X_{2r+s},$$
$$1 \leq k_1 \leq k-2r-1,\ k_1 < s \leq k-2r,$$
$$c_2 = \frac{P_{2r}-P_{2r+s}}{P_{2r+k_1}-P_{2r+s}},\quad c_1 = 1-c_2$$

(b) *For degree* 5,

$$X_{2r+k_1}Y_{3r+s} - c'_{k,s}X_{2r}Y_{3r+s} - c''_{k,s}X_lY_m - c'''_{k,s}X_{2r}Y_{3r}$$

where $l = 2r$, $m = 2r+k_1$ *if* $k_1 \geq r$ *and* $l = 2r+k_1$, $m = 3r$ *if* $k_1 < r$. *Here* $1 \leq k_1 \leq k-2r$; $0 \leq s \leq k-3r$ *where* $s \neq 0$ *if* $k_1 \leq r$ *and* $k_1 - r - s \neq 0$ *if* $k_1 > r$.

(c) *For degree* 6,

$$Y_{3r+s}Y_{3r+t} - \sum_{m=1}^{2}\sum_{l=1}^{r-1} c_{ml}^{st} X_{2r}^{3-m} X_{2r+l}^{m} - c_{m}^{st} X_{2r}^{2} X_{3r+s} - c_{0m}^{st} X_{2r}^{2} X_{3r+t}$$

where $0 \leq s \leq k - 3r$ *and* $s \leq t \leq k - 3r$.

We leave the proof to the reader.

LEMMA 3.8. *Let D be a divisor on a Riemann surface of genus* 0 *such that*

$$D = -rP + \sum_{i=1}^{k} \frac{\beta_i}{\alpha_i} P_i, \quad k \geq 3r, \ \frac{\beta_i}{\alpha_i} \geq \frac{1}{2}.$$

Then $G_D(t)$, $P_D(t)$ and $R_D(t)$ are as given in Table 2.

Proof. Lemma 3.6 proves this lemma for $D_1 = -rP + \Sigma_{i=1}^{k} 1/2P_i$, $k \geq 3r$. Let D be any other divisor satisfying the conditions of the lemma. The conditions of Theorem 2.10 are shown to be satisfied. For conditions 1 and 2, $L(2D)$, $L(3D)$ nontrivial implies $L(nD) \supset L(2D) \otimes L((n-2)D)$ is nontrivial for all $n \geq 4$ which implies $\deg nD > 0$ for all $n \geq 2$. Condition 3 follows from the fact that all new convergents are of degree 3 or more. Theorem 2.12 and Corollary 2.11 finish the proof. □

Finally, we show that if D_1 and D fulfill the hypothesis of Theorem 2.10 where $D_1 < D$, $\mathscr{L}(D)$ is rarely isomorphic to the coordinate ring of a complete intersection. Let I and I_1 be the ideal of relations corresponding to $\mathscr{L}(D)$ and $\mathscr{L}(D_1)$ respectively. Let $Z(I)$ be the affine variety with coordinate ring isomorphic to $\mathscr{L}(D)$. Suppose D_1 and D fulfill the hypotheses of Theorem 2.10 where $D_1 < D$.

LEMMA 3.9 (1). *If $G_D(1) \geq G_{D_1}(1) + 2$ and $G_{D_1}(1) \geq 2$ then $Z(I)$ is not a complete intersection.*

(2) *Suppose $G_D(1) - G_{D_1}(1) = 1$ and call the new generator $x_{i,j}$. Then $Z(I)$ is a complete intersection only if $I_{D_1} \subset \langle QM - f^{QM} | QM$ is a quadratic monomial such that $x_{i,j} | QM \rangle$.*

Proof. (1) Given $G_D(1) - G_{D_1}(1) = k$, Theorem 2.12 and Corollary 2.14A imply

$$R_D(1) \geq G_{D_1}(1) - 1 + G_{D_1}(1) + \cdots + G_{D_1}(1) + k - 2.$$

Since $G_D(1) = G_{D_1}(1) + k$, $Z(I)$ is a complete intersection only if

$$R_D(1) = G_{D_1}(1) + k - 2.$$

Now if $k \geq 2$ and $G_{D_1}(1) \geq 2$ we get $R_D(1) \geq 1 + G_{D_1}(1) + k - 2$.

(2) Since $G_D(1) = G_{D_1}(1) + 1$, $Z(I)$ is a complete intersection if and only if $R_D(1) = G_{D_1}(1) - 1$. With the new generator $x_{i,j}$ one gets $G_{D_1}(1) - 1$ nontrivial relations $QM - F^{QM}$ where QM is a quadratic monomial such that $x_{i,j} | QM$. By Corollary 2.14A these are necessary as generators for I. It follows that $Z(I)$ is a complete intersection only if these relations are sufficient to generate I. □

Tables 1 and 2 now follow. The results in Table 2 have all been obtained by applying the theorems of Chapter 2 to the established results listed in Table 1. In Table 1 the entries in 1-3 are due to Mumford (see [4]) and Saint-Donat (see [7] and [8]) while the rest of the entries can be found in Wagreich's papers [10] and [11]. The number of relations of degrees 3 is not determined for the divisors given in 2 and 3. In Table 2 we consider divisors of the form

$$D = D_0 + \sum_{i=1}^{k} \frac{\beta_i}{\alpha_i} P_i$$

where $0 < \beta_i/\alpha_i < 1$ and either $\deg D_0 \leq 0$ or D_0 is one of the divisors considered in Table 1. Recall that in Theorem 3.1 it was shown that for the divisor

$$D^* = D_0 + \sum_{i=1}^{k} \frac{\beta_i}{\alpha_i} P_i \quad \text{where } \deg D_0 \geq 2g + 2 \text{ and } 0 < \frac{\beta_i}{\alpha_i} < 1,$$

we have the following results. We let

$$\varphi_{\alpha_i, \beta_i} = \sum_{j=1}^{k_i} t^{p_{ij}}$$

and let $S = \{(i, j, i', j')\colon 1 \leq i \leq i' \leq k, 1 \leq j \leq k_i, 1 \leq j' \leq k_{i'}$, and finally $j - j' \geq 2$ if $i = i'\}$. It was then shown that

$$G_{D^*}(t) = G_{D_0}(t) + \sum_{i=1}^{k} \varphi_{\alpha_i, \beta_i},$$

$$R_{D^*}(t) = R_{D_0}(t) + \sum_{i=1}^{k} \left(G_{D_0}(t)\varphi_{\alpha_i, \beta_i} - t^{p_{i1}+1} \right) + \sum_{(i, j, i', j') \in S} t^{p_{ij}+p_{i'j'}},$$

$$G_{D^*}(1) = G_{D_0}(1) + \sum_{i=1}^{k} k_i,$$

$$R_{D^*}(1) = R_{D_0}(1) + \frac{(\Sigma k_i)\left(2G_{D_0}(1) + \sum_{i=1}^{k} k_i - 3\right)}{2}$$

$$P_{D^*}(t) = P_{D_0}(t) + \sum_{i=1}^{k} \sum_{j=0}^{k_i} \left(\frac{1}{(1 - t^{p_{ij}})(1 - t^{p_{ij+1}})} - \frac{1}{(1 - t^{p_{ij}})} \right)$$

Table 1

$D = D_0 + \frac{\beta}{\alpha} P_1$ $D_0 = \Sigma n_p P,\ n_p \in Z$ Extra Conditions on D	$G_D(t)$	$R_D(t)$	$P_D(t)$
1. $\deg D_0 \geq 2g+2,\ \frac{\beta}{\alpha} = 0$	$(\deg D_0 + 1 - g)t$	$\frac{(G_{D_0}(t))^2 - tG_{D_0}(t) - 2t^2 deg D_0}{2}$	$\frac{t \deg D_0}{(1-t)^2} + \frac{1-gt}{(1-t)}$
2. $\deg D_0 = 2g+1,\ \frac{\beta}{\alpha} = 0$	$(g+2)t$	$\frac{g(g-1)}{2} t^2 + dt^3$	$\frac{t \deg D_0}{(1-t)^2} + \frac{1-gt}{(1-t)}$
3. $D_0 = K,\ g > 3,\ \frac{\beta}{\alpha} = 0$, non-hyperelliptic curve. ($d \neq 0$ implies curve is non-singular plane quintic or a triagonal covering of P^1)	gt	$\frac{(g-3)(g-2)}{2} t^2 + dt^3$	$\frac{t(2g-2)}{(1-t)^2} + \frac{1-gt+t-t^2}{(1-t)}$
4. $D_0 = 0,\ g = 1,\ \frac{\beta}{\alpha} = \frac{1}{2}$	$t + t^4 + t^6$	t^{12}	$\frac{1-t^{12}}{(1-t)(1-t^4)(1-t^6)}$
5. $D_0 = 0,\ g = 1,\ \frac{\beta}{\alpha} = \frac{2}{3}$	$t + t^3 + t^5$	t^{10}	$\frac{1-t^{10}}{(1-t)(1-t^3)(1-t^5)}$
6. $D_0 = 0,\ g = 1,\ \frac{\beta}{\alpha} = \frac{3}{4}$	$t + t^3 + t^4$	t^9	$\frac{1-t^9}{(1-t)(1-t^3)(1-t^4)}$
7. $D_0 = 0,\ g = 1,\ \frac{\beta}{\alpha} = \frac{4}{5}$	$t + t^3 + t^4 + t^5$	$t^6 + t^8$	$\frac{1-t^9}{(1-t)(1-t^3)(1-t^4)} + \frac{t^5}{(1-t^4)(1-t^5)}$

Table 2

$D = D_0 + \sum_{i=1}^{k} \frac{\beta_i}{\alpha_i}, 0 < \frac{\beta_i}{\alpha_i} < 1$ Extra Conditions on D	$f(t)$	$g(t)$	$h(t)$	$f(1)$	$g(1)$
1. $\deg D_0 \geq 2g + 1$					
2. $D_0 = K$ non-hyperelliptic curve $g > 3$					
3. $D_0 = 0$ $g = 0$					
4. $g = 1$, $D = \frac{\beta_1}{\alpha_1} P$, $\frac{\beta_1}{\alpha_1} \geq \frac{3}{4}$, For δ_{ij}, $i = j$ if and only if $\frac{3}{4} \leq \frac{\beta_1}{\alpha_1} < \frac{4}{5}$	$-t^2$	$\delta_{ij} t^9 - \varphi_{\alpha_1 \beta_1} t^2$	$\frac{-t^2}{(1-t)}$	-1	$\delta_{ij} - k_1$
5. $g = 0$, $D = -P + \frac{\beta_1}{\alpha_1} P_1 + \frac{\beta_2}{\alpha_2} P_2$ $\frac{\beta_i}{\alpha_i} \geq \frac{1}{2}$ for all i	$-t^2$	$2t^3 + t^4 - \sum_{i=1}^{2} t^2 \varphi_{\alpha_i, \beta_i}$	$1 + \frac{-t^2}{(1-t)^2}$	-1	$3 - k_1 - k_2$
6. $g = 0$, $D = -rP + \sum_{i=1}^{k} \frac{\beta_i}{\alpha_i} P_i$ $k \geq 3r$, $\frac{\beta_i}{\alpha_i} \geq \frac{1}{2}$ for all i $r \in Z^+$	$-(2r-1)t^2$ $+(k-3r+1)t^3$	$[-(2r-1)t^2$ $+(k-3r+1)t^3]$ $\left(\sum_{i=1}^{k} \varphi_{\alpha_i, \beta_i}\right)$ $+(2r^2+r-k)t^4$ $+(6r^2-2rk-3r)t^5$ $+\left(\frac{k-3r+2}{2}\right)t^6$ $+kt^3$	$1 + \frac{(-2r+1)t^2}{(1-t)^2}$ $+\frac{(r-1)t^3}{(1-t)^2}$	$k - 5r + 2$	$(k-5r+2)\sum_{i=1}^{k} k_i$ $+\frac{(k-5r)^2 - 13r + k + 2}{2} + 2$

For each divisor D in Table 2 we give the polynomials $f(t)$ and $g(t)$ such that $G_D(t) = G_{D^*}(t) + f(t)$ and $R_D(t) = R_{D^*}(t) + g(t)$. The rational function $h(t)$, where $P_D(t) = P_{D^*}(t) + h(t)$ is also provided and the numbers $f(1)$ and $g(1)$ are computed. In the case

$$(*) \qquad g = 0, \quad D = D_0 + \sum \frac{\beta_i}{\alpha_i} P_i \text{ where } \deg D_0 \geq 0 \text{ or}$$

$$\deg D_0 = -1 \text{ and } \frac{1}{2} \leq \frac{\beta_i}{\alpha_i} \leq 1,$$

Henry Pinkham has shown that $\mathscr{L}(D)$ is isomorphic to the coordinate ring of an affine surface with a single isolated rational singularity at zero. Jonathan Wahl has shown that this implies $R_D(1) = 1/2(k-1)(k-2)$ whenever $G_D(1) = k$. A simple computation shows that for all the divisors listed in Table 2 which fulfill the conditions of $(*)$ the entries satisfy Wahl's result.

Appendix

The Appendix gives all the facts about the convergents of the simple continued fraction α/β which are used in the paper. Given $0 < \beta/\alpha < 1$, consider the decomposition

$$\frac{\alpha}{\beta} = a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \cfrac{}{\ddots - \cfrac{1}{a_k}}}} = [a_1, \ldots, a_k].$$

Let $p_0 = 1$, $q_0 = 0$ and let $p_i/q_i = [a_1, \ldots, a_i]$ be the convergents of the above decomposition. By induction one can prove the following two properties.

Fact 1. $p_i + p_{i+2} = a_{i+2} p_{i+1}$, $q_i + q_{i+2} = a_{i+2} q_{i+1}$.

Fact 2. $p_i q_{i+1} - p_{i+1} q_i = 1$.

(See [2] for the standard proofs for 1 and 2). Adding and subtracting multiples of various p_j's in 1, one can see:

Fact 3. $p_i + p_{i+3} = (a_{i+2} - 1)p_{i+1} + (a_{i+3} - 1)p_{i+2}$.

Fact 4.

$$\begin{aligned} p_i + p_{i+k'} = {} & (a_{i+2} - 1)p_{i+1} + (a_{i+3} - 2)p_{i+2} + (a_{i+4} - 2)p_{i+3} \\ & + \cdots + (a_{i+(k'-1)} - 2)p_{i+k'-2} + (a_{i+k'} - 1)p_{i+k'-1} \end{aligned}$$

This holds for all k' such that $3 < k' \leq k - i$. (The same property holds for the q_i's.)

From these the following can be deduced.

Fact 5. $p_i/q_i > p_{i+j}/q_{i+j}$ for $j \geq 1$ so $p_i q_{i+j} - p_{i+j} q_i > 0$ for $j \geq 1$. (This follows from fact 2).

Fact 6. Given $\Sigma_i b_i p_i$ and $\Sigma_i b_i q_i$, $b_i \geq 0$, $b_i \in Z$, there exists $j, m, n \in Z^+ \cup 0$ such that

$$\sum_{i=l}^{s} b_i p_i = m p_j + n p_{j+1}$$

and

$$\sum_{i=l}^{s} b_i q_i = m q_j + n q_{j+1}.$$

Proof. We use induction on $s - l$. If $s - l = 1$, there is nothing to show. Suppose the fact is true whenever $s - l < t$ and $s - l = t$. Without loss of generality say $\min(b_l, b_s)$ is b_l. Then

$$\sum_{i=l}^{s} b_i p_i = b_l p_l + b_l p_s + \sum_{i=l+1}^{s-1} b_i p_i + (b_s - b_l) p_s$$

Now, using Fact 1, 3 or 4, replace the first two terms on the right hand side by

$$b_l \sum_{i=+1}^{s-1} c_i p_i$$

and then apply the induction hypothesis to the new right hand side. The proof is identical for the q_i's.

Fact 7. For $n < p_k$, $[nq_k/p_k] = [nq_{k-1}/p_{k-1}]$.

Proof. Suppose there exists an integer s such that

$$n\frac{q_{k-1}}{p_{k-1}} < s \leq n\frac{q_k}{p_k}.$$

Then $nq_{k-1} < sp_{k-1} \leq nq_{k-1} + n/p_k$ as

$$\frac{q_k}{p_k} = \frac{q_{k-1}}{p_{k-1}} + \frac{1}{p_k p_{k-1}}$$

follows from Fact 2. Since $sp_{k-1} \in Z$ and is larger than nq_{k-1}, $sp_{k-1} \geq nq_{k-1} + 1$ which implies $n \geq p_k$, a contradiction.

Fact 8. If $mp_i + np_{i+1} = t$ where $m, n \in Z^+ \cup 0$ then $mq_i + nq_{i+1} \leq t\beta/\alpha$.

Proof. Suppose not. Then

$$mp_i q_k + np_{i+1} q_k = tq_k \tag{1}$$

and

$$mq_i + nq_{i+1} > t\frac{q_k}{p_k}. \tag{2}$$

Multiplying by p_k in (2) and then subtracting (1) from (2) we get

$$m(p_k q_i - p_i q_k) + n(p_k q_{i+1} - p_{i+1} q_k) > 0.$$

This contradicts Fact 5.

Using the above one can show the following theorem.

THEOREM A.1. *Given any fraction β/α, $0 < \beta/\alpha < 1$, and any positive integer j, consider j and $[j\beta/\alpha]$. Choose any k' in the set $\{0, 1, 2, \ldots [j\beta/\alpha]\}$. Then there exists non-negative integers m, n, i such that*

$$mq_i + nq_{i+1} = k' \quad \textit{and} \quad mp_i + np_{i+1} = j$$

where p_i and q_i are as above. Furthermore, whenever $mn \neq 0$ the integers m, n, i are unique.

Proof. The existence is shown by induction on the length of the decomposition of α/β. If the length is 1 then $\beta/\alpha = 1/s$, $s \in Z^+$. We have $p_0 = 1$, $q_0 = 0$, $p_1 = s$, $q_1 = 1$. Choose any $j \in Z^+$ and $l \in \{0, 1, \ldots [j1/s]\}$. Now $l \leq j/s$ so $ls \leq j$ and one can write $j = nls + r$, $0 \leq r < ls$, $n \geq 1$. It follows that

$$((n-1)ls + r)q_0 + lq_1 = l$$

and

$$((n-1)ls + r)p_0 + lp_1 = j$$

Suppose existence holds whenever the decomposition is of length $k - 1$ and α/β has decomposition of length k. It should be noted that $\beta/\alpha = q_k/p_k$ and has the same first $k - 1$ convergents as q_{k-1}/p_{k-1}. Choose the smallest j for which one can find no proper m, n, and i. If $j < p_k$, $[jq_k/p_k] = [jq_{k-1}/p_{k-1}]$ by Fact 7. In these cases there exists m, n, i such that

$$mp_i + np_{i+1} = j \quad \text{and} \quad mq_i + nq_{i+1} = l$$

since by the induction hypothesis existence holds for q_{k-1}/p_{k-1}. Therefore we can assume $j \geq p_k$. If $l \neq [j\beta/\alpha]$ then $l \in \{0, 1, \ldots [(j-1)\beta/\alpha]\}$ so there exists m, n, i such that

$$mp_i + np_{i+1} = j - 1, \quad mq_i + nq_{i+1} = l.$$

But then

$$p_0 + mp_i + np_{i+1} = j, \qquad q_0 + mq_i + nq_{i+1} = l.$$

Using Fact 6 one can get the desired integers. Therefore assume $l = [j\beta/\alpha]$.

Now $j = n'p_k + r$, $n' \in Z^+$, $0 \leq r < p_k$, and $[jq_k/p_k] = n'q_k + [rq_k/p_k]$. But we know there exists m, n, i such that

$$mp_i + np_{i+1} = r, \quad mq_i + nq_{i+1} = [rq_k/p_k].$$

Then

$$n'p_k + mp_i + np_{i+1} = j, \quad n'q_k + mq_i + nq_{i+1} = [jq_k/p_k]$$

and again use Fact 6 to get the desired integers. Therefore there exists no smallest j and we can always find integers to satisfy the theorem.

We now show uniqueness.

Suppose there exists m', n', i' and $m, n, i \in Z^+ \cup \{0\}$ such that

$$mq_i + nq_{i+1} = J, \quad mp_i + np_{i+1} = K$$

and

$$m'q_{i'} + n'q_{i'+1} = J, \quad m'p_{i'} + n'p_{i'+1} = K.$$

Without loss of generality assume $i' \geq i$ and suppose $J > 0$ and $J, K \in Z^+$. Now

$$\frac{mq_i + nq_{i+1}}{mp_i + np_{i+1}} = \frac{m'q_{i'} + n'q_{i'+1}}{m'p_{i'} + n'p_{i'+1}}$$

Cross multiply and subtract the left hand side from the right hand side. We have

$$(*)\quad mm'(p_iq_{i'} - p_{i'}q_i) + m'n(p_{i+1}q_{i'} - q_{i+1}p_{i'}) + mn'(p_{i'}q_{i'+1} - p_{i'+1}q_i) + nn'(p_{i+1}q_{i'+1} - p_{i'+1}q_{i+1}) = 0$$

If $i' > i + 1$ and $J > 0$ we claim that the left hand side of $(*)$ is strictly positive. Suppose not. By Fact 5 the expressions $p_jq_k - p_kq_j$ are all positive so we must have $mm' = m'n = mn' = nn' = 0$. One gets $m = n = 0$ or $m' = n' = 0$. This is impossible if $J > 0$. The claim then holds and $\Rightarrow i = i'$ or $i + 1 = i'$. If $i = i'$ one gets

$$(3)\qquad (m - m')q_i(n - n')q_{i+1} = 0,$$

$$(4)\qquad (m - m')p_i + (n - n')p_{i+1} = 0.$$

Multiply (3) by p_i and (4) by q_i, subtract (4) from (3) to get $n = n'$. Then $m = m'$ follows. If $i' = i + 1$ then

$$mm' + nn' + mn'(p_iq_{i+2} - p_{i+2}q_i) = 0$$

implies $m = n' = 0$ if $J > 0$. Note that the expression for J and K is still unique in this case. The labeling is just different. One can write

$$0p_i + np_{i+1} = J, \quad 0q_i + nq_{i+1} = K$$

or

$$m'p_{i+1} + 0p_{i+2} = J, \quad m'q_{i+1} + 0q_{i+2} = K.$$

It is clear that $m' = n$.

Fact 10. If $k/(k + 1) \leq \beta/\alpha < 1$, $k \in Z^+$, then

$$\frac{\alpha}{\beta} = [2, \ldots, 2, a_{k+1}, \ldots, a_n]$$

and has first k convergents $p_j/q_j = (j + 1)/j$, $j = 1, 2, \ldots, k$.

Proof. Let $k = 1$. If $1/2 \leq \beta/\alpha < 1$ then $1 < \alpha/\beta \leq 2$ so

$$\frac{\alpha}{\beta} = [2, a_2, \ldots, a_n]$$

Suppose the fact holds for $k \leq t - 1$ and $k = t$ where $t > 1$. Now $t/(t+1) \leq \beta/\alpha < 1$ implies $1 < \alpha/\beta \leq (t+1)/t$. We have

$$\frac{\alpha}{\beta} = 2 - \frac{1}{\dfrac{\beta}{2\beta - \alpha}} \quad \text{and} \quad \frac{2\beta - \alpha}{\beta} \geq 2 - \frac{t+1}{t} = \frac{t-1}{t}$$

By the induction hypothesis, $\beta/(2\beta - \alpha) = [2, \dots, 2, a_t, \dots]$. The fact now follows.

Fact 11. For $0 < \beta_i/\alpha_i < 1$, $p_j < p_{j+1}$, $q_j < q_{j+1}$ for all $j \geq 0$.

Proof. The fact certainly holds for fractions with decomposition of length 1. Suppose it holds whenever the fraction has decomposition of length $k - 1$ and α/β has decomposition of length $k \geq 2$. The fact holds for p_{k-1}/q_{k-1} so we must only show that $p_k > p_{k-1}$, $q_k > q_{k-1}$. We have $\alpha/\beta = [a_1, \dots, a_k]$, $a_i \geq 2$. Now $p_{k-2} + p_k = a_k p_{k-1}$ by Fact 1. So $p_k = a_k p_{k-1} - p_{k-2}$, and $p_k > (a_k - 1)p_{k-1}$ as $p_{k-1} > p_{k-2}$. Therefore $p_k > p_{k-1}$ as $a_k > 1$.
The same property holds for the q_i's.

LEMMA A.2. *Assume* $0 < p_l/q_l < 1$ *and* $0 < p_l'/q_l' < 1$. *Suppose*

$$\frac{p_l}{q_l} = [a_1, \dots, a_l], \; \frac{p_{l-1}}{q_{l-1}} = [a_1, \dots, a_{l-1}],$$

and

$$\frac{p_l'}{q_l'} = [a_1, \dots, a_{l-1}, b].$$

Then

$$\frac{p_l'}{q_l'} = \frac{p_l + np_{l-1}}{q_l + nq_{l-1}} \quad \textit{for} \quad n \in Z$$

where $p_l + np_{l-1} > p_{l-1}$.

Proof. By Fact 2, $p_{l-1}q_l' - q_{l-1}p_l' = 1$ and $p_{l-1}q_l - q_{l-1}p_l = 1$. By elementary number theory, if x_0 and y_0 are a particular integral solution to the equation $p_{l-1}x - q_{l-1}y = 1$ the general integral solution is

$$x = (x_0 + nq_{l-1}), \quad y = y_0 + np_{l-1},$$

where $n \in Z$. By Fact 11, $p_l' > p_{l-1}$. The lemma now follows.

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