ILLINOIS JOURNAL OF MATHEMATICS
Volume 34, Number 3, Fall 1990

BOUNDARY LOCALIZATION OF THE NORMAL FAMILY OF HOLOMORPHIC MAPPINGS AND REMARKS ON EXISTENCE OF BOUNDED HOLOMORPHIC FUNCTIONS ON COMPLEX MANIFOLDS

BY

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1. Basic definitions and statements of results

A modern treatment of the normal family of holomorphic mappings between complex manifolds was given in [6]. We shall recall some of the basic definitions here.

Let M and N be two metric spaces. A subset F of $C(M, N)$, the set of continuous mappings between M and N, is called normal if every sequence of F contains a subsequence which is either relatively compact in $C(M, N)$ or compactly divergent. A sequence $\{f_i\} \subset C(M, N)$ is called compactly divergent if for any compact sets $K \subset M$ and $K' \subset N$ there exists n_0 such that $f_i(K) \cap K' = \emptyset$ for all $i \geq n_0$.

DEFINITION. A complex manifold N is said to be *taut* if for every complex manifold M, the set of all holomorphic mappings from M to N, denoted by $\mathrm{Hol}(M, N)$, is a normal family.

A subset $F \subseteq C(M, N)$ is called an equicontinuous family if for any $\varepsilon > 0$ and any point $x \in M$ there is a neighborhood U of x such that if $x' \in U$, then $d_N(f(x), f(x')) < \varepsilon$ for all $f \in F$. Here N is a metric space equipped with a metric d_N inducing its underlying topology.

DEFINITION. Let N be a complex manifold equipped with a metric d inducing its underlying topology. (N, d) is called a *tight* manifold if for every complex manifold M, $\mathrm{Hol}(M, N)$ is equicontinuous.

An equivalent theory to the normal family of holomorphic mappings concerning intrinsic measures on complex manifolds has been developed by Eisenman, Kobayashi, Royden [1], [3], [5] and others. We recall its definition here for future use.

Received May 19, 1988.
[1]Research partially supported by a grant from the National Science Foundation.

DEFINITION. Let M be a complex manifold of dimension n, $x \in M$, k an integer between 1 and n. The Eisenman differential k-measure on M is a function

$$E_M^k \colon \wedge^k T_x(M) \to R$$

such that for all $(x, v) \in \wedge^k T(M)$,

$$E_M^k(x, v) = \inf\left\{ R^{-2k} \mid \text{there exists a } f \in \operatorname{Hol}(B_k(R), M) \text{ such that } f(0) = x,\ df\left(\left(\frac{\partial}{\partial w_1} \wedge \cdots \wedge \frac{\partial}{\partial w_k}\right)(0)\right) = v \right\},$$

where $B_k(R) = \{w = (w_1, w_2, \dots, w_k) \in C^k \mid |w| < R\}$.

When $k = n$, it associates with the Eisenman-Kobayashi volume form denoted by the same symbol

$$E_M^n = |E_M^n| dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n,$$

where $|E_M^n|$ is a local function on M. Here we identify

$$E_M^n(x, v) = E_M^n(x, v \wedge \bar{v}),$$
$$\bar{v} = \text{complex conjugate of the vector } v \in \wedge^n T_n(M).$$

When $k = 1$, it corresponds to the Kobayashi-Royden differential metric, denoted by $K_M = \sqrt{E_M^1}$. Its integrated form is called the Kobayashi distance function on M, denoted by d_M^K [3], [5]. Both E_M^K and d_M^K are decreasing under holomorphic mappings between complex manifolds. As a consequence, they are invariant under biholomorphisms. If d_M^K is a metric on M, then M is called a hyperbolic manifold. If d_M^K is Cauchy complete, then M is called a completely hyperbolic manifold. Cauchy completeness of d_M^K on a hyperbolic manifold M is equivalent to compact completeness, i.e., for every $r > 0$ the level set

$$M_r = \{y \in M | d_M^K(x, y) < r\}$$

is relatively compact on M, where x is a fixed point in M. It follows from the definitions that taut manifolds are always tight. Tight manifolds are equivalent to hyperbolic manifolds. Completely hyperbolic manifolds are always taut, but the converse is not necessarily true. It is a well-known fact that a tight manifold does not admit any non-trivial holomorphic curve [3].

The differential Caratheodory k-measure C_M^k, $1 \leq k \leq n$, on a complex manifold M is defined as follows:

$$C_M^k \colon \wedge^k T(M) \to R, (x,v) \in \wedge^k T(M),$$

$$C_M^k(x,v) = \sup\left\{ \frac{1}{R^{2k}} \,\middle|\, \text{there exists an } f \in \mathrm{Hol}(M, B_k(R)) \text{ such that } f(x) = 0, df_x(v) = \frac{\partial}{\partial w_1} \wedge \cdots \wedge \frac{\partial}{\partial w_k}(0) \right\},$$

where

$$B_k(R) = \{ w = (w_1, w_2, \ldots, w_k) \in C^k \colon |w| < R \}.$$

C_M^k has the important property of being measure-decreasing under holomorphic mappings. For this reason, C_M^k is invariant under biholomorphisms. When $k = n$, it associates with a volume form

$$C_M^n = |C_M^n| dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n,$$

where $|C_M^n|$ is a local function defined on M. When $k = 1$, it corresponds to the Caratheodory-Reiffen differential metric, denoted by

$$C_M = \sqrt{C_M^1}.$$

The Caratheodory distance function d_M^C is defined as follows: Let $x, y \in M$,

$$d_M^C(x,y) = \sup\{ \delta(\bar{x}, \bar{y}) | \text{ there exists a holomorphic mapping } f \in \mathrm{Hol}(M, B_1), \text{so that } \bar{x} = f(x), \bar{y} = f(y) \}.$$

Here $\delta(\bar{x}, \bar{y})$ denotes the distance between $\bar{x}$ and $\bar{y}$ with respect to the Poincare metric on $B_1 = \{ w \in C^1 | \, |w| < 1 \}$. The Caratheodory distance function is decreasing under holomorphic mappings and invariant under biholomorphisms.

Let N be a complex manifold with d_N^C nontrivial everywhere. Suppose d_N^C is compactly complete, in the sense that for any $r > 0$ and $x \in N$, the level set

$$N_r = \{ y | d_N^C(x,y) \leq r \}$$

is compact in N. It is easy to prove that N is holomorphically convex with respect to the set of bounded holomorphic functions (for the proof of this standard fact, see [3]).

Let D be a domain on a complex manifold M and $p \in \partial D$ a fixed boundary point. A boundary neighborhood $\hat{D}$ of D at p means an open set $\hat{D} = U \cap D$, where U is an open set in M containing p. A boundary point $p \in \partial D$ is called *totally real* if there exists no complex analytic variety containing p of positive dimension lying on ∂D. We remark here that for a relatively compact domain D on a complex manifold which admits a strictly plurisubharmonic function, the set of totally real points on ∂D is nonempty.

A domain D on a complex manifold is said to be admitting a compact quotient if $D/\mathrm{Aut}(D)$ is compact, where $\mathrm{Aut}(D)$ is the group of biholomorphisms of D. One should notice that $D/\mathrm{Aut}(D)$ is compact if D covers a compact complex manifold. When D is either a taut manifold or a relatively compact set of a tight manifold, $\mathrm{Aut}(D)$ is a Lie group [6]. Furthermore, $\mathrm{Aut}(D)$ acts properly on D if D is taut.

DEFINITION. Let D_1 and D_2 be two domains on two complex manifolds respectively. D_1 is said to be *locally biholomorphic* to D_2 at two boundary points $p_1 \in \partial D_1$ and $p_2 \in \partial D_2$ if:

(i) There exist boundary neighborhoods $\hat{D}_1$ of p_1 and $\hat{D}_2$ of p_2 with a biholomorphism $f\colon \hat{D}_1 \to \hat{D}_2$.

(ii) There is a sequence $\{x_i\} \subset \hat{D}_1$ converging to p_1 such that $\{f(x_i) \subset \hat{D}_2\}$ will converge to p_2.

THEOREM 1. *Let D_1 and D_2 be domains on two taut manifolds X_1 and X_2 respectively. Suppose both D_1 and D_2 admit compact quotients and D_1 is locally biholomorphic to D_2 at two totally real boundary points $p_1 \in \partial D_1$, $p_2 \in \partial D_2$. Then D_1 is biholomorphic to D_2.*

The second part of this paper is concerning an open problem to characterize those Stein manifolds with nontrivial bounded holomorphic functions. The following problem is of interest to both Kähler geometry and several complex variables.

Main Problem. Let M be a compact Kähler manifold with negative sectional curvature; does its universal cover admit a non-trivial bounded holomorphic function?

We have the following observations.

THEOREM 2. *Let $D \subset M$ be a domain admitting a compact quotient on a taut manifold M. Suppose there is a boundary neighborhood $\hat{D} = U \cap D$ of a totally real boundary point $p \in \partial D$ satisfying one of the following local conditions:*

(1) $C_{\hat{D}}/K_{\hat{D}} \geq c^2 > 0$ *on $\hat{D}$ for some constant $c^2 > 0$;*

(2) $C^n_{\hat{D}}/E^n_{\hat{D}} \geq d^2 > 0$ *on $\hat{D}$ for some constant $d^2 > 0$.*

Then D admits a lot of bounded holomorphic functions which give local coordinate functions at each point of D.

THEOREM 3. *Let D be a domain admitting a compact quotient on a taut manifold M. Suppose there exists a boundary neighborhood $\hat{D} = V \cap D$ of a totally real boundary point $p \in \partial D$ with the condition*

$$\frac{d^C_{\hat{D}}(x, y)}{d^K_{\hat{D}}(x, y)} \geq e^2 \geq 0,$$

for all distinct points x, y in D, where e is a constant. Then D is a holomorphically convex with respect to the set of all bounded holomorphic functions.

It is hoped that the localization procedure developed here would be helpful to solve our main problem stated above.

2. Remarks on the normal family of holomorphic mappings, biholomorphic group actions and proof of theorem 1

LEMMA 2.1. *Let D be a domain on a taut manifold such that $D/\mathrm{Aut}(D)$ is compact. Then there exists a compact subset K on D with the property that for every $y \in D$ there is $x \in K$ and $g \in \mathrm{Aut}(D)$ such that $g(x) = y$ (i.e., $\mathrm{Aut}(D) \cdot K = D$; K is sometimes called a fundamental domain of $\mathrm{Aut}(D)$).*

Proof. One can always exhaust D by a sequence of relatively compact open set $\{D_i\}_{i=1}^{\infty}$ such that $D_i \subset\subset D_{i+1}$ and $\bigcup_{i=1}^{\infty} D_i = D$. Let

$$\pi: D \to D/\mathrm{Aut}(D)$$

be the canonical projection.

π is an open map and $D/\mathrm{Aut}(D)$ is compact. Hence there is a positive integer m such that

$$\pi(D_m) = D/\mathrm{Aut}(D).$$

We can take K to be the closure of D_m.

LEMMA 2.2. *Let X and Y be complex manifolds and let Y be taut. Suppose:*

(1) f_i: $X_i \to Y$ is holomorphic and $\{X_i\}$ is an increasing family of open sets in X with $X_i \subset\subset X_{i+1}$ and $X = \bigcup_{i=1}^{\infty} X_i$

(2) There exist compact subsets $K \subset X$ and $L \subset Y$ such that $f_i(K) \cap L \neq \emptyset$ for all sufficiently large i.

Then there is a subsequence of $\{f_i\}$ converging uniformly on compact sets to a holomorphic mapping f: $X \to Y$.

Proof. Consider only those j so large that $K \subset X_j$. Fix such a j, then a subsequence of $\{f_i\}$ converges uniformly on compact sets on X_j to a holomorphic map $X_j \to Y$ because of the tautness of Y and the assumption that $f_i(K) \cap L \neq \emptyset$ for sufficiently large i. Now let $j \to \infty$. By passing to subsequences repeatedly and by the usual diagonal process, we arrive at the desired holomorphic map f: $X \to Y$.

LEMMA 2.3. *Let D be a domain on a taut manifold with $D/\mathrm{Aut}(D)$ compact. Then D is taut.*

Remark. We shall actually prove that D is completely hyperbolic.

Proof. By Lemma 2.1, there is a compact set $K \subset D$ such that for any point $x \in D$ there exist $y \in K$ and $g \in \mathrm{Aut}(D)$ with $g(y) = x$. Let $\varepsilon > 0$ be a sufficiently small number such that

$$L = \{z \in D | d_D^K(w, z) \leq \varepsilon, w \in K\}$$

is a compact set. Let $\{x_i\}$ be a Cauchy sequence in D with respect to d_D^K. We can find a positive integer m such that for all $i \geq m$, $d_D^K(x_m, x_i) < \varepsilon$. Moreover there is a $g \in \mathrm{Aut}(D)$ such that $g(x_m) \in K$. Clearly, $g(x_i) \in L$ for all $i \geq m$ because d_D^K is invariant under $\mathrm{Aut}(D)$. Passing through a subsequence if necessary, $\{g(x_i)\}$ will converge to a point $q \in L$ because L is compact. It is easy to see $\{x_i\}$ must converge to $g^{-1}(q)$, for the same reason that g is an isometry with respect to d_D^K.

LEMMA 2.4. *Let D be a domain on a taut manifold X with $D/\mathrm{Aut}(D)$ compact. Let $\{x_i\}$ be a sequence of points in D converging to a boundary point $p \in \partial D$. Then there exists $\{m_i\} \subset \mathrm{Aut}(D)$ such that $\{\bar{x}_i = m_i^{-1}(x_i)\}$, through a subsequence if necessary, converges to a point $x \in D$. Furthermore, $\{z_i = m_i(x)\}$ will also converge to p.*

Proof. Let K be a compact subset of D as in Lemma 2.1 and let m_i be an element in $\mathrm{Aut}(D)$ such that $\bar{x}_i = m_i^{-1}(x_i) \in K$. Through a subsequence, $\{\bar{x}_i\}$ will converge to a point $x \in K \subset D$. To prove $\{z_i = m_i(x)\}$ is convergent to p, we consider the distance with respect to d_D^K as follows:

$$d_D^K(z_i, x_i) = d_D^K(m_i(x), x_i) = d_D^K(x, m_i^{-1}(x_i)) = d_D^K(x, \bar{x}_i).$$

The following inequality is clear by distance decreasing property:

$$(*) \qquad d_D^K \geq d \text{ on } D, \text{ where } d \text{ is the Kobayashi metric on } X.$$

We observe that $d_D^K(z_i, x_i) \to 0$ as $i \to \infty$ because $d_D^K(x, \bar{x}_i) \to 0$ as $\{\bar{x}_i\} \to x$. By $(*)$, $d(z_i, x_i) \to 0$ as $i \to \infty$. Since d is finite around an open set of

$p \in \partial D$, hence $d(x_i, p) \to 0$, as $\{x_i\} \to p$. By the triangle inequality for d, one has $d(z_i, p) \to 0$ as $i \to \infty$. Thus $\{z_i\} \to p$ as a limit.

LEMMA 2.5. *Let D be a domain of a taut manifold X. Suppose there is a totally real point $p \in \partial D$. Let $\{m_i\} \subset \mathrm{Aut}(D)$ be a sequence such that $\{m_i(x)\} \to p$ for some $x \in D$. Then for any compact subset $K \subset D$ and any boundary neighborhood $\hat{D}$ of p, $m_i(K) \subset \hat{D}$ for sufficiently large i. In particular, $\{m_i(y)\} \to p$ for any $y \in D$.*

Proof. By normal family argument, through a subsequence, $\{m_i\}$ will converge on compacta to a holomorphic mapping m: $D \to X$ such that $m(D) \subset \partial D$ and $m(x) = p$. Nevertheless, there is no complex analytic variety of positive dimension lying on ∂D through p. This implies m must be a constant map such that $m(D) = p$. Our lemma now follows easily from this fact.

Proof of Theorem 1. Since D_1 and D_2 are locally biholomorphic at $p_1 \in \partial D_1$ and $p_2 \in \partial D_2$, there is a biholomorphism f between two boundary neighborhood $\hat{D}_1$ and $\hat{D}_2$ of p_1 and p_2 respectively. Choose two sequences of relatively compact open subsets $\{X_i\}$ and $\{Y_i\}$ in D_1 and D_2 respectively so that

(i) $X_i \subset\subset X_{i+1}$, $Y_i \subset\subset Y_{i+1}$,
(ii) $\bigcup_{i=1}^{\infty} X_i = \hat{D}_1$, $\bigcup_{i=1}^{\infty} Y_i = \hat{D}_2$,
(iii) X_i is biholomorphic to Y_i under f (i.e. $f(X_i) = Y_i$).

Clearly the sequence of relatively compact open subsets

$$D_1^i = g_i^{-1}(X_i),\ D_2^i = h_i^{-1}(Y_i)$$

will satisfy the following properties, where

$$\{g_i\} \subset \mathrm{Aut}(D_1) \quad \text{and} \quad \{h_i\} \subset \mathrm{Aut}(D_2)$$

are the corresponding sequences obtained in Lemmas 2.4, 2.5 (with respect to the sequences $\{X_i\}$ and $\{f(X_i)\}$ in the definition of local biholomorphism at two boundary points p_1 and p_2):

(i) $D_1^i \subset\subset D_1^{i+1}$, $D_2^i \subset\subset D_2^{i+1}$,
(ii) $\bigcup_{i=1}^{\infty} D_1^i = D_1$, $\bigcup_{i=1}^{\infty} D_2^i = D_2$,
(iii) the composition of mappings $F_i = h_i^{-1} \circ f \circ g_i$ is a biholomorphism between D_1^i and D_2^i.

Let K and L be the compact subsets (i.e., fundamental domains) in D_1 and D_2 respectively which are obtained in Lemma 2.1. It is easy to show that

$\{F_i, D_1^i, K, L\}$ satisfies the non-divergent condition in Lemma 2.2. Thus $\{F_i\}$, through a subsequence, will converge to a holomorphic mapping F: $D_1 \to D_2$. On the other hand, one can repeat the same argument to $\{F_i^{-1}, D_2^i, L, K\}$. In this way, one can then prove F_i^{-1} converge to a holomorphic map G: $D_2 \to D_1$. Let a be a fixed point in K. By taking subsequences and readjustments of indices, one can assume $g_i(a) \in X_i$, $f \circ g_i(a) \in Y_i$ for all i. It is clear from the proofs of Lemmas 2.4 and 2.5; one can choose the above sequence $\{h_i\} \subset \mathrm{Aut}(D_2)$ satisfying further property, namely, $(h_i^{-1} \circ f \circ g_i)(a) \in L$. Let's say $\{(h_i^{-1} \circ f \circ g_i)(a)\}$ converge to $b \in L$. Apparently we obtain the following two conclusions:

(i) $G \circ F(a) = b$,
(ii) $|\det(G \circ F)(a)| = 1$.

To see that F is a biholomorphism, we apply the following result due to Dektyarev-Graham-Wu [2] which is a generalization of a theorem of H. Cartan [4].

THEOREM (Dektyarev-Graham-Wu). *Let M and N be complex manifolds of dimension n, where M is taut. Let $a \in M$ and $b \in N$ be two fixed points. Consider the set T of all holomorphic mappings h: $M \to N$ and g: $N \to M$ such that $h(a) = b$ and $g(b) = a$. If*

$$\sup_T |\det(g \circ h)(a)| = 1$$

then M and N are biholomorphic.

3. Proofs of Theorem 2 and 3

LEMMA 3.1. *Let D be a domain admitting a compact quotient on the taut manifold M. Then K_D does not vanish everywhere on D.*

Proof. By Lemma 2.3, D is taut. It follows immediately from Lemma 2.2, definition of K_D and normal family, that K_D cannot vanish everywhere on D, otherwise D would admit a nontrivial holomorphic curve.

LEMMA 3.2. *Under the assumptions of Theorem* 2(1), *C_D does not vanish everywhere on D.*

Proof. Let $x \in D$, be an arbitrary point. Suppose $\{m_i\}$ is the sequence in Lemma 2.5 such that $\{m_i(x)\} \to p$.

D can always be exhausted by a sequence of open sets $\{D_k\}_{k=1}^{\infty}$ with $D_k \subset\subset D_{k+1}$. Let (x, v) be a nonzero vector at x. For a fixed k and sufficiently large i, from Lemma 2.5, we have $m_i(D_k) \subset \hat{D}$. We therefore have the following chain of inequality:

$$\frac{C_{D_k}(x,v)}{K_D(x,v)} = \frac{C_{m_1(D_k)}(m_i(x), dm_i(v))}{K_D(m_i(x), dm_i(v))} \geq \frac{C_{\hat{D}}(m_i(x), dm_i(v))}{K_D(m_i(x), dm_i(v))}.$$

The above inequality follows from the fact that the Caratheodory and Eisenman measures are decreasing under holomorphic mappings (note: the inclusion $m_i(D_k) \hookrightarrow \hat{D}$ is holomorphic) and they are invariant under biholomorphisms. If one considers the inclusion map $\hat{D} \hookrightarrow D$, then it is clear that

$$K_{\hat{D}}(m_i(x), dm_i(v)) \geq K_D(m_i(x), dm_i(v))$$

by the volume decreasing property again. Thus we have

$$\frac{C_{D_k}(x,v)}{K_D(x,v)} \geq \frac{C_{\hat{D}}(m_i(x), dm_i(v))}{K_{\hat{D}}(m_i(x), dm_i(v))} \geq c^2 > 0.$$

Finally, letting $k \to \infty$, by the normal family argument as in Lemma 2.2, it is easy to prove that $C_{D_k}(x,v)$ is convergent to $C_D(x,v)$. Then $C_D(x,v) \geq c^2 K_D(x,v)$ and, from Lemma 3.1, C_D does not vanish everywhere on D.

The proof of Theorem 2(1) follows from Lemma 3.2 and the definition of C_D. The proof of Theorem 2(2) is similar to Theorem 2(1).

As for the proof of Theorem 3, one can go through the same argument as for Theorem 2 above and conclude that $d_D^C \geq c^2 d_D^K$. Since d_D^K is a complete hyperbolic metric on D (Lemma 2.3), d_D^C is compactly complete. By a remark in the first section of this paper, D is therefore holomorphically convex with respect to bounded holomorphic functions.

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