ILLINOIS JOURNAL OF MATHEMATICS
Volume 34, Number 3, Fall 1990

THE McSHANE INTEGRAL OF BANACH-VALUED FUNCTIONS

BY

RUSSELL A. GORDON

The generalized Riemann integral has, as the name suggests, a definition similar to the Riemann integral. The difference lies in the class of partitions that are used to form the Riemann sums. Two such generalizations have been studied for real-valued functions. One of these generalizations leads to an integral, often called the Henstock integral, that is equivalent to the restricted Denjoy integral while the other yields an integral, which we will refer to as the McShane integral, that is equivalent to the Lebesgue integral. We shall confine our attention to the latter definition and develop the properties of this integral for the case in which the function has values in a Banach space. The main result of this paper is that every measurable, Pettis integrable function is generalized Riemann integrable.

Throughout this paper X will denote a real Banach space and X^* its dual. We first extend the notion of partition of an interval.

DEFINITION 1. Let $\delta(\cdot)$ be a positive function defined on the interval $[a, b]$. A tagged interval $(s, [c, d])$ consists of an interval $[c, d] \subset [a, b]$ and a point s in $[a, b]$. The tagged interval $(s, [c, d])$ is subordinate to δ if

$$[c, d] \subset (s - \delta(s), s + \delta(s)).$$

Note that this may not be a point in $[c, d]$. Script capital letters such as $\mathscr{P}$ and $\mathscr{D}$ will be used to denote finite collections of non-overlapping tagged intervals. Let

$$\mathscr{P} = \{(s_i, [c_i, d_i]): 1 \le i \le N\}$$

be such a collection in $[a, b]$.

(a) The points $\{s_i: 1 \le i \le N\}$ are called the tags of $\mathscr{P}$.
(b) The intervals $\{[c_i, d_i]: 1 \le i \le N\}$ are called the intervals of $\mathscr{P}$.
(c) If $(s_i, [c_i, d_i])$ is subordinate to δ for each i, then we write $\mathscr{P}$ is sub δ.
(d) If $[a, b] = \bigcup_{i=1}^{N} [c_i, d_i]$, then $\mathscr{P}$ is called a tagged partition of $[a, b]$.

Received April 6, 1988

(e) If $\mathscr{P}$ is a tagged partition of $[a, b]$ and if $\mathscr{P}$ is sub δ, then we write $\mathscr{P}$ is sub δ on $[a, b]$.

(f) If $f\colon [a, b] \to X$, then $f(\mathscr{P}) = \sum_{i=1}^{N} f(s_i)(d_i - c_i)$.

(g) If F is defined on the intervals of $[a, b]$, then

$$F(\mathscr{P}) = \sum_{i=1}^{N} F([c_i, d_i]).$$

(h) We will write $\mu(\mathscr{P})$ for $\sum_{i=1}^{N}(d_i - c_i)$ and $\int_{\mathscr{P}} f$ for $\sum_{i=1}^{N}\int_{c_i}^{d_i} f$.

DEFINITION 2. The function $f\colon [a, b] \to X$ is McShane integrable on $[a, b]$ if there exists a vector z in X with the following property: for each $\varepsilon > 0$ there exists a positive function δ on $[a, b]$ such that $\|f(\mathscr{P}) - z\| < \varepsilon$ whenever $\mathscr{P}$ is sub δ on $[a, b]$. The function f is McShane integrable on the set $E \subset [a, b]$ if the function $f\chi_E$ is McShane integrable on $[a, b]$.

For real-valued functions the McShane integral and the Lebesgue integral are equivalent. See McShane [6] and Davies and Schuss [2].

The next three theorems record some of the basic properties of the McShane integral. The proofs of these facts are virtually identical to the proofs for real-valued functions and the reader is referred to McLeod [5] for the details.

THEOREM 3. *The function $f\colon [a, b] \to X$ is McShane integrable on $[a, b]$ if and only if for each $\varepsilon > 0$ there exists a positive function δ on $[a, b]$ such that $\|f(\mathscr{P}_1) - f(\mathscr{P}_2)\| < \varepsilon$ whenever $\mathscr{P}_1$ and $\mathscr{P}_2$ are sub δ on $[a, b]$.*

THEOREM 4. *Let f and g be functions mapping $[a, b]$ into X.*

(a) *If f is McShane integrable on $[a, b]$, then f is McShane integrable on every subinterval of $[a, b]$.*

(b) *If f is McShane integrable on each of the intervals $[a, c]$ and $[c, b]$, then f is McShane integrable on $[a, b]$ and*

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

(c) *If f and g are McShane integrable on $[a, b]$ and if α and β are real numbers, then $\alpha f + \beta g$ is McShane integrable on $[a, b]$ and*

$$\int_a^b (\alpha f + \beta g) = \alpha \int_a^b f + \beta \int_a^b g.$$

THEOREM 5. *Let* $f\colon [a,b] \to X$ *be McShane integrable on* $[a,b]$, *let* $F(t) = \int_a^t f$, *and consider* F *as a function of intervals. Given* $\varepsilon > 0$, *let* δ *be a positive function defined on* $[a,b]$ *such that* $\|f(\mathscr{P}) - F(b)\| < \varepsilon$ *whenever* $\mathscr{P}$ *is sub* δ *on* $[a,b]$. *If* $\mathscr{D}$ *is sub* δ, *then* $\|f(\mathscr{D}) - F(\mathscr{D})\| \leq \varepsilon$.

THEOREM 6. *Let* $f\colon [a,b] \to X$ *be McShane integrable on* $[a,b]$. *If* $f = g$ *almost everywhere on* $[a,b]$, *then* g *is McShane integrable on* $[a,b]$ *and* $\int_a^b f = \int_a^b g$.

Proof. It is sufficient to prove that if $f = \theta$ (the zero of X) almost everywhere on $[a,b]$, then f is McShane integrable on $[a,b]$ and $\int_a^b f = \theta$. Since $\|f\| = 0$ almost everywhere on $[a,b]$, the function $\|f\|$ is McShane integrable on $[a,b]$ since it is Lebesgue integrable and $\int_a^b \|f\| = 0$. Let $\varepsilon > 0$ and choose a positive function δ on $[a,b]$ such that $\|f\|(\mathscr{P}) < \varepsilon$ whenever $\mathscr{P}$ is sub δ on $[a,b]$. Let $\mathscr{P}$ be sub δ on $[a,b]$ and compute

$$\|f(\mathscr{P}) - \theta\| = \|f(\mathscr{P})\| \leq \|f\|(\mathscr{P}) < \varepsilon.$$

This shows that f is McShane integrable on $[a,b]$ and that $\int_a^b f = \theta$.

DEFINITION 7. Let $F\colon [a,b] \to X$ and let $E \subset [a,b]$. The function $f\colon E \to X$ is a scalar derivative of F on E if for each x^* in X^* the function x^*F is differentiable almost everywhere on E and $(x^*F)' = x^*f$ almost everywhere on E.

The next theorem follows easily from the known properties of the McShane integral of real-valued functions. See Gordon [4].

THEOREM 8. *Let* $f\colon [a,b] \to X$ *be McShane integrable on* $[a,b]$ *and let* $F(t) = \int_a^t f$.

(1) *For each* x^* *in* X^* *the function* x^*f *is McShane integrable on* $[a,b]$ *and* $\int_a^t x^*f = x^*F(t)$.

(2) *The function* F *is continuous on* $[a,b]$ *and* f *is a scalar derivative of* F *on* $[a,b]$.

In order to prove that every measurable, Pettis integrable function is McShane integrable we need a convergence theorem for the McShane integral. The convergence theorem that we will prove is essentially an iterated limits theorem. The next definition and the proof of the theorem that follows can be found in McLeod [5].

DEFINITION 9. (a) Let V be a set. A direction $\mathscr{S}$ in V is a nonempty collection of nonempty subsets of V that is directed downward by inclusion.

That is, for each pair of sets S_1 and S_2 in $\mathscr{S}$ there exists a set S_3 in $\mathscr{S}$ such that $S_3 \subset S_1$ and $S_3 \subset S_2$.

(b) Let $f: V \to X$. The vector z is the limit of f with respect to $\mathscr{S}$ if for each $\varepsilon > 0$ there exists a set S_ε in $\mathscr{S}$ such that $\|f(v) - z\| < \varepsilon$ for all v in S_ε. In this case we write $z = \lim_{\mathscr{S}} f(v)$.

(c) The function f is Cauchy with respect to $\mathscr{S}$ if for each $\varepsilon > 0$ there exists a set S_ε in $\mathscr{S}$ such that $\|f(v_1) - f(v_2)\| < \varepsilon$ for all v_1 and v_2 in S_ε.

(d) Let W be a set and let $g: V \times W \to X$. Suppose that

$$h(w) = \lim_{\mathscr{S}} g(v, w)$$

exists for all w in W. Then $h(w) = \lim_{\mathscr{S}} g(v, w)$ uniformly for w in W if for each $\varepsilon > 0$ there exists a set S_ε in $\mathscr{S}$ such that $|g(v, w) - h(w)| < \varepsilon$ for all v in S_ε and for all w in W. That is, the set S_ε works for all w in W.

It is a standard exercise to prove that the limit of f with respect to $\mathscr{S}$ is unique when it exists and that the limit of f with respect to $\mathscr{S}$ exists if and only if f is Cauchy with respect to $\mathscr{S}$.

THEOREM 10. *Let $\mathscr{S}$ be a direction in V, let $\mathscr{T}$ be a direction in W, and let $f: V \times W \to X$. If $g(v) = \lim_{\mathscr{T}} f(v, w)$ exists for each v in V and if $h(w) = \lim_{\mathscr{S}} f(v, w)$ exists uniformly for w in W, then each of the iterated limits exists and the values are equal. That is,*

$$\lim_{\mathscr{S}} g(v) = \lim_{\mathscr{S}} \lim_{\mathscr{T}} f(v, w) = \lim_{\mathscr{T}} \lim_{\mathscr{S}} f(v, w) = \lim_{\mathscr{T}} h(w).$$

In order to apply the theorem on iterated limits to obtain a convergence theorem for the McShane integral we introduce the concept of uniform McShane integrability. The idea behind the uniform McShane integrability of a family $\{f_\alpha\}$ of McShane integrable functions is that for each $\varepsilon > 0$ there exists a single positive function δ that works for all of the functions f_α. However, some care in the definition is required. Consider the sequence $\{f_n\}$ of functions defined on $[0, 1]$ by $f_n(t) = 0$ for $t \in (0, 1]$ and $f_n(0) = n$. All of these functions belong to the same equivalence class but there is no positive function δ on $[0, 1]$ for which $|f_n(\mathscr{P})| < 1$ for all n and for all $\mathscr{P}$ sub δ on $[0, 1]$.

DEFINITION 11. Let $\{f_\alpha\}$ be a family of McShane integrable functions defined on $[a, b]$. The family $\{f_\alpha\}$ is uniformly McShane integrable on $[a, b]$ if there exists a set E in $[a, b]$ such that $\mu(E) = b - a$ and for each $\varepsilon > 0$ there exists a positive function δ on $[a, b]$ such that $\|f_\alpha \chi_E(\mathscr{P}) - \int_a^b f_\alpha\| < \varepsilon$ for all α and whenever $\mathscr{P}$ is sub δ on $[a, b]$.

Now we are ready to state two convergence theorems for the McShane integral that are direct consequences of Theorem 10. We first set up the necessary notation. Let V be the set of positive integers and let $\mathscr{S}$ be the direction in V defined by

$$\mathscr{S} = \{S_n = \{n, n+1, \ldots\} : n \in V\}.$$

Let W be the set of all tagged partitions of $[a, b]$ and let $\mathscr{T}$ be the direction in W defined by

$$\mathscr{T} = \{T_\delta = \{\mathscr{P} : \mathscr{P} \text{ is sub } \delta \text{ on } [a, b]\} : \delta \text{ is a positive function on } [a, b]\}.$$

Let $\{f_n\}$ be a sequence of McShane integrable functions defined on $[a, b]$ and suppose that $f_n \to f$ pointwise on $[a, b]$. Define

$$g : V \times W \to X$$

by

$$g(n, \mathscr{P}) = f_n(\mathscr{P}).$$

Note that

$$f(\mathscr{P}) = \lim_{\mathscr{S}} g(n, \mathscr{P})$$

exists for every $\mathscr{P}$ in W and that

$$\int_a^b f_n = \lim_{\mathscr{T}} g(n, \mathscr{P})$$

exists for every n in V. Suppose that one of these limits exists uniformly. Then by Theorem 10 the function f is McShane integrable on $[a, b]$ since $\lim_{\mathscr{S}} f(\mathscr{P})$ exists and we have

$$\int_a^b f = \lim_{\mathscr{T}} \lim_{\mathscr{S}} g(n, \mathscr{P}) = \lim_{\mathscr{S}} \lim_{\mathscr{T}} g(n, \mathscr{P}) = \lim_{n \to \infty} \int_a^b f_n.$$

This observation proves the next two theorems.

THEOREM 12. *Let $f_n : [a, b] \to X$ be a McShane integrable function on $[a, b]$ for each positive integer n. If $f_n \to f$ uniformly on $[a, b]$, then f is McShane integrable on $[a, b]$ and*

$$\int_a^b f = \lim_{n \to \infty} \int_a^b f_n.$$

THEOREM 13. *Let* f_n: $[a, b] \to X$ *be a McShane integrable function on* $[a, b]$ *for each positive integer n and suppose that* $f_n \to f$ *pointwise on* $[a, b]$. *If the family* $\{f_n\}$ *is uniformly McShane integrable on* $[a, b]$, *then f is McShane integrable on* $[a, b]$ *and*

$$\int_a^b f = \lim_{n \to \infty} \int_a^b f_n.$$

We now proceed to prove that every measurable, Pettis integrable function is McShane integrable.

THEOREM 14. *If* f: $[a, b] \to X$ *is a simple function, then f is McShane integrable on* $[a, b]$.

Proof. Since the McShane integral is linear it is sufficient to consider the case $f(t) = \chi_E(t)x$ where E is a measurable set in $[a, b]$ and $x \in X$. Let $\varepsilon > 0$ be given. Choose an open set G such that

$$E \subset G \quad \text{and} \quad \mu(G) < \mu(E) + \varepsilon$$

and choose a closed set H such that

$$H \subset E \quad \text{and} \quad \mu(H) > \mu(E) - \varepsilon.$$

Define a positive function δ on $[a, b]$ as follows. Let

$$B_t = (t - \delta(t), t + \delta(t)).$$

(i) If $t \in H$, then choose $\delta(t) > 0$ so that $B_t \subset G$.
(ii) If $t \in G - H$, then choose $\delta(t) > 0$ so that $B_t \subset G - H$.
(iii) If $t \in [a, b] - G$, then choose $\delta(t) > 0$ so that $B_t \cap H = \varnothing$.

Let $\mathscr{P}$ be sub δ on $[a, b]$ and let $\mathscr{P}_E$ be the subset of $\mathscr{P}$ that has tags in E. Since

$$-\varepsilon < \mu(H) - \mu(E) \le \mu(\mathscr{P}_E) - \mu(E) \le \mu(G) - \mu(E) < \varepsilon$$

we find that

$$\| f(\mathscr{P}) - \mu(E)x \| = \|x\| \, | \mu(\mathscr{P}_E) - \mu(E) | \le \|x\| \varepsilon.$$

Therefore, the function f is McShane integrable on $[a, b]$ and $\int_a^b f = \mu(E)x$.

THEOREM 15. *Let $\{E_n\}$ be a sequence of disjoint measurable sets in $[a, b]$, let $\{x_n\}$ be a sequence in X, and let $f\colon [a, b] \to X$ be defined by*

$$f(t) = \sum_n x_n \chi_{E_n}(t).$$

If the series $\sum_n \mu(E_n) x_n$ is unconditionally convergent, then the function f is McShane integrable on $[a, b]$ and

$$\int_a^b f = \sum_n \mu(E_n) x_n.$$

Proof. For each positive integer n let

$$f_n(t) = \sum_{k=1}^{n} x_k \chi_{E_k}(t).$$

By Theorem 14 each f_n is McShane integrable on $[a, b]$ and

$$\int_a^b f_n = \sum_{k=1}^{n} \mu(E_k) x_k.$$

Note that $f_n \to f$ pointwise on $[a, b]$ and that

$$\lim_{n\to\infty} \int_a^b f_n = \sum_{n=1}^{\infty} \mu(E_n) x_n.$$

By Theorem 13 it is sufficient to prove that the family $\{f_n\}$ is uniformly McShane integrable on $[a, b]$.

Let $\varepsilon > 0$ and for each n choose a positive function δ_n on $[a, b]$ such that

$$\left\| f_n(\mathscr{P}) - \sum_{k=1}^{n} \mu(E_k) x_k \right\| < \varepsilon 2^{-n-2}$$

whenever $\mathscr{P}$ is sub δ_n on $[a, b]$. Since the series $\sum_n \mu(E_n) x_n$ is unconditionally convergent there exists a positive integer N such that

$$\sum_{k=N}^{\infty} |x^*(x_k)| \mu(E_k) < \varepsilon/4$$

for all x^* with $\|x^*\| \le 1$. Let

$$H_N = \bigcup_{k=1}^{N} E_k \cup \left\{[a,b] - \bigcup_{k=1}^{\infty} E_k\right\}$$

and let $H_n = E_n$ for $n > N$. Now the H_n's are disjoint and $[a,b] = \bigcup_{n=1}^{\infty} H_n$. Define a positive function δ on $[a,b]$ by $\delta(t) = \min\{\delta_1(t), \ldots, \delta_n(t)\}$ for $t \in H_n$. We will show that $\|f_n(\mathscr{P}) - \sum_{k=1}^{n} \mu(E_k)x_k\| < \varepsilon$ for all n whenever $\mathscr{P}$ is sub δ on $[a,b]$.

Let $\mathscr{P}$ be sub δ on $[a,b]$. Then $\mathscr{P}$ is sub δ_n on $[a,b]$ for each $n \le N$ and we find that

$$\left\| f_n(\mathscr{P}) - \sum_{k=1}^{n} \mu(E_k)x_k \right\| < \varepsilon 2^{-n-2} < \varepsilon \quad \text{for each } n \le N.$$

Now fix $n > N$. For each $i \ge N$ let $\mathscr{P}_i$ be the subset of $\mathscr{P}$ that has tags in H_i. Let $\mathscr{D}_1 = \bigcup_{i=N}^{n-1} \mathscr{P}_i$, let $\mathscr{D}_2 = \bigcup_{i=n}^{\infty} \mathscr{P}_i$, and let $F_i(t) = \int_a^t f_i$. Then $\mathscr{D}_2$ is sub δ_n and for $N \le i < n$ we see that $f_n = f_i$ on H_i and $\mathscr{P}_i$ is sub δ_i. Using Theorem 5 and treating the F_i's as functions of intervals we obtain

$$\begin{aligned}
\left\| f_n(\mathscr{P}) - \sum_{k=1}^{n} \mu(E_k)x_k \right\| &\le \|f_n(\mathscr{D}_1) - F_n(\mathscr{D}_1)\| + \|f_n(\mathscr{D}_2) - F_n(\mathscr{D}_2)\| \\
&\le \sum_{i=N}^{n-1} \|f_n(\mathscr{P}_i) - f_i(\mathscr{P}_i)\| \\
&\quad + \sum_{i=N}^{n-1} \|f_i(\mathscr{P}_i) - F_i(\mathscr{P}_i)\| \\
&\quad + \left\| \sum_{i=N}^{n-1} (F_i(\mathscr{P}_i) - F_n(\mathscr{P}_i)) \right\| + \varepsilon 2^{-n-2} \\
&< \sum_{i=N}^{n-1} \varepsilon 2^{-i-2} + \left\| \sum_{i=N}^{n-1} (F_n - F_i)(\mathscr{P}_i) \right\| + \frac{\varepsilon}{4} \\
&< \left\| \sum_{i=N}^{n-1} (F_n - F_i)(\mathscr{P}_i) \right\| + \frac{\varepsilon}{2}.
\end{aligned}$$

By the choice of N we find that

$$\begin{aligned}\left\|\sum_{i=N}^{n-1}(F_n - F_i)(\mathscr{P}_i)\right\| &= \left\|\sum_{i=N}^{n-1}\sum_{k=i+1}^{n} x_k\mu(E_k \cap \mathscr{P}_i)\right\| \\ &= \left\|\sum_{k=N+1}^{n}\sum_{i=N}^{k-1} x_k\mu(E_k \cap \mathscr{P}_i)\right\| \\ &= \left\|\sum_{k=N+1}^{n} x_k\mu\left(E_k \cap \bigcup_{i=N}^{k-1}\mathscr{P}_i\right)\right\| \\ &= \sup_{\|x^*\|\le 1}\left|\sum_{k=N+1}^{n} x^*(x_k)\mu\left(E_k \cap \bigcup_{i=N}^{k-1}\mathscr{P}_i\right)\right| \\ &\le \sup_{\|x^*\|\le 1}\sum_{k=N+1}^{\infty}|x^*(x_k)|\mu(E_k) \\ &\le \frac{\varepsilon}{4}.\end{aligned}$$

Combining the last two inequalities we obtain $\|f_n(\mathscr{P}) - \sum_{k=1}^{n}\mu(E_k)x_k\| < \varepsilon$. Since this is valid for all n we conclude that the family $\{f_n\}$ is uniformly McShane integrable on $[a, b]$. This completes the proof.

THEOREM 16. *If f: $[a, b] \to X$ is Bochner integrable on $[a, b]$, then f is McShane integrable on $[a, b]$.*

Proof. Since f is measurable there exist $E \subset [a, b]$ with $\mu(E) = b - a$ and a sequence $\{f_n\}$ of countably-valued functions such that for each n the inequality $\|f_n(t) - f\chi_E(t)\| \le 1/n$ holds for all t in $[a, b]$. It is clear that each f_n is Bochner integrable on $[a, b]$. For each n let

$$f_n = \sum_{k=1}^{\infty} x_k^n \chi_{E_k^n}$$

where the sets $\{E_k^n$: $k \ge 1\}$ are disjoint and measurable. The series $\sum_k \mu(E_k^n)x_k^n$ is absolutely convergent and hence unconditionally convergent for each n. By Theorem 15 each of the functions f_n is McShane integrable on $[a, b]$. Since $f\chi_E$ is the uniform limit of $\{f_n\}$ on $[a, b]$, the function $f\chi_E$ is McShane integrable on $[a, b]$ by Theorem 12. By Theorem 6 the function f is McShane integrable on $[a, b]$.

THEOREM 17. *Let f: $[a, b] \to X$ be measurable. If f is Pettis integrable on $[a, b]$, then f is McShane integrable on $[a, b]$.*

Proof. Since f is measurable there exist $E \subset [a,b]$ with $\mu(E) = b - a$ and a countably-valued function $g\colon [a,b] \to X$ such that

$$\| g(t) - f\chi_E(t) \| \leq 1$$

for all t in $[a,b]$. It is easy to see that $g - f\chi_E$ is Bochner integrable on $[a,b]$ and that g is Pettis integrable on $[a,b]$. By Theorem 16 the function $g - f\chi_E$ is McShane integrable on $[a,b]$. Let

$$g = \sum_n x_n \chi_{E_n}$$

where the E_n's are disjoint, measurable sets in $[a,b]$. Since g is Pettis integrable on $[a,b]$ every subseries of $\Sigma_n \mu(E_n) x_n$ is weakly convergent. By a theorem of Orlicz and Pettis (see Diestel and Uhl [3, p. 22]) the series $\Sigma_n \mu(E_n) x_n$ is unconditionally convergent. By Theorem 15 the function g is McShane integrable on $[a,b]$ and it follows that the function $f\chi_E = g - (g - f\chi_E)$ is McShane integrable on $[a,b]$. By Theorem 6 the function f is McShane integrable on $[a,b]$.

Hence, in separable spaces every Pettis integrable function is McShane integrable. In order to determine conditions on a Banach space for every McShane integrable function to be Pettis integrable we need the result that appears next. Recall that a function $f\colon [a,b] \to X$ is Dunford integrable on $[a,b]$ if x^*f is Lebesgue integrable on $[a,b]$ for each x^* in X^*. The proof of the theorem below is a consequence of the Bessaga-Pelczyński characterization of Banach spaces that do not contain a copy of c_0 (see Diestel and Uhl [3, p. 22]).

THEOREM 18. *Suppose that X contains no copy of c_0 and let $f\colon [a,b] \to X$ be Dunford integrable on $[a,b]$. If $\int_I f \in X$ for every interval $I \subset [a,b]$, then f is Pettis integrable on $[a,b]$.*

The fact that every McShane integrable function is Dunford integrable and X-valued on intervals yields the following result.

THEOREM 19. *Suppose that X contains no copy of c_0. If $f\colon [a,b] \to X$ is McShane integrable on $[a,b]$, then f is Pettis integrable on $[a,b]$.*

Putting Theorems 17 and 19 together we obtain:

THEOREM 20. *Suppose that X is separable and contains no copy of c_0. A function $f\colon [a,b] \to X$ is McShane integrable on $[a,b]$ if and only if f is Pettis integrable on $[a,b]$.*

Unfortunately, we have been unable to prove containment in either direction for arbitrary Banach spaces or to find an example of a function integrable in one sense but not the other.

We end this paper with two comments. A McShane integrable function need not be measurable. It is easy to show that the function

$$f\colon [0,1] \to l_\infty[0,1]$$

defined by $f(t) = \chi_{[0,t]}$ is McShane integrable and not measurable. Furthermore, even for measurable functions the collection of McShane integrable functions properly includes the collection of Bochner integrable functions: simply take any measurable, Pettis integrable function that is not Bochner integrable. This invalidates a statement by Artstein [1] that the Bochner integral and McShane integral are equivalent.

References

1. Z. Artstein, *Lyapounov convexity theorem and Riemann-type integrals*, Indiana Math. J., vol. 25 (1976), pp. 717–724.
2. R.O. Davies and Z. Schuss, *A proof that Henstock's integral includes Lebesgue's*, J. London Math. Soc. (2), vol. 2 (1970), pp. 561–562.
3. J. Diestel and J.J. Uhl, *Vector measures*, American Mathematical Society, Providence, R.I., 1977.
4. R.A. Gordon, *Equivalence of the generalized Riemann and restricted Denjoy integrals*, Real Anal. Exchange, vol. 12 (1987), pp. 551–574.
5. R.M. McLeod, *The generalized Riemann integral*, Carus Mathematical Monographs, No. 20, Mathematical Association of America, 1980.
6. E.J. McShane, *A unified theory of integration*, Amer. Math. Monthly, vol. 80 (1973), pp. 349–359.

Whitman College
Walla Walla, Washington