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RELATIVE CHOW GROUPS

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We shall propose a definition for the relative Chow groups of a scheme with respect to a closed subscheme and establish some basic properties. The definition generalizes to provide a definition of relative *higher* Chow groups as well.

In Section 1 we recall for reference the most important relationships between the classical (absolute) Chow groups and algebraic K-theory. In Section 2 we describe analogous relationships for the higher Chow groups introduced by Bloch in [B]. (These are canonically isomorphic to the higher "PreChow groups" of [L].) The results here are of independent interest. In Section 3 we introduce relative analogues of many important constructions. In Section 4 we define the relative Chow groups and relative higher Chow groups. We establish their basic properties and their relationship to K-theory, emphasizing the analogies between this material and that of Sections 1 and 2.

1. Absolute Chow groups and algebraic K-theory

We begin by recalling, for later reference, some of the main properties of the usual (absolute) Chow groups, particularly those that relate the Chow groups to algebraic K-theory.

1.1. Let X be a regular scheme essentially of finite type over a field k. (Regularity can be relaxed in much of what follows.) We have the following invariants:

$Z^m(X)$, the group of codimension-p algebraic cycles on X. That is, $Z^m(X)$ is free abelian on those reduced and irreducible closed subschemes of X that have codimension m.

$Ch^m(X) = Z^m(X)/R^m(X)$, the mth Chow group of X. Here $R^m(X) \subset Z^m(X)$ is the subgroup consisting of cycles rationally equivalent to zero.

$K_m(X)$, the mth Quillen K-group of X.

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$\mathcal{K}_m$, the sheafification of the presheaf $U \to K_m(U)$ on X.

$E_2^{p,q}(X)$, the $E_2^{p,q}$ term in the Gersten-Quillen spectral sequence converging to $K_{-p-q}(X)$.

$gr^m K_0(X)$, the mth graded piece of the Grothendieck group $K_0(X)$, under the filtration induced by codimension of support (using the natural isomorphism between $K_0(X)$ and K_0 of the category of coherent sheaves on X).

1.2. The objects above are related as follows:

$$gr^m K_0(X) \underset{(1)}{\longleftarrow} Ch^m(X) \underset{(2)}{\approx} H^m(X, \mathcal{K}_m) \underset{(3)}{\approx} E_2^{m,-m}(X)$$

and the arrow (1) is an isomorphism up to torsion.

2. Higher analogues

We continue to assume that X is regular and essentially of finite type over a field.

2.1. Spencer Bloch has defined higher Chow groups $Ch^m(X, n)$ in [B]. The author has given an alternative definition in [L], where the groups are called $PreCh^m(X, n)$. By Proposition 3.5 of [L], the two constructions are canonically isomorphic. (For a detailed proof, see Corollary 1.9 of [L2].) We recall Bloch's definition. Let $X^\cdot$ be the cosimplicial scheme that is defined by

$$X^{(n)} = X \times_k \operatorname{Spec}\Big(k[t_0, \ldots, t_n]/\Big(\sum t_i - 1\Big)\Big),$$

with the obvious cofaces and codegeneracies. Let $Z^m(X, n)$ be the subgroup of $Z^m(X^{(n)})$ consisting of those cycles that meet properly (i.e. in the correct codimension) with the images of all of the cofaces and compositions of cofaces. Make $Z^m(X, \cdot\,)$ into a simplicial group by defining the faces and degeneracies by pullback along the cofaces and codegeneracies of $X^\cdot$. Then $Ch^m(X, n) = \pi_n(Z^m(X, \cdot\,))$.

2.2. We view $Ch^m(X, n)$ as a higher analogue of $Ch^m(X)$ (note that $Ch^m(X, 0) = Ch^m(X)$) and list the higher analogues of the relations in 1.2 above. First, Bloch has shown in [B] that $Ch^m(X, n)$ is isomorphic to $gr^m K_n(X)$ up to torsion, where $gr^m K_n(X)$ is the mth graded piece associated to the gamma filtration on higher K theory.[1] In [L] it is shown that this map

[1]This statement relies on Theorem 3.1 of [B], although the proof given in [B] is inadequate. The gap appears to be filled by the widely circulated preprint [B2]. Bloch has indicated in private correspondence that because he finds [B2] extraordinarily complicated and difficult to understand, he has no plans to publish it at the present time.

Similar comments apply to the proof of Theorem 2.5 of the present paper, and the associated lemmas, which make use of results in [B] that depend on the crucial Theorem 3.1.

arises from an integrally defined map

$$Ch^m_\infty(X,n) \to F^m K_n(X)/F^{m+1}K_n(X),$$

where $Ch^m_\infty(X,n)$ is a certain subgroup of $Ch^m(X,n)$ and $F^{\cdot}$ is an appropriate filtration. This map is the higher analogue of the map (1) in 1.2.

In [L] it is conjectured that there is a spectral sequence with $E_2^{p,q} = Ch^{-q}(X, -p-q)$, converging to the filtration F on $K_{-p-q}(X)$. This should be the higher analogue of the composition of isomorphisms (2) and (3) in 1.2.

Alternatively, the Gersten-Quillen spectral sequence has $E_2^{p,q} = H^p(X, \mathscr{K}_{-q})$, which can be viewed as a higher analogue of the isomorphism (3) in 1.2.

The higher analogue of (2) should be a map $Ch^m(X, m-p) \to H^p(X, \mathscr{K}_m)$. In the next subsection, we construct such a map.

2.3. We define a map $\Psi_{m,p}$: $Ch^m(X, m-p) \to H^p(X, \mathscr{K}_m)$. Recall that $Ch^m(X, m-p)$ is the homology of the complex

$$\cdots \to Z^m(X, m-p+1) \to Z^m(X, m-p) \\ \to Z^p(X, m-p-1) \to \cdots$$

and that $H^p(X, \mathscr{K}_m)$ can be computed as the homology of the Gersten complex

$$\cdots \to \coprod_{x\in X^{p-1}} K_{m-p+1}(k(x)) \to \coprod_{x\in X^p} K_{m-p}(k(x)) \\ \to \coprod_{x\in X^{p+1}} K_{m-p-1}(k(x)) \to \cdots$$

(Here X^p represents the set of points in X whose closures have codimension p). Thus it will suffice to construct maps

$$Z^m(X, m-p) \xrightarrow{\alpha_{m,p}} \coprod_{x\in X^p} K_{m-p}(k(x))$$

that are compatible with the differentials in these complexes.

To construct $\alpha_{m,p}$ we construct a map

$$Z^m(X, m-p) \xrightarrow{\alpha^M_{m,p}} \coprod_{x\in X^p} K^M_{m-p}(k(x))$$

where K^M is Milnor's K-theory; $\alpha_{m,p}$ will then be the composition of $\alpha^M_{m,p}$ with the natural map from Milnor's K-theory to Quillen's.

We construct $\alpha^M_{m,p}$ as follows: If $[\bar{x}]$ is a generator of $Z^m(X, m-p)$, and if x is the generic point of $\bar{x}$, then $\alpha^M_{m,p}([\bar{x}])_y = 0$ unless $\bar{y} = \overline{\Pi_*(x)}$. If y is the generic point of $\Pi_*(x)$, then set

$$\alpha^M_{m,p}([\bar{x}])_y = N_{k(x)/k(y)}\{t_1/t_0, t_2/t_0, \ldots, t_{m-p}/t_0\}$$

where the t_i are the coordinate functions as in the definition of the cosimplicial set $X^{\cdot}$, the curly brackets denote a Steinberg symbol, and N is the norm map partially constructed by Bass and Tate in [BT] and confirmed to exist by Kato in [K].

We claim that $\alpha_{m,p}$ is compatible with the differentials in the sense that the following diagram commutes:

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & Z^m(X, m-p+1) & \longrightarrow & Z^m(X, m-p) & \longrightarrow & Z^m(X, m-p-1) & \longrightarrow \cdots \\
& \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & \coprod_{x \in X^{p-1}} K_{m-p+1}(k(x)) & \longrightarrow & \coprod_{x \in X^{p}} K_{m-p}(k(x)) & \longrightarrow & \coprod_{x \in X^{p+1}} K_{m-p-1}(k(x)) & \longrightarrow \cdots
\end{array}$$

Taking homology gives the desired map $\Psi_{m,p}$.

In verifying the claim of commutativity, it will be convenient to write $n = m - p$. Let z be any element of X^{p+1}. Then it suffices to demonstrate the commutativity of the square

$$\begin{array}{ccc}
Z^m(X, n) & \xrightarrow{d} & Z^m(X, n-1) \\
\alpha \downarrow & & \downarrow \alpha_z \\
\coprod_{x \in X^p} K^M_n k(x) & \xrightarrow{\partial} & K^M_{n-1} k(z)
\end{array} \tag{2.4.1}$$

Let $[\bar{x}]$ be a generator of $Z^m(X, n)$ (that is, x is the generic point of a variety of codimension m in $X^{(n)}$). Let F_i be the locus of $t_i = 0$ in $X^{(n)}$. Let $\{z_{ij}\}$ be the generic points of those components of $\bar{x} \cap F_i$ which project to z.

Note that $d([\bar{x}]) = \sum_{i,j} d_{ij}([\bar{x}])$, where

$$d_{ij}([\bar{x}]) = (-1)^i I_{z_{ij}}(\bar{x}, F_i)[\bar{z}_{ij}]$$

and I denotes an intersection number.

(It is possible that the classes $[\bar{z}_{ij}]$ do not remain distinct after they are identified with classes in $Z^m(X, n-1)$, but we continue to maintain a separate summand for each of the original z_{ij}.)

Let ∂_{ij}: $K^M_n k(x) \to K^M_{n-1} k(z_{ij})$ be the Milnor K-theory differential.

We will prove the following two lemmas:

LEMMA A. $\partial(\alpha([\bar{x}])) = \Sigma_{i,j} N_{k(z_{ij})/k(z)}(\partial_{ij}\{t_1/t_0, \ldots, t_n/t_0\})$

LEMMA B. $N_{k(z_{ij})/k(z)}(\partial_{ij}\{t_1/t_0, \ldots, t_n/t_0\}) = \alpha_z(d_{ij}([\bar{x}])$.

From these will follow immediately:

THEOREM. *The square* (2.4.1) *commutes; therefore* $\Psi_{m,p}$ *is well-defined.*

Proof of the theorem. $(\alpha \circ d)([\bar{x}]) = (\alpha \circ (\Sigma d_{ij}))([\bar{x}])$. Apply Lemma B and then Lemma A to see that this is equal to $\partial(\alpha([\bar{x}]))$. Q.E.D.

Proof of Lemma A. It follows from a proposition of Suslin [Su, 4.3] that $\{t_1/t_0, \ldots, t_n/t_0\}$ maps to the same image under either composition in the diagram

$$\begin{array}{ccc} K_n^M k(x) & \xrightarrow{N} & K_n^M k(\Pi(x)) \\ \partial\big\downarrow & & \big\downarrow\partial \\ \bigoplus_{i,j} K_{n-1}^M k(z_{ij}) & \xrightarrow[N]{} & K_{n-1}^M k(z)\ . \end{array}$$

(According to Suslin, the diagram becomes commutative if the lower left corner is replaced with a sum over *all* points that project to z. But the components of $\partial\{t_1/t_0, \ldots, t_n/t_0\}$ are trivial for all of these points except the z_{ij}.)

Now

$$\partial(N(\{t_1/t_0, \ldots, t_n/t_0\}))$$

is easily seen to be

$$\partial(\alpha([\bar{x}]))$$

and

$$N(\partial(\{t_1/t_0, \ldots, t_n/t_0\}))$$

is easily seen to be

$$\sum_{i,j} N_{k(z_{ij})/k(z)}(\partial_{ij}\{t_1/t_0, \ldots, t_n/t_0\}). \quad Q.E.D.$$

Proof of Lemma B. From the definitions,

$$\alpha_z\big(d_{ij}([\bar{x}])\big) = (-1)^i \sum_j I_{z_{ij}}(\bar{x}, F_i)\, N_{k(z_{ij})/k(z)}\{t_1/t_0, \ldots, \widehat{t_i/t_0}, \ldots, t_n/t_0\}$$

when $i \neq 0$; and

$$\alpha_z\big(d_{0j}([\bar{x}])\big) = \sum_j I_{z_{0j}}(\bar{x}, F_0)\, N_{k(z_{0j})/k(z)}\{t_2/t_1, t_3/t_1, \ldots, t_n/t_1\}.$$

It therefore suffices to demonstrate that

$$\partial_{ij}\{t_1/t_0, \ldots, t_n/t_0\} = (-1)^i \sum_j I(\bar{x}, F_i)\{t_1/t_0, \ldots, \widehat{t_i/t_0}, \ldots, t_n/t_0\}$$

for $i \neq 0$ and that

$$\partial_{0j}\{t_1/t_0, \ldots, t_n/t_0\} = \sum_j I_{z_{0j}}(\bar{x}, F_0)\{t_2/t_1, t_3/t_1, \ldots, t_n/t_1\}.$$

Both of these follow from repeated used of the formulas in [BT, 4.5], together with the multilinearity of the Steinberg symbols. Q.E.D.

2.4. *Remark.* In Section 10 of [B], Bloch introduces the filtration

$$F^n Z^m(X, \cdot) = \{z \in Z^m(X, \cdot) \mid \text{the projection of } z \text{ on } X \text{ has codimension} \geq n\}.$$

Clearly the map $\Psi_{m,p}$ induces zero on $F^{p+1}Z^m(X, m-p)$.

2.5. THEOREM. *$\Psi_{m,p}$ is an isomorphism for $p = m$ or $m - 1$.*

Proof. For $p = 0$, $\Psi_{m,p}$ is precisely the isomorphism from $Ch^m(X, 0) = Ch^m(X)$ to $H^m(X, \mathscr{K}_m)$ constructed by Quillen in [Q, 5.19].

For $p = 1$, we need a sequence of lemmas. Lemma E in 2.8 will complete the proof.

2.6. LEMMA C. *Let $X = \mathrm{Spec}(k)$. Then $\Psi_{m,0}$: $Ch^m(X, m) \to K_m(k)$ is an isomorphism for $m = 0$ or 1 and a surjection for $m = 2$.*

Proof. For $m = 0$, $\Psi_{0,0}$ is the identity map on $\mathbf{Z}$. For $m = 1$, the unit a $(\neq -1) \in K_1(k)$ is the image of the cycle given by the rational point

$$\left(\frac{1}{1-a}, \frac{-a}{1-a}\right) \in X^1 = X \times \mathrm{Spec}(k[t_0, t_1]/(t_0 + t_1 + 1)).$$

This demonstrates that $\Psi_{1,0}$ is surjective. For $m = 2$, the symbol $\{a, b\}$ $(a + b \neq -1)$ is the image of the cycle defined by the rational point

$$\left(\frac{1}{1+a+b}, \frac{a}{1+a+b}, \frac{b}{1+a+b}\right) \in X^2.$$

If $a + b = -1$, then $a - ab \neq 1$ (since b is a unit), and it is either the case that $a - 1/a \neq 0$ or that $a + 1/a \neq 0$. Write either

$$\{a, b\} = \{a, -1/a\} \cdot \{a, -ab\}$$

or

$$\{a, b\} = \{a, -1\} \cdot \{a, 1/a\} \cdot \{a, -ab\}$$

so that each factor in the product can be lifted. This shows that $\Psi_{2,0}$ is also surjective.

It remains to demonstrate injectivity of $\Psi_{1,0}$. First we show that $Ch^1(X, 1)$ is generated by the classes of rational points. Because

$$R = k[t_0, t_1]/(t_0 + t_1 - 1)$$

is a principal ideal domain, $Z^1(X, 1)$ is generated by the classes of modules of the form $R/f(t_0)$, where f_0 is an irreducible monic polynomial. Given such an f, write $f(T) = Tg(T) - \alpha$ for some $\alpha \in k$. By definition of $Z^1(X, 1)$, we have $\alpha \neq 0$ and $g(1) \neq \alpha$. Now define a function h on the boundary of X^2 as follows:

On the set $\{t_2 = 0\}$, $h(t_0, t_1) = g(1) - \alpha$.
On the set $\{t_1 = 0\}$, $h(t_0, t_2) = t_0 g(t_0) - \alpha = f(t_0)$.
On the set $\{t_0 = 0\}$, $h(t_1, t_2) = t_1 g(1) - \alpha$.

These descriptions are compatible at the three vertices and so h is really well-defined. Lift h arbitrarily to a function $\tilde{h}$ on X^2. The cycle defined by $\tilde{h}$ in $Z^1(X, 2)$ has as its boundary the cycle defined by the rational point $(1 - \alpha/g(1), \alpha/g(1))$ minus the cycle defined by f. It follows that $Ch^1(X, 1)$ is generated by the classes of rational points.

Now let a and b be arbitrary units. Consider the cycle defined by the line $a \cdot t_2 + b \cdot t_1 = a \cdot b \cdot t_0$ in $Z^1(X, 2)$. The boundary of this cycle is

$$\left(\frac{a}{a+b}, \frac{b}{a+b}\right) - \left(\frac{1}{1+b}, \frac{b}{1+b}\right) + \left(\frac{1}{1+a}, \frac{a}{1+a}\right),$$

giving a relation in $Ch^1(X, 1)$ that lifts the relation $(ba^{-1}) \cdot a = b$ in k^*. Since all of the relations among elements of k^* are generated by relations of this form, it follows that the map $\Psi_{1,0}$ is injective.

2.7. LEMMA D. *Let $\mathcal{CH}^m(q)$ be the sheafified higher Chow group defined in [B]. Then*

$$H^{q-1}(X, \mathcal{CH}^m(q)) = \begin{cases} 0 & \text{if } q \neq m \\ H^{q-1}(X, \mathcal{K}_q) & \text{if } q = m. \end{cases}$$

Proof. By theorem 10.1 of [B], we can compute $H^{q-1}(X, \mathcal{CH}^m(q))$ as the homology of the complex

$$\coprod_{X^{q-2}} Ch^{m-q+2}(k(x),2) \to \coprod_{X^{q-1}} Ch^{m-q+1}(k(x),1) \to \coprod_{X^q} Ch^{m-q}(k(x),0).$$

The term in the middle is zero by definition when $m < q - 1$ and easily seen to be zero when $m = q - 1$. It is zero for reasons of dimension when $m > q$. In case $m = q$, we apply Lemma C to see that Ψ induces a map of complexes

$$\begin{array}{ccccc} \coprod_{X^{m-2}} Ch^2(k(x),2) & \longrightarrow & \coprod_{X^{m-1}} Ch^1(k(x),1) & \longrightarrow & \coprod_{X^m} Ch^0(k(x),0). \\ \downarrow & & \downarrow\approx & & \downarrow\approx \\ \coprod_{X^{m-2}} K_2(k(x)) & \longrightarrow & \coprod_{X^{m-1}} K_1(k(x)) & \longrightarrow & \coprod_{X^m} K_0(k(x)). \end{array}$$

where the bottom complex is the Gersten-Quillen complex that computes $H^{m-1}(X, \mathcal{K}_m)$. It is immediate that the induced map on cohomology is an isomorphism.

2.8. LEMMA E. *$\Psi_{m,m-1}$: $Ch^m(X,1) \to H^{m-1}(X, \mathcal{K}_m)$ is an isomorphism for every m.*

Proof. From Section 10 of [B], there is a spectral sequence

$$E_2^{p,-q} = H^r(X, \mathcal{CH}^m(q)) \Rightarrow Ch^m(X, q - r).$$

The spectral sequence arises from the filtration of 2.4 above. We invoke Theorem 10.1 of [B] to see that $E_2^{p,-q}$ can be computed as the homology of a complex

$$\coprod_{X^{p+1}} Ch^{m-p+1}(k(x), q-p+1) \to \coprod_{X^p} Ch^{m-p}(k(x), q-p)$$
$$\to \coprod_{X^{p+1}} Ch^{m-p-1}(k(x), q-p-1).$$

which shows that the spectral sequence is concentrated in degrees $(p, -q)$ such that $p \geq m \geq q$. (In all other cases the middle term of the complex is zero). By Lemma D, the only E_2-term along the diagonal $q - r = 1$ is

$$E_2^{m-1,-m} = H^{m-1}(X, \mathscr{C}\mathscr{H}^m(m)) = H^{m-1}(X, \mathscr{K}_m),$$

and by the observation of the preceding sentence, there are no non-zero differentials mapping into or out of this term. Therefore the map to the abutment gives an isomorphism φ: $H^{m-1}(X, \mathscr{K}_m) \to Ch^m(X, 1)$.

We claim that this isomorphism is in fact inverse to $\Psi_{m,m-1}$. For this, let $F^{\cdot}Z^m(X, \cdot\,)$ be the filtration described in 2.4. It is shown in [B] that $F^{m-1}Z^m(X, \cdot\,)/F^m(X, \cdot\,)$ is quasi-isomorphic to the complex $\coprod_{X^{m-1}} Z^1(X, \cdot\,)$ giving an exact sequence

$$H^1(F^{m-1}Z^m(X, \cdot)) \to \coprod_{X^{m-1}} Ch^1(\operatorname{Sp} k(x), 1)$$
$$\to H^0(F^m Z^m(X, \cdot)) = Z^m(X, 0).$$

From this and the construction of the spectral sequence, we see that φ can be described as follows: given an element of $H^{m-1}(X, \mathscr{K}_m)$, represent it by a sum

$$\sum (x_i, f_i) \in \coprod_{X^{m-1}} Ch^1(\operatorname{Sp} k(x), 1) = \coprod_{X^{m-1}} k(x)^*.$$

(The x_i are points of codimension $m - 1$ and the f_i are rational functions in their residue fields); then map this to the class of (Σ graph(f_i)) in $Ch^m(X, 1)$. It is thus manifest that φ is a right inverse to $\Psi_{m,m-1}$; since φ is an isomorphism it is a left inverse as well.

2.9. The maps $\Psi_{m,p}$ relate higher Chow groups to higher K-theory. The cycle map in [L] also relates higher Chow groups to higher K-theory. We can ask in what sense these maps are compatible. For example, consider the case $X = \operatorname{Spec}(k)$, where k is a field. Then both the map $\Psi_{m,0}$ and the cycle map in [L] take $Ch^m(X, m)$ to $K_m(k)$. Here we will show that the two maps agree up to a sign in the case $m = 2$. The case $m = 1$ is an easy exercise.

THEOREM. *When* $X = \mathrm{Spec}(k)$ *with* k *a field*, $\Psi_{2,0}\colon Ch^2(X,2) \to K_2(k)$ *is the same as the cycle map constructed in* [L].

Proof. We recall the cycle map $\varphi_{2,0}$ from [L]. Identify

$$X^2 = \mathrm{Spec}(k[z,x,y]/(x+y+z-1))$$

with $X \times \mathbf{A}^2$, where x and y are identified with the coordinates on $\mathbf{A}^2$. Let C consist of two copies of $X \times \mathbf{A}^2$ pasted together along the set

$$S = \{xy(1-x-y) = 0\}.$$

Then a generator for $Ch^2(X,2)$ can be represented by a prime cycle $[V]$ of dimension zero on $X \times \mathbf{A}^2$ that does not meet S. Identifying $X \times \mathbf{A}^2$ with the first copy of $X \times \mathbf{A}^2$ in C, we view the structure sheaf of V as a module on C. That module necessarily has finite projective dimension, and so determines a class in $K_0(C) \approx K_2(k) \oplus K_0(k)$. Projecting onto $K_2(k)$ gives the image of $[V]$.

Arguing as in the proof of Lemma C in 2.6, we conclude that $Ch^2(X,2)$ is generated by the classes of rational points. Let (a,b) be such a point. Then $\Psi_{2,0}(a,b) = \{a/(1-a-b), b/(1-a-b)\}$, where the curly brackets stand for the Steinberg symbol. Next we compute $\varphi_{2,0}(a,b)$.

Let g be the function on C which is identically 0 on the first copy of $X \times \mathbf{A}^2$ and equal to xyz on the second copy. Then the module associated to (a,b) is $M = \Gamma(C)/(x-a, y-b, g)$. We can resolve M as follows:

$$0 \to N \to \Gamma(C)^3 \xrightarrow{(x-a,\, y-b,\, g)} \Gamma(C) \to M \to 0.$$

where N is defined to be the kernel. The image of M in $K_2(k)$ is the same as the image of the rank two projective module N in $K_2(k)$. Let N_1 and N_2 be the restrictions of N to the two copies of $X \times \mathbf{A}^2$ in C. Then N_1 is generated by the two column vectors

$$\begin{pmatrix} y-b \\ -(x-a) \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and N_2 is generated by the two column vectors

$$\begin{pmatrix} (y-b)(ab-yz) \\ -(x-a)(ab-yz) - xyz \\ y-b \end{pmatrix}, \quad \begin{pmatrix} (y-b)(ab-az) + xyz \\ -(x-a)(ab-az) \\ -(x-a) \end{pmatrix}.$$

(To verify this, note that both vectors are in N_2, and that when combined with the column vector

$$\begin{pmatrix} (ab - yz) \\ (ab - az) \\ 1 \end{pmatrix}$$

they form a matrix whose determinant is a unit.)

On the overlap $xyz = 0$, these two pairs of generators are related by the patching matrix

$$\rho = \begin{pmatrix} ab - yz & ab - az \\ y - b & a - x \end{pmatrix}.$$

That is, N is constructed from free rank two modules on the two components of C, patched together by ρ along the overlap. The groups $K_0(C)/K_0(k)$ and $K_1(\{xyz = 0\}/K_1(k)$ are isomorphic by Karoubi-Villamayor theory, and the isomorphism takes the class of N to the class of the matrix ρ.

It remains to pass from $K_1(\{xyz = 0\})/K_1(k)$ to $K_2(k \times k)/K_2(k)$. To do so, we first perform some elementary transformations (one with rows and one with columns) to bring ρ to the more convenient form

$$\rho' = \begin{pmatrix} (ab - yz) & (bzy - yz^2)/ab^2(b + a - 1) \\ (y^2 - by)z/ab & * \end{pmatrix}.$$

where $*$ is an unpleasant expression. Without affecting the K_1 class, we can alter ρ' to

$$\rho'' = \begin{pmatrix} \rho'_{11}/(ab) & -\rho'_{12}(a + b - 1) \\ -\rho'_{21}/(a + b - 1) & \rho'_{22}ab \end{pmatrix}.$$

Then ρ'' restricts to the identity on $\{yz = 0\}$, and to an elementary matrix $\bar{\rho}''$ on the set $\{x = 0\}$. To find the corresponding class in $K_2(k)$ we must lift $\bar{\rho}''$ to the Steinberg group, and then reduce to the Steinberg groups of the two vertices $(y = 0,\ z = 1)$ and $(y = 1,\ z = 0)$.

The reader with access to a good symbolic manipulation program (or a lot of time on his hands) may verify that $\bar{\rho}''$ can be written as the following product of elementary matrices:

$$\bar{\rho}'' = e_{21}\left(\frac{b - y}{a + b - 1}\right)e_{12}\left(\frac{(b + a - 1)y}{ab}\right)e_{21}\left(\frac{ab}{(b - 1)(a + b - 1)}\right)$$

$$\times e_{12}\left(\frac{-(b - 1)^2 y}{ab^2}\right)e_{21}\left(\frac{-b}{b - 1}\right)e_{12}\left(\frac{-y}{b}\right)$$

Let s be the element of the Steinberg group $\mathrm{St}(k[x, y]/(x))$ corresponding to this presentation. It is easy to check that the map $y \to 0$ takes s to the identity in $\mathrm{St}(k)$, so we only need to check the image of s under $y \to 1$. One verifies that this image is given by

$$\langle 0, 1\rangle^{-1}\left\langle \frac{b+a-1}{ab}, \frac{ab}{(b+a-1)(b-1)}\right\rangle\left\langle \frac{1}{b}, \frac{b}{b-1}\right\rangle$$

where the pointy brackets denote the Dennis-Stein symbol. (The easiest way to verify this calculation is directly from the definition of the pointy brackets on page 8 of [DS]. Much cancellation occurs.) Now using the formulas in [DS] to convert to Steinberg symbols, we discover that this is the same as

$$\left\{\frac{b}{a+b-1}, \frac{a}{a+b-1}\right\},$$

which is $\Psi_{2,0}(a, b)$ up to a sign.

Remark. The cycle map $\varphi_{2,0}$ can be defined "cubically" instead of simplicially, by patching two copies of $X \times \mathbf{A}^2$ together along the set $xy(x-1)(y-1)$. In this case, it seems natural to expect that

$$\varphi_{2,0}^{\text{cubical}}(a, b) = \left\{\frac{a}{(a-1)(b-1)}, \frac{b}{(a-1)(b-1)}\right\}$$

up to a sign. However, a calculation similar to the one in the proof of the theorem reveals that in actuality

$$\varphi_{2,0}^{\text{cubical}}(a, b) = \left\{\frac{a-1}{a}, \frac{b}{b-1}\right\}$$

which does not appear to be the same thing. Similar maps exist for any configuration of lines that is topologically a one-sphere, so it should be possible to write down a general formula of which this and the theorem are special cases.

3. Relative Invariants

Here we introduce the relative analogues of the objects and theorems in Sections 1 and 2. Throughout, X is a regular scheme essentially of finite type over a field and Y is a regular closed subscheme. As in earlier sections, regularity can be relaxed in much of what follows.

3.1. We begin with the relative analogue of the group $Z^m(X)$. Define $Z^m_Y(X)$ to be free abelian on those closed subvarieties of X that have codimension m and meet Y properly. An obvious candidate for a relative cycle group is the kernel of the restriction map

$$Z^m_Y(X) \xrightarrow{r} Z^m(Y).$$

In Section 4 we will discover that this definition is not quite right and will have to modify it. Nevertheless, it will be convenient to have a name for this "naive" relative cycle group for the time being. Thus we define

$$Z^m_{\text{naive}}(X,Y) = \ker(r).$$

3.2. We turn now to the relative analogue of the Gersten-Quillen spectral sequence. Let $\mathscr{M}^m(Y)$ be the category of those coherent Y-modules that have codimension of support $\geq m$. Let $\mathscr{C}^m(Y)$ be the category of all complexes $C_.$ of coherent Y-modules that satisfy these three conditions:

(a) $C_.$ is bounded below.

(b) There is a surjective quasi-isomorphism $Q_. \twoheadrightarrow C_.$ where $Q_.$ is a complex of locally free coherent modules that is bounded in both directions.

(c) $C_.$ has all of its homology objects in $\mathscr{M}^m(Y)$.

Let $\mathscr{M}^m(X)$ be the category of those coherent X-modules that have codimension of support $\geq m$, and restrict to modules in $\mathscr{M}^m(Y)$. (Thus $\mathscr{M}^m(X)$ depends on Y, but we suppress this dependence in the notation.) Define $\mathscr{C}^m(X)$ to be the category of those complexes of X-modules that satisfy conditions (a) through (c) above, with $\mathscr{M}^m(Y)$ replaced by $\mathscr{M}^m(X)$ in condition (c).

In $\mathscr{C}^m(X)$ or $\mathscr{C}^m(Y)$ we define a cofibration to be a degree-wise split monomorphism and a weak equivalence to be a quasi-isomorphism of complexes. Then $\mathscr{C}^m(X)$ and $\mathscr{C}^m(Y)$ are categories with cofibrations and weak equivalences in the sense of Waldhausen [W]. By the methods of [W], we construct spectra $K(\mathscr{C}^m(X))$ and $K(\mathscr{C}^m(Y))$ whose homotopy groups are defined to be the K-groups of $\mathscr{C}^m(X)$ and $\mathscr{C}^m(Y)$.

Let i_m: $\mathscr{C}^{m+1}(Y) \to \mathscr{C}^m(Y)$ be the inclusion, let $F(i_m)$ be the homotopy fiber of i_m, and let $K(\mathscr{C}^{m/m+1}(Y)) = \Omega^{-1}(F(i_m))$. Define $K(\mathscr{C}^{m/m+1}(X))$ similarly. (Note that we jump directly to the construction of the K-theory; we do not construct a category $\mathscr{C}^{m/m+1}(X)$.) Thus there are homotopy fibrations

$$K(\mathscr{C}^{m+1}(X)) \to K(\mathscr{C}^m(X)) \to K(\mathscr{C}^{m/m+1}(X))$$

$$K(\mathscr{C}^{m+1}(Y)) \to K(\mathscr{C}^m(Y)) \to K(\mathscr{C}^{m/m+1}(Y)).$$

We construct a map of fibration sequences as follows: From Waldhausen's Approximation Theorem (see [W]; also [T, 1.9.1]), $K(\mathscr{C}^m(X))$ is homotopy equivalent to $K(\mathscr{P}^m(X))$, where $\mathscr{P}^m(X) \subset \mathscr{C}^m(X)$ is the subcategory consisting of bounded complexes of locally free sheaves. Similarly for $\mathscr{P}^m(Y)$. Thus the exact functor $\mathscr{P}^m(X) \to \mathscr{P}^m(Y)$ given by restriction induces a map $K(\mathscr{C}^m(X)) \to K(\mathscr{C}^m(Y))$. The same construction gives a compatible map $K(\mathscr{C}^{m+1}(X)) \to K(\mathscr{C}^{m+1}(Y))$ and hence a map $K(\mathscr{C}^{m/m+1}(X)) \to K(\mathscr{C}^{m/m+1}(Y))$. These fit together to give the bottom two rows of the following diagram:

$$\begin{array}{ccccc} F^{m+1} & \longrightarrow & F^m & \longrightarrow & F^{m/m+1} \\ \downarrow & & \downarrow & & \downarrow \\ K(\mathscr{C}^{m+1}(X)) & \longrightarrow & K(\mathscr{C}^m(X)) & \longrightarrow & K(\mathscr{C}^{m/m+1}(X)) \\ \downarrow & & \downarrow & & \downarrow \\ K(\mathscr{C}^{m+1}(Y)) & \longrightarrow & K(\mathscr{C}^m(Y)) & \longrightarrow & K(\mathscr{C}^{m/m+1}(Y)) \end{array}$$

The top row is constructed by taking fibers of the vertical maps. The top sequence is a fibration by the Quetzlcoatl Lemma.

For each m we get a diagram as above and a corresponding long exact homotopy sequence

$$\cdots \to \pi_{n+1}(F^{m/m+1}) \to \pi_n(F^{m+1}) \to \pi_n(F^m) \to \pi_n(F^{m/m+1}) \\ \to \pi_{n-1}(F^{m+1}) \to \cdots,$$

and these sequences fit together to form an exact couple. From this we deduce a spectral sequence

$$E_1^{p,q} = \pi_{-p-q}(F^{p/p+1}) \Rightarrow K_{-p-q}(X,Y).$$

Perhaps a word more should be said on the definition and determination of the abutment. Let $K(X) = K(\mathscr{C}^0(X))$ and $K(Y) = K(\mathscr{C}^0(Y))$. Then the homotopy groups of $K(X)$ and $K(Y)$ are the Quillen K-groups of X and Y by [W, 1.9]. (See also [T, 1.12.].) Thus there is a homotopy fibration

$$F^0 \to K(X) \to K(Y)$$

and we *define* the relative groups $K_*(X,Y)$ to be $\pi_*(F^0)$. The abutment of the spectral sequence is $\pi_{-p-q}(F^0) = K_{-p-q}(X,Y)$.

3.3. We want analogues of the maps in 1.2. Here we focus on the composition of the isomorphisms (2) and (3) from that subsection. That is, we would like to interpret the elements of $E_2^{m,-m}$ as cycle classes. Since $E_2^{m,-m}$

is a subquotient of $E_1^{m,-m}$, we would like to interpret the elements of $E_1^{m,-m}$ as cycles, for example by showing that $E_1^{m,-m} = Z^m_{\text{naive}}(X, Y)$, where the latter group is as defined in 3.1.

From the diagram of fibrations in 3.2, we have an exact sequence

$$K_1(\mathscr{C}^{m/m+1}(X)) \to K_1(\mathscr{C}^{m/m+1}(Y)) \to E_1^{m,-m} \to K_0(\mathscr{C}^{m/m+1}(X)) \to K_0(\mathscr{C}^{m/m+1}(Y))$$

To understand $E_1^{m,-m}$, we can attempt to understand the other groups in the exact sequence.

Note that $K_i(\mathscr{C}^{m/m+1}(Y)) = K_i(\mathscr{M}^m(Y)/\mathscr{M}^{m+1}(Y))$ where the group on the right is the ith Quillen K-group of the quotient category. (To see this, note that Walhausen's Approximation Theorem ([W, 1.9] or [T, 1, 12.1]) gives a homotopy equivalence of K-theory spaces $K(\mathscr{C}^m(Y)) \approx K(\mathscr{M}^m(Y))$, and that the fiber of $K(\mathscr{M}^{m+1}(Y)) \to K(\mathscr{M}^m(Y))$ can be identified with the loop space of $K(\mathscr{M}^m(Y)/\mathscr{M}^{m+1}(Y))$ by Quillen's Localization Theorem from [Q].)

Thus

$$K_0(\mathscr{C}^{m/m+1}(Y)) = Z^m(Y) \quad \text{and} \quad K_1(\mathscr{C}^{m/m+1}(Y)) = \coprod_{Y^{m-1}} k(y)^*.$$

As for the groups $K_i(\mathscr{C}^{m/m+1}(X))$, we also have

$$K_i(\mathscr{C}^{m/m+1}(X)) = K_i(\mathscr{M}^m(X))/K_i(\mathscr{M}^{m+1}(X)),$$

but the latter group is more difficult to interpret. Suppose for the moment that we could establish the following two statements:

(*) $K_0(\mathscr{C}^{m/m+1}(X)) = Z^m_Y(X)$ (as defined in 3.1);
(* *) $K_1(\mathscr{C}^{m/m+1}(X)) \to K_1(\mathscr{C}^{m/m+1}(Y))$ is onto.

Given (*) and (* *), the exact sequence allows us to conclude that $E_1^{m,-m} \approx Z^m_{\text{naive}}(X, Y)$ (as defined in 3.1).

In the next subsection, we will establish some cases in which conditions (*) and (* *) hold.

3.4. Theorem. *Suppose that either the codimension of Y in X is 1 or the dimension of Y is 0. Then conditions (*) and (* *) of 3.3 hold. Thus in either of these two cases, $E_1^{m,-m}$ is isomorphic to $Z^m_{\text{naive}}(X, Y)$.*

(In the case of codimension 1, compare this result to Lemma 1.2 of [Lev], where the categories are defined slightly differently but the analogous result holds.)

Proof. We make use of our earlier observation that

$$K_*(\mathscr{C}^{m/m+1}(X)) = K_*(\mathscr{M}^m(X)/\mathscr{M}^{m+1}(X))$$

where the latter is the Quillen K-group of the quotient category. Let $\mathscr{M}^m_{\text{full}}(X)$ be the category of all coherent X-modules with codimension of support $\geq m$. Write $\mathscr{M}_{\text{fl}}(R)$ for the category of all finitely generated modules of finite length over the ring R. We have

$$\mathscr{M}^m_{\text{full}}(X)/\mathscr{M}^m(X) = \begin{cases} 0 & \text{if } m = 0, \\ \coprod_{Y^{m-1}} \mathscr{M}_{\text{fl}}(\mathscr{O}_{X,y}) & \text{if } m > 0 \text{ and } \operatorname{cod}_X(Y) = 1, \\ \mathscr{M}^m(\mathscr{O}_{X,Y}) & \text{if } m > 0 \text{ and } \dim(Y) = 0, \end{cases}$$

by applying [S, Thm. 5.11]. (We have written $\mathscr{M}^m(\mathscr{O}_{X,Y})$ for $\mathscr{M}^m(\operatorname{Spec}(\mathscr{O}_{X,Y}))$.
First assume $m > 0$ and $\operatorname{cod}_X(Y) = 1$. Then we have a diagram:

$$\begin{array}{ccccc}
\Omega K(\mathscr{M}^{m/m+1}(X)) & \longrightarrow & \Omega K(\mathscr{M}^{m/m+1}_{\text{full}}(X)) & \longrightarrow & \coprod_{Y^{m-1}} \Omega K(\mathscr{M}_{\text{fl}}(\mathscr{O}_{X,y})) \oplus \coprod_{Y^m} K(\mathscr{M}_{\text{fl}}(\mathscr{O}_{X,y})) \\
\downarrow & & \downarrow & & \downarrow \\
K(\mathscr{M}^{m+1}(X)) & \longrightarrow & K(\mathscr{M}^{m+1}_{\text{full}}(X)) & \longrightarrow & \coprod_{Y^m} K(\mathscr{M}_{\text{fl}}(\mathscr{O}_{X,y})) \\
\downarrow & & \downarrow & & \downarrow \\
K(\mathscr{M}^m(X)) & \longrightarrow & K(\mathscr{M}^m_{\text{full}}(X)) & \longrightarrow & \coprod_{Y^{m-1}} K(\mathscr{M}_{\text{fl}}(\mathscr{O}_{X,y}))
\end{array}$$

(The categories appearing in the top row are the obvious quotients).

The bottom two rows and first two columns are fibrations by Quillen's localization theorem. The rightmost column is a fibration in which the second map induces zero on all homotopy groups. It follows that the top row is a fibration, yielding a long exact homotopy sequence

$$\begin{aligned} K_1(\mathscr{M}^{m/m+1}(X)) \to \coprod_{X^m} k(x)^* \to \coprod_{Y^{m-1}} k(y)^* \oplus \coprod_{Y^m} \mathbf{Z} \\ \to K_0(\mathscr{M}^{m/m+1}(X)) \to \coprod_{X^m} \mathbf{Z} \to \coprod_{Y^{m-1}} \mathbf{Z} \end{aligned}$$

from which we get

$$\begin{aligned} K_1(\mathscr{M}^{m/m+1}(X)) \to \coprod_{(X-Y)^m} k(x)^* \xrightarrow{f} \coprod_{Y^m} \mathbf{Z} \to K_0(\mathscr{M}^{m/m+1}(X)) \\ \to \coprod_{(X-Y)^m} \mathbf{Z} \to 0. \end{aligned}$$

The map f is onto by an application of Chow's moving lemma. Indeed, the generators of $\coprod_{Y^m}\mathbf{Z}$ are closed subvarieties, each of which can be moved to a cycle that meets Y properly. The functions that do the moving constitute an inverse image for the cycle under f. This demonstrates $(*)$. For $(**)$, let $y \in Y^m$. A typical generator of $K_1(\mathscr{C}^{m/m+1}(Y))$ is a unit $u \in k(y)^*$. Choose any codimension m subvariety $V \subset X$ that meets Y in $\bar{y} \cup W_1 \cup \cdots \cup W_r$ for some codimension m subvarieties W_j of Y. Let R be the coordinate ring of V semilocalized at y and the generic points of the W_i. Then u lifts to a unit in R, which can be viewed as an element of $\ker(k(V)^* \to \coprod_{Y^m}\mathbf{Z})$. By the Chinese Remainder Theorem, the lifting can be chosen to restrict to 1 on each W_i. Then by the exact sequence, u lifts to $K_1(\mathscr{C}^{m/m+1}(Y))$, establishing $(**)$.

Next assume $m > 0$ and $\dim(Y) = 0$. Then we have a diagram:

$$\begin{array}{ccccc}
K(\mathscr{M}^{m+1}(X)) & \longrightarrow & K(\mathscr{M}_{\text{full}}^{m+1}(X)) & \longrightarrow & K(\mathscr{M}^{m+1}(\mathscr{O}_{X,Y})) \\
\downarrow & & \downarrow & & \downarrow \\
K(\mathscr{M}^{m}(X)) & \longrightarrow & K(\mathscr{M}_{\text{full}}^{m}(X)) & \longrightarrow & K(\mathscr{M}^{m}(\mathscr{O}_{X,Y})) \\
\downarrow & & \downarrow & & \downarrow \\
K(\mathscr{M}^{m/m+1}(X)) & \longrightarrow & K(\mathscr{M}_{\text{full}}^{m/m+1}(X)) & \longrightarrow & K(\mathscr{M}^{m/m+1}(\mathscr{O}_{X,y}))
\end{array}$$

The bottom row is then a fibration, giving the exact sequence

$$K_1(\mathscr{M}^{m/m+1}(X)) \to \coprod_{X^m} k(x)^* \to \coprod_{\substack{x \in X^m \\ y \in \bar{x}}} k(x)^* \to K_0(\mathscr{M}^{m/m+1}(X)) \to \coprod_{X^m} \mathbf{Z} \to \coprod_{\substack{x \in X^m \\ y \in \bar{X}}} \mathbf{Z}$$

from which we get

$$K_1(\mathscr{M}^{m/m+1}(X)) \to \coprod_{\substack{x \in X^m \\ y \notin \bar{x}}} k(x)^* \to 0 \to K_0(\mathscr{M}^{m/m+1}(X)) \to \coprod_{\substack{x \in X^m \\ y \notin \bar{x}}} \mathbf{Z} \to 0.$$

The rightmost four terms establish $(*)$. Note that $(**)$ is vacuous.

Finally, suppose that $m = 0$. Then $\mathcal{M}^0(X) = \mathcal{M}^0_{\text{full}}(X)$. In either case ($\text{cod}(Y) = 1$ or $\dim(Y) = 0$) we get a square

$$\begin{array}{ccccc} \Omega K(\mathcal{M}^{0/1}(X)) & \longrightarrow & \Omega K(\mathcal{M}^{0/1}_{\text{full}}(X)) & \longrightarrow & K(\mathcal{M}^1(\mathcal{O}_{X,Y}) \\ \downarrow & & \downarrow & & \downarrow \\ K(\mathcal{M}^1(X)) & \longrightarrow & K(\mathcal{M}^1_{\text{full}}(X)) & \longrightarrow & K(\mathcal{M}^1(\mathcal{O}_{X,Y}) \\ \downarrow & & \downarrow & & \downarrow \\ K(\mathcal{M}^0(X)) & \longrightarrow & K(\mathcal{M}^0_{\text{full}}(X)) & \longrightarrow & * \end{array}$$

in which the top row is a fibration because everything else is.

If $\text{cod}(Y) = 1$, then $K(\mathcal{M}^1(O_{X,Y}) = K(k(Y))$ and we get

$$K_1(\mathcal{M}^{0/1}(X)) \to K_1(k(X)) \to K_0(k(Y)) \to K_0(\mathcal{M}^{0/1}(X)) \to \mathbf{Z} \to 0.$$

whence $K_0(\mathcal{M}^{0/1}(X)) = \mathbf{Z}$ and $K_1(\mathcal{M}^{0/1}(X))$ maps onto $\mathcal{O}^*_{X,Y}$, which in turn maps onto $k(Y)^* = K_1(\mathcal{M}^{0/1}(Y))$. This gives $(*)$ and $(**)$.

If $\dim(Y) = 0$, then $K_0(\mathcal{M}^1(O_{X,Y})) = k(X)/\mathcal{O}^*_{X,Y}$ and we get

$$K_1(\mathcal{M}^{0/1}(X)) \to \underset{\substack{\| \\ k(X)^*}}{K_1(k(X))} \to k(X)^*/\mathcal{O}^*_{X,Y} \to K_0(\mathcal{M}^{0/1}(X)) \to \mathbf{Z} \to 0$$

so that $K_0(\mathcal{M}^{0/1}(X)) = \mathbf{Z}$ and $K_1(\mathcal{M}^{0/1}(X))$ maps onto $\mathcal{O}^*_{X,Y}$ (and consequently onto $k(Y)^*$) as needed.

3.5. *Remark.* Theorem 3.4 is substantially generalized by Theorem 1 of [L3], which was written after this paper was submitted.

3.6. Next we introduce the relative analogue of the sheaf $\mathcal{K}_m$. Write $\mathcal{K}_m(X)$ for the K-theory sheaf on X (that is, $\mathcal{K}_m(X)$ is the sheaf that was called $\mathcal{K}_m$ in 1.1.) Write $\mathcal{K}_m(Y)$ for the analogous sheaf on Y. Let i: $Y \to X$ be the inclusion. Let $\mathbf{K}_m$ be the (cohomological) complex of sheaves on X:

$$\mathbf{K}_m = \mathcal{K}_m(X) \to i_* \mathcal{K}_m(Y)$$

with $\mathcal{K}_m(X)$ in the degree zero and $i_* \mathcal{K}_m(Y)$ in degree 1. $\mathbf{K}_m$ will serve as a relative analogue of $\mathcal{K}_m$.

3.7. We shall construct a map $E_2^{p,-q} \to \mathbf{H}^p(X, \mathbf{K}_q)$ that generalizes the classical isomorphism $E_2^{p,-q}(X) \approx H^p(X, \mathcal{K}_q)$ where $E_2^{p,-q}(X)$ is the E_2-term in the Gersten-Quillen spectral sequence. In particular, when $m = p = q$, this gives the relative analogue of the isomorphism (3) in subsection 1.1.

Write the Gersten-Quillen resolutions of the sheaves $\mathscr{K}_m(X)$ and $\mathscr{K}_m(Y)$ as

$$\begin{array}{l}\mathscr{K}_m(X) \longrightarrow \mathscr{G}_m^{0,0} \longrightarrow \mathscr{G}_m^{0,1} \longrightarrow \mathscr{G}_m^{0,2} \longrightarrow \cdots \\ i_*\mathscr{K}_m(Y) \longrightarrow \mathscr{G}_m^{1,0} \longrightarrow \mathscr{G}_m^{1,1} \longrightarrow \mathscr{G}_m^{1,2} \longrightarrow \cdots\end{array}$$

Also, take injective resolutions

$$\begin{array}{l}\mathscr{K}_m(X) \longrightarrow \mathscr{I}_m^{0,0} \longrightarrow \mathscr{I}_m^{0,1} \longrightarrow \mathscr{I}_m^{0,2} \longrightarrow \cdots \\ \mathrm{i}_*\mathscr{K}_m(Y) \longrightarrow \mathscr{I}_m^{1,0} \longrightarrow \mathscr{I}_m^{1,1} \longrightarrow \mathscr{I}_m^{1,2} \longrightarrow \cdots\end{array}$$

There are maps of resolutions $\mathscr{G}_m^{j,*} \to \mathscr{I}_m^{j,*}$ for $j = 0, 1$. Also, the map $\mathscr{K}_m(X) \to i_*\mathscr{K}_m(Y)$ induces a map of resolutions $\mathscr{I}_m^{0,*} \to \mathscr{I}_m^{1,*}$.

In the notation of subsection 3.2, write

$$E_1^{p,q}(X,Y) = \pi_{-p-q}(F^{p/p+1})$$

(which has been denoted simply $E_1^{p,q}$ until now),

$$\hat{E}_1^{p,q}(X) = K_{-p-q}(\mathscr{C}^{p/p+1}(X)) \quad \text{and} \quad \hat{E}_1^{p,q}(Y) = K_{-p-q}(\mathscr{C}^{p/p+1}(Y)).$$

Also write $E_1^{p,q}(X) = \Gamma(\mathscr{G}_{-q}^{0,p})$ and $E_1^{p,q}(Y) = \Gamma(\mathscr{G}_{-q}^{1,p})$ for the E_1 terms in the Gersten-Quillen spectral sequences for X and Y. There are obvious maps of complexes

$$\begin{array}{ccccc} E_1^{*,q}(X,Y) & \longrightarrow & \hat{E}_1^{*,q}(X) & \longrightarrow & \hat{E}_1^{*,q}(Y) \\ & & \downarrow & & \downarrow \approx \\ & & E_1^{*,q}(X) & & E_1^{*,q}(Y) \\ & & \downarrow & & \downarrow \\ & & \Gamma(\mathscr{I}_{-q}^{0,*}) & \longrightarrow & \Gamma(\mathscr{I}_{-q}^{0,*}) \end{array}$$

and the composition in the first row is zero. Thus there is an induced map from $E_1^{*,q}(X,Y)$ to the shifted mapping cone of $[\hat{E}_1^{*,q}(X) \to \hat{E}_1^{*,q}(Y)]$ and from there to the shifted mapping cone of $[\Gamma(\mathscr{I}_{-q}^{0,*}) \to \Gamma(\mathscr{I}_{-q}^{0,*})]$. Since the latter computes the hypercohomology $\mathbf{H}^p(X, \mathbf{K}_{-q})$, there is an induced map $E_2^{p,q}(X,Y) \to \mathbf{H}^p(X, \mathbf{K}_{-q})$, as promised.

3.8. We shall construct a map $Z^m_{\text{naive}}(X,Y) \to \mathbf{H}^m(X, \mathbf{K}_m)$ that is the analogue of the isomorphism (2) in 1.2. Suppose that we are given a cycle $z \in Z^m_{\text{naive}}(X,Y)$, and write φ for the support of z. Let $\mathscr{F}_m^0 = \mathscr{K}_m(X)$ and

$\mathcal{F}_m^1 = i_* \mathcal{K}_m(Y)$, and consider the spectral sequence of hypercohomology

$$E_1^{p,q} = H_\varphi^q(X, \mathcal{F}_m^p) \Rightarrow \mathbf{H}_\varphi^{p+q}(X, \mathbf{K}_m)$$

From the Gersten-Quillen resolution, it is easy to check that

$$E_1^{p,q} = \begin{cases} 0 \text{ when } q < m \\ \text{free abelian on the components of } z \text{ when } q = m \text{ and } p = 0 \\ \text{free abelian on the components of } z \cap Y \text{ when } q = m \text{ and } p = 1. \end{cases}$$

From this we easily compute that

$$H_\varphi^m(X, K_m) = \ker\left(\coprod_{\substack{\text{components} \\ \text{of } z}} \mathbf{Z} \to \coprod_{\substack{\text{components} \\ \text{of } z \cap Y}} \mathbf{Z} \right)$$

and in particular that there is a canonical class representing z itself in $\mathbf{H}_\varphi^m(X, \mathbf{K}_m)$. Now the natural map $\mathbf{H}_\varphi^m(X, \mathbf{K}_m) \to \mathbf{H}^m(X, \mathbf{K}_m)$ carries this class to the image of z in $\mathbf{H}^m(X, \mathbf{K}_m)$.

3.9. To summarize: Writing $E^{p,q}$ for the terms in the spectral sequence of 3.3, we have constructed maps

$$\begin{array}{ccc} E_1^{m,-m} & \xrightarrow[\text{sometimes iso (3.4)}]{\text{exists(3.5)}} & Z^m_{\text{naive}}(X,Y) \\ {\scriptstyle \text{from construction of spectral sequence (3.3)}} \downarrow & & \downarrow {\scriptstyle (3.8)} \\ E_2^{m,-m} & \xrightarrow[(3.7)]{} & \mathbf{H}^m(X, \mathbf{K}_m) \end{array}$$

Moreover, it is an easy exercise to verify that the square commutes. The arrows in the square generalize the maps (2) and (3) in subsection 1.2, and the map to the abutment in the spectral sequence is a generalization of map (1). Also, the bottom arrow is defined more generally in the case where the two indices differ.

The square suggests that one could hope to define $Ch^m(X,Y) = \mathbf{H}^m(X, \mathbf{K}_m)$ and to prove that this is a quotient of $Z^m_{\text{naive}}(X,Y)$. However, this is not the course that we will take. In the next section, we will give a more concrete definition for the relative Chow group $Ch^m(X,Y)$ as a quotient of a relative cycle group $Z^m(X,Y)$. $Z^m(X,Y)$ contains $Z^m_{\text{naive}}(X,Y)$ as a subgroup. Then we will show that the right-hand map in the square above extends to an isomorphism $Ch^m(X,Y) \approx \mathbf{H}^m(X, \mathbf{K}_m)$.

4. Relative Chow groups

In this section we will define the relative Chow groups and the higher relative Chow groups. The assumptions of Section 3 remain in force.

4.1 DEFINITION. Let $Z^m(X, \cdot)$ denote the Bloch complex of 2.1. Let $Z^m_Y(X, \cdot) \subset Z^m(X, \cdot)$ be the subcomplex consisting of cycles that meet all of the faces of $Y \times \Delta^m$ properly, where $\Delta^m \subset \mathbf{A}^{m+1}$ is the set $\{\Sigma t_i = 1\}$. Then by [B, 2.3], the inclusion $Z^m_Y(X, \cdot) \subset Z^m(X, \cdot)$ is a quasi-isomorphism. There is a restriction map r: $Z^m_Y(X, \cdot) \to Z^m(Y, \cdot)$, and we define

$$Z^m(X, Y, \cdot) = (\mathrm{Cone}(r))[-1].$$

Then we set $Ch^m(X, Y, n) = \pi_n(Z^m(X, Y, \cdot))$. $Ch^m(X, Y, 0)$ will be abbreviated as just $Ch^m(X, Y)$. We call $Ch^m(X, Y)$ the mth *relative Chow group* of the pair (X, Y), and the groups $Ch^m(X, Y, n)$ $(n > 0)$ are called the *higher relative Chow groups*.

4.2. By construction, there is a long exact sequence

$$\begin{aligned} \cdots &\to Ch^m(X, Y, m) \to Ch^m(X, m) \to Ch^m(Y, m) \\ &\to Ch^m(X, Y, m-1) \to \cdots \end{aligned}$$

4.3. As in 3.7, let $\mathscr{G}^{0,*}_m$ and $\mathscr{G}^{1,*}_m$ be the Gersten resolutions of $\mathscr{K}_m(X)$ and $i_*\mathscr{K}_m(Y)$. Let $\mathscr{I}^{0,*}_m$ and $\mathscr{I}^{1,*}_m$ be injective resolutions. The map $\mathscr{K}_m(X) \to i_*\mathscr{K}_m(Y)$ induces a map of complexes $\mathscr{I}^{0,*}_m \to \mathscr{I}^{1,*}_m$. From 2.3, we have maps of complexes $Z^m(X, *) \to \Gamma(\mathscr{G}^{0,*}_m)$ and $Z^m(Y, *) \to \Gamma(\mathscr{G}^{1,*}_m)$. These induce a commutative square

$$\begin{array}{ccc} Z^m_Y(X, \cdot) & \longrightarrow & Z^m(Y, \cdot) \\ \downarrow & & \downarrow \\ \Gamma(\mathscr{I}^{0,*}_m) & \longrightarrow & \Gamma(\mathscr{I}^{1,*}_m) \end{array}$$

and hence a map between the mapping cones of the two horizontal arrows. Taking cohomology of these mapping cones, we get a map

$$\Psi_{m,p}: Ch^m(X, Y, m-p) \to \mathbf{H}^p(X, \mathbf{K}_m).$$

When Y is the empty set, this reduces to the same map $\Psi_{m,p}$ constructed in 2.3 above.

4.4. In general, we do not expect $\Psi_{m,p}$ to be an isomorphism for the higher relative Chow groups. However, for the lower Chow groups, it is an isomorphism:

THEOREM (Bloch's Formula). $\Psi_{m,m}$: $Ch^m(X,Y) \to \mathbf{H}^m(X, \mathbf{K}_m)$ *is an isomorphism.*

Proof. From the construction of 4.3 we have a commutative diagram

$$\begin{array}{ccccccccc} Ch^m(X,1) & \longrightarrow & Ch^m(Y,1) & \longrightarrow & Ch^m(X,Y) & \longrightarrow & Ch^m(X) & \longrightarrow & Ch^m(Y) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^{m-1}(X,\mathcal{K}_1) & \longrightarrow & H^{m-1}(Y,\mathcal{K}_1) & \longrightarrow & \mathbf{H}^m(X,\mathbf{K}_m) & \longrightarrow & H^m(X,\mathcal{K}_m) & \longrightarrow & H^m(Y,\mathcal{K}_m) \end{array}$$

in which all vertical arrows except the middle one are isomorphisms by 2.4. The five lemma completes the proof.

4.4. We want to give a geometric interpretation of the elements of $Ch^m(X,Y)$. By construction, $Ch^m(X,Y)$ is the homology of the complex

$$Z^m(Y,2) \oplus Z^m_Y(X,1) \xrightarrow[\begin{pmatrix} d & -r \\ 0 & d \end{pmatrix}]{} Z^m(Y,1) \oplus Z^m_Y(X) \xrightarrow[(d,r)]{} Z^m(Y)$$

where the differential d comes from the complex $Z^m(-,\cdot)$ and r is restriction. Set $Z^m(X,Y) = \text{kernel}(Z^m(Y,1) \oplus Z^m_Y(X) \to Z^m(Y))$, so that $Ch^m(X,Y)$ is a quotient of $Z^m(X,Y)$. Thus we can think of each class in $Ch^m(X,Y)$ as being represented by a pair consisting of a cycle z on X and a choice of trivialization for $z|_Y$ in $Ch^m(Y)$.

Let $Z^m_{\text{naive}}(X,Y)$ be as in 3.1. Then there is an inclusion $Z^m_{\text{naive}}(X,Y) \to Z^m(X,Y)$ given by $z \to (0, z)$.

4.5. We can easily compute the relative and relative higher Chow groups in codimension 1.

THEOREM.

$$Ch^1(X,Y,m) = \begin{cases} \operatorname{Pic}\left(X \coprod_Y X\right) \Big/ \operatorname{Pic}(X) & \textit{if } m = 0, \\ \ker(\Gamma\mathcal{O}^*_X) \to \Gamma(\mathcal{O}^*_Y) & \textit{if } m = 1, \\ 0 & \textit{if } m > 1. \end{cases}$$

Proof. By 4.3, $Ch^1(X,Y,0) = Ch^1(X,Y) = \mathbf{H}^1(X,\mathbf{K}_1)$. A straightforward Cech calculation identifies the latter with $\operatorname{Pic}(X \coprod_Y X)/\operatorname{Pic}(X)$. The remaining two statements follow from the exact sequence of 4.2, together with Bloch's theorem [B, 6.1] that $Ch^1(X,m)$ is $\Gamma(\mathcal{O}_X^*)$ for $m = 1$ and 0 for $m > 1$.

4.6. In the spirit of the map (1) in 1.2, one expects a filtration on higher relative K-theory such that $gr^m K_n(X,Y) \approx Ch^m(X,Y,n)$ up to torsion. Karoubi-Villamayor calculations show that

$$K_n(X,Y) \approx K_n\Big(X \coprod_Y X\Big)\Big/K_n(X) \approx K_n\Big(X \coprod_{(Y\times 0)} (Y \times \mathbf{A}^1) \coprod_{(Y\times 1)} X\Big)\Big/K_n(X).$$

Thus 4.5 can be viewed as a first result in this direction. In general, I expect that patching techniques such as those of [L] can be used to define a cycle map. For example, consider the case $n = 0$. Then a class in $Ch^m(X,Y,0) = Ch^m(X,Y)$ can be represented by the difference of two positive cycles $z_+ - z_-$ on X, together with a cycle w on $Y \times \mathbf{A}^1$ such that

$$w \cdot (Y \times 0) = z_+ \cdot Y \quad \text{and} \quad w \cdot (Y \times 1) = z_- \cdot Y.$$

One hopes to represent these cycles by modules that can be patched together to give a class in

$$K_0\Big(X \coprod_{(Y\times 0)} (Y \times \mathbf{A}^1) \coprod_{(Y\times 1)} X\Big).$$

Actually, for reasons that will be clear to readers of [L], one only expects that to work for classes in a certain subgroup of $Ch^m(X,Y,n)$. I expect to return to this subject in a forthcoming paper.

4.7. Let $B \subset A \subset X$ be closed inclusions. One can ask when there is an excision isomorphism $Ch^m(X,A) \approx Ch^m(X-B, A-B)$. A simple counterexample is given by $\{0\} \subset \mathbf{A}^1 \subset \mathbf{A}^2$, with $m = 1$. In this case a quick calculation with the exact sequence of 4.2 shows that

$$Ch^1(\mathbf{A}^2, \mathbf{A}^1) = 0$$

but

$$Ch^1(\mathbf{A}^2 - \{0\}, \mathbf{A}^1 - \{0\}) = \mathbf{Z}.$$

Of course this failure of excision is due to the failure of excision for K_0 in the same example.

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