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CLOSURE PROPERTIES OF THE CLASS OF UNIFORM SWEEPING-OUT TRANSFORMATIONS

BY

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A measure-preserving transformation $(S\colon X, \mu)$ on a probability space is *uniform sweeping-out* if for any set A of positive mass and any ε there exists N such that: Any collection $\mathbf{K}$ of integers will satisfy

$$\mu\Big(\bigcup_{k \in \mathbf{K}} S^k A\Big) > 1 - \varepsilon$$

if $\sharp\mathbf{K} \geq N$. Nat Friedman introduced this property in [F]. Our goal here is to affirmatively answer a question of Friedman by showing that the class of uniform sweeping-out transformations is closed under countable cartesian product. The proof is a second application of the conditional expectation argument of [K] followed by a counting argument. I am indebted to Nat Friedman and Dan Rudolph who pointed out that uniform sweeping-out has a "mixinglike" characterization. This provoked the "lightly-mixing" characterization which is (C0) below and suggested dusting off the argument which shows that the class of lightly-mixing maps is closed under cartesian product. It is not known (in the category of weak-mixing transformations) whether uniform sweeping-out is implied by the existence of a dense family of sets A each of which sweeps-out uniformly.

Our cartesian product result appears now, rather than in 1988 when it was done, because it is now known that uniform sweeping-out is strictly weaker than mixing. (That mixing implies u.s.o appears in [F].) Terry Adams [A] has recently announced that the lightly mixing example of [F, K] has the stronger uniform sweeping-out property. Yet it is not mixing; indeed, not even partial-mixing.

A "lightly-mixing" characterization. Each of the following two properties is equivalent to uniform sweeping-out. For a $\beta \in [0, 1]$ let $\mathrm{Indices}_\beta(A, B)$ represent the set of indices k satisfying $\mu(S^k A \cap B) \leq \beta$. The function

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Zero$(\cdot,\cdot)$ below is a uniform bound on the cardinality of such a set of indices. In the sequel δ and ε are numbers in $(0,1)$. The phrase $a \stackrel{\triangleleft}{=} b$ means that the expression b defines the (new) symbol a.

(C0) *For any set A of positive mass, any ε, there exists $M \stackrel{\triangleleft}{=} \mathrm{Zero}(A,\varepsilon)$ such that for any set B with $\mu(B) \geq \varepsilon$,*

$$\sharp \mathrm{Indices}_0(A,B) < M.$$

If S satisfies (C0) then, given any collection $\mathbf{K}$ with $\sharp\mathbf{K} \geq M$, let B be the complement of the union $\bigcup_{k\in\mathbf{K}} S^k A$. Were $\mu(B)$ at least ε we could apply (C0) to obtain a contradiction. Hence any M iterates of A sweep out more than $1-\varepsilon$ of the space.

A similar argument shows the converse, that uniform sweeping-out implies (C0).

(C1) *For any set A of positive mass, any ε, there exists $M \stackrel{\triangleleft}{=} \mathrm{Small}(A,\varepsilon)$ and positive number $\delta \stackrel{\triangleleft}{=} \mathrm{Size}(A,\varepsilon)$ such that for any set B with $\mu(B) \geq \varepsilon$,*

$$\sharp \mathrm{Indices}_\delta(A,B) < M.$$

Evidently (C1) implies (C0). Conversely, fix ε, A and $M \stackrel{\triangleleft}{=} \mathrm{Zero}(A,\varepsilon/2)$. Set $\delta \stackrel{\triangleleft}{=} \varepsilon/2M$. Fix any B of mass at least ε. Suppose there were a collection $\mathbf{K}$, $\sharp\mathbf{K} = M$, of indices k such that $\mu(S^kA \cap B) \leq \delta$. Then the difference set

$$B' \stackrel{\triangleleft}{=} B \sim \bigcup_{k\in\mathbf{K}} S^k A$$

has mass at least $\varepsilon/2$. Yet $\mathrm{Indices}_0(A,B')$ contains $\mathbf{K}$, a contradiction. We conclude that the quantity $\mathrm{Small}_\delta(A,\varepsilon)$ is dominated by M.

Remark. For the properties above, to emphasize the dependence on the transformation S we may write $\mathrm{Zero}(A,\varepsilon;S)$, etc.

The class of uniform sweeping-out transformations is evidently closed under powers and roots since

$$\mathrm{Zero}(A,\varepsilon;S^n) \leq \mathrm{Zero}(A,\varepsilon;S) \leq |n| \cdot \mathrm{Zero}(A,\varepsilon;S^n)$$

for any non-zero integer n.

Cartesian product. Fix $(S\colon X,\mu)$ and $(T\colon \hat{X},\hat{\mu})$, two uniform sweeping-out transformations. Our goal is to show that $S\times T$ is uniform sweeping-out by showing it to satisfy (C0). Fix some set $\mathbf{V} \subset X \times \hat{X}$ of positive mass. We shall compute an upper bound for

$$\text{Zero}(\mathbf{V}, 2\varepsilon; S \times T)$$

in terms of Small(, ε; S) and Small(, ε; T).

Given a point $z \in X$ let $\mathbf{V}_z$ denote the cross-section of $\mathbf{V}$ above z; thus $\mathbf{V}_z$ is the subset of $\hat{X}$ such that $\{z\} \times V_z$ equals $[\{z\} \times \hat{X}] \cap \mathbf{V}$. By standard measurability arguments, the following holds for μ-a.e. z. Set $\hat{A} \stackrel{\triangleleft}{=} \mathbf{V}_z$. Then for any positive $\hat{\delta}$ the set

$$V \stackrel{\triangleleft}{=} \left\{x \;\middle|\; \hat{\mu}\left(\mathbf{V}_x \Delta \hat{A}\right) \leq \hat{\delta}\right\} \tag{1}$$

has positive μ-mass. Consider z and $\hat{A}$ as henceforth fixed. Define the quantities

$$\hat{M} \stackrel{\triangleleft}{=} \text{Small}(\hat{A}, \varepsilon; T) \quad \text{and} \quad \hat{\delta} \stackrel{\triangleleft}{=} \text{Size}(\hat{A}, \varepsilon; T).$$

For this $\hat{\delta}$, define V as in (1). Finally, set

$$M \stackrel{\triangleleft}{=} \text{Small}(V, \varepsilon; S) \quad \text{and} \quad \delta \stackrel{\triangleleft}{=} \text{Size}(V, \varepsilon; S).$$

Wishing to establish (C0) for $S \times T$, it suffices to show that

$$\text{Zero}(V, 2\varepsilon; S \times T) \leq M + \hat{M}/\delta.$$

Fix any set $\mathbf{W} \subset X \times \hat{X}$ with mass at least 2ε. Set

$$W \stackrel{\triangleleft}{=} \left\{x \;\middle|\; \hat{\mu}(\mathbf{W}_x) \geq \varepsilon\right\}$$

and note that $\mu(W) \geq \varepsilon$ follows by a Fubini argument. Define a function $f\colon \mathbf{Z} \to [0, 1]$ by

$$f(k) \stackrel{\triangleleft}{=} \mu\left\{x \in W \;\middle|\; \hat{\mu}\left(T^k\hat{A} \cap \mathbf{W}_x\right) \leq \hat{\delta}\right\}.$$

This function measures the probability that a fiber $\mathbf{W}_x$ has k in its bad set $\text{Indices}_{\hat{\delta}}(\hat{A}, \mathbf{W}_x)$. Let $\mathbf{1}[\cdot]$ denote the Dirac function where $\mathbf{1}[\mathbf{true}] = 1$ and

1[false] $= 0$. By Fubini, the sum $\sum_{k \in \mathbf{Z}} f(k)$ equals

$$\sum_k \int_W \mathbf{1}\left[\hat{\mu}\left(T^k \hat{A} \cap \mathbf{W}_x\right) \leq \hat{\delta}\right] d\mu(x)$$

$$= \int_W \sum_k \mathbf{1}\left[\hat{\mu}\left(T^k \hat{A} \cap \mathbf{W}_x\right) \leq \hat{\delta}\right] d\mu(x)$$

$$\leq \int_W \sharp \operatorname{Indices}_{\hat{\delta}}\left(\hat{A}, \mathbf{W}_x\right) d\mu(x) \leq \mu(W) \cdot \hat{M}.$$

This yields the inequality

$$\sum_{k \in \mathbf{Z}} f(k) \leq \hat{M}$$

whose usefulness arises from the fact that although $f(\cdot)$ depends on the set $\mathbf{W}$, the bound $\hat{M}$ does not.

Counting the set of bad k. Suppose k is such that $[S \times T]^k \mathbf{V} \cap \mathbf{W}$ has zero mass. For μ-a.e. x then $\hat{\mu}(T^k(\mathbf{V}_{S^{-k}x}) \cap \mathbf{W}_x)$ equals zero. Thus if $x \in S^k V$ then

$$\hat{\mu}\left(T^k \hat{A} \cap \mathbf{W}_x\right) \leq \hat{\delta} \tag{2}$$

by (1). In particular, (2) holds for every $x \in S^k V \cap W$. Thus $f(k) \geq \mu(S^k V \cap W)$. This last quantity will exceed δ if k is chosen outside of $\mathbf{K} \stackrel{\triangleleft}{=} \operatorname{Indices}_\delta(V, W; S)$. As a consequence

$$\sharp(\operatorname{Indices}_0(\mathbf{V}, \mathbf{W}; S \times T) \sim \mathbf{K}) \leq \sum_{k \in \mathbf{Z}} \frac{f(k)}{\delta}.$$

Since the righthand quantity is dominated by $\hat{M}/\delta$ we may conclude that

$$\sharp \operatorname{Indices}_0(\mathbf{V}, \mathbf{W}; S \times T) \leq \sharp \mathbf{K} + \hat{M}/\delta \leq M + \hat{M}/\delta$$

as desired. □

Countable Cartesian products. In order to pass from finite to countable cartesian products we need to show that the class "Uniform Sweeping-out" is closed under inverse limits.

Given $(T\colon X, \mu)$ and a factor algebra $\mathscr{F}$ recall that the *conditional probability* function $\mathscr{P}[\cdot|\mathscr{F}]$ is canonically defined by the equality

$$\int_F \mathscr{P}[B|\mathscr{F}](x)\,d\mu(x) = \mu(B \cap F)$$

for all $F \in \mathscr{F}$ and measurable B.

INVERSE LIMIT LEMMA. *Given $(T\colon X, \mu)$ and an increasing tower $\mathscr{F}_1 \subset \mathscr{F}_2 \subset \ldots$ of factor algebras whose join is the entire σ-algebra. Then*

$$T|_{\mathscr{F}_n} \text{ uniform sweeping-out for all } n \quad \Rightarrow T \text{ uniform sweeping-out.}$$

Proof. Fix ε and a set A of positive mass. Pick $\mathscr{F} \in \{\mathscr{F}_n\}_n$ sufficiently far out in the sequence that A is nearly $\mathscr{F}$-measurable: Choose it so that $\mu(F)$ is positive, where

$$F \stackrel{\triangleleft}{=} \Big\{x \;\Big|\; \mathscr{P}[A|\mathscr{F}](x) > 1 - \varepsilon\Big\}.$$

Let N be the constant arising from the uniform sweeping-out of $T|_{\mathscr{F}}$; thus for any collection $\sharp\mathbf{K} \geq N$ of integers, $\mu(\mathbf{F}) > 1 - \varepsilon$ where $\mathbf{F}$ denotes the union $\bigcup_{k \in \mathbf{K}} T^k F$. Let $\mathbf{A} \stackrel{\triangleleft}{=} \bigcup_{k \in \mathbf{K}} T^k A$. Consider a point $x \in \mathbf{F}$, say, $x \in T^k F$. Then

$$\mathscr{P}[\mathbf{A}|\mathscr{F}](x) \geq \mathscr{P}[T^k A|\mathscr{F}](x) = \mathscr{P}[A|\mathscr{F}](T^{-k}x) > 1 - \varepsilon$$

where the last inequality follows from the definition of F. Consequently

$$\begin{aligned}\mu\Big(\bigcup_{k \in \mathbf{K}} T^k A\Big) &= \int \mathscr{P}[\mathbf{A}|\mathscr{F}]\,d\mu \geq \int_{\mathbf{F}} \mathscr{P}[\mathbf{A}|\mathscr{F}]\,d\mu \\ &\geq \int_{\mathbf{F}} 1 - \varepsilon\,d\mu \\ &= \mu(\mathbf{F}) \cdot (1 - \varepsilon) > (1 - \varepsilon)^2 > 1 - 2\varepsilon.\end{aligned}$$

Thus any N iterates of A sweep out all but 2ε of the space. □

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