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THE GROWTH OF $\Phi\Phi''/\Phi'^2$ FOR CONVEX FUNCTIONS

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1. Introduction

In W.K. Hayman's survey of the Wiman-Valiron method, the following growth lemma is proved:

LEMMA A [2, Lemma 9]. *Let $\Phi(r)$ be a positive, increasing and convex function of r for $r \geq r_0$ and suppose that*

$$\text{(1.1)} \qquad \liminf_{r\to\infty} \frac{\log \Phi(r)}{\log r} \leq \rho \leq \limsup_{r\to\infty} \frac{\log \Phi(r)}{\log r},$$

where $\rho > 1$. Let $\alpha(\rho) = (\rho - 1)/\rho$ if $\rho < \infty$; $\alpha(\rho) = 1$ if $\rho = \infty$. Suppose that a, K are constants such that $K > 1$ and $a < \alpha(\rho)$. Then if E is the set of all r such that

$$\text{(1.2)} \qquad \textit{either } (\text{a}) \ \frac{\Phi(r)\Phi''(r)}{\Phi'(r)^2} > K\alpha(\rho) \ \textit{or } (\text{b}) \ \Phi'(r) \leq \Phi(r)^a,$$

we have $\underline{\text{dens}}\, E \leq K^{-1}$, where "$\underline{\text{dens}}$" is the lower (linear) density.

Hayman applies this in a context the details of which need not detain us here, save to say that his results suggest that, when ρ is the upper limit in (1.1), it would be desirable to strengthen the part of the conclusion of the lemma that concerns E, from lower to upper density. This is evidently not possible, however, since Φ may be linear for arbitrarily long stretches, and (1.2b) itself may therefore hold on a set of upper density 1. In Hayman's argument it is (1.2a) that plays the vital role, (1.2b) being subsidiary in the sense that it is used only to show that an error term is inessential. What can be said about the set on which (1.2a) holds? It is entirely with this question that the remainder of the present note is concerned.

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We first introduce some notation. Let E be a set on the part $r \geq 1$ of the real axis, and let E_R be the intersection of E with $[1, R]$. We define $m(R) = m(R, E)$ to be the Lebesgue measure of E_R and

$$\text{(1.3)} \quad \log \operatorname{meas} E_R = \int_{E_R} \frac{dt}{t} = \int_1^R \frac{dm(t)}{t} = \frac{m(R)}{R} + \int_1^R \frac{m(t)\,dt}{t^2}$$

to be the logarithmic measure of E_R. Further

(1.4)
$$\underline{\operatorname{dens}}\, E = \liminf_{R\to\infty} m(R)/(R-1), \quad \overline{\operatorname{dens}}\, E = \limsup_{R\to\infty} m(R)/(R-1)$$

are the lower and upper densities of E and

$$\text{(1.5)} \quad \begin{aligned} \underline{\log \operatorname{dens}}\, E &= \liminf_{R\to\infty} (\log \operatorname{meas} E_R)/\log R \\ \overline{\log \operatorname{dens}}\, E &= \limsup_{R\to\infty} (\log \operatorname{meas} E_R)/\log R \end{aligned}$$

are the lower and upper logarithmic densities. We recall [1, 446–47] that

$$\text{(1.6)} \quad \underline{\operatorname{dens}}\, E \leq \underline{\log \operatorname{dens}}\, E \leq \overline{\log \operatorname{dens}}\, E \leq \overline{\operatorname{dens}}\, E.$$

Now suppose that Φ is the function of Lemma A and define

(1.7)
$$p = \liminf_{R\to\infty} (\log \Phi(R))/\log R, \quad P = \limsup_{R\to\infty} (\log \Phi(R))/\log R.$$

We may assume without loss of generality that $r_0 < 1$ and that $\Phi'(r) > 0$ for $r \geq 1$, so that $1 \leq p \leq P \leq \infty$. We then have the following:

THEOREM. *Suppose that $K > 0$ and that E is the set of r at which*

$$\frac{\Phi(r)\,\Phi''(r)}{\Phi'(r)^2} \geq K.$$

Then

(i) *If $K > 1$, $\overline{\operatorname{dens}}\, E \leq 1/K$.*
(ii) *If $p < \infty$ and $K > \alpha(p) = 1 - 1/p$, we have $\underline{\log \operatorname{dens}}\, E \leq \alpha(p)/K$.*
(iii) *If $P < \infty$ and $K > \alpha(P)$, we have $\overline{\log \operatorname{dens}}\, E \leq \alpha(P)/K$.*

We note that, by (1.6), (i) is stronger than the corresponding statement in (iii) for the case $P = \infty$. We shall show by an example at the end of the paper that in general upper logarithmic density cannot be replaced by upper density in (iii).

I am most grateful for the referee's comments on the original draft. The theorem has been recast and its proof considerably shortened as a result of his suggestions.

2. Proof of the theorem

We define

$$p(r) = \frac{\log\{\Phi(r)/\Phi(1)\}}{\log r}, \ q(r) = \frac{r\Phi'(r)}{\Phi(r)},$$

$$\xi(r) = r - \frac{\Phi(r)}{\Phi'(r)} = r\left(1 - \frac{1}{q(r)}\right).$$

Since $\Phi(r)$ is convex, the right hand derivative $\Phi'(r)$ exists for all r, is non-decreasing with r and is equal to the left hand derivative outside a countable set. Also $\Phi''(r)$ exists outside a set of measure zero and $\Phi''(r) \geq 0$. Further $\xi(r)$, which is the abscissa of the point at which the tangent to the graph of $y = \Phi(r)$ meets the r-axis, is nondecreasing; in fact $\xi'(r) = \Phi(r)\Phi''(r)/\Phi'(r)^2 \geq 0$. It follows [3, p. 96] that

$$\xi(r) - \xi(1) \geq \int_1^r \xi'(t)\,dt = \int_1^r \frac{\Phi(t)\Phi''(t)}{\Phi'(t)^2}\,dt \geq \int_{E_r} \frac{\Phi(t)\Phi''(t)}{\Phi'(t)^2}\,dt \geq Km(r).$$

Thus

$$(2.1)\quad m(r) \leq \frac{\xi(r) - \xi(1)}{K} = \frac{1}{K}\left\{r - \xi(1) - \frac{r}{q(r)}\right\} \leq \frac{r + |\xi(1)|}{K},$$

and now (1.4) yields (i). To prove (ii) and (iii) we note that by (1.3),

(2.2)

$$\log \operatorname{meas} E_R \leq 1 + \int_1^R \frac{m(t)}{t^2}\,dt \leq 1 + \frac{1}{K}\left\{|\xi(1)| + \log R - \int_1^R \frac{dt}{tq(t)}\right\}.$$

Also by Schwarz's inequality,

$$(\log R)^2 = \left\{\int_1^R \frac{dt}{t}\right\}^2 \leq \left\{\int_1^R \frac{q(t)\,dt}{t}\right\}\left\{\int_1^R \frac{dt}{tq(t)}\right\} = \left\{\log\frac{\Phi(R)}{\Phi(1)}\right\}\int_1^R \frac{dt}{tq(t)},$$

so that

$$\int_1^R \frac{dt}{tq(t)} \geq \frac{(\log R)^2}{\log\{\Phi(R)/\Phi(1)\}} = \frac{\log R}{p(R)}.$$

Now (2.2) yields

$$\log \operatorname{meas} E_R \leq \frac{1}{K}\log R\left(1 - \frac{1}{p(R)}\right) + 1 + \frac{|\xi(1)|}{K}. \tag{2.3}$$

Using (1.5) and (1.7) we obtain (ii) and (iii) of our Theorem and the proof is complete.

3. Two examples

The first example shows that the estimates in (ii) and (iii) of the theorem are sharp.

Given $p_0 > 1$ and $K > \alpha(p_0)$, let J be the set consisting of all intervals $(n - \alpha(p_0)/K, n)$, for $n = 2, 3, \ldots$. Write J_r for the part of J in $[1, r]$ and let Φ be defined by

$$\Phi(r) = \exp\left(\int_1^r \frac{1}{t - Km(J_t)}\,dt\right) \quad \text{for } r \geq 1,$$

where m denotes Lebesgue measure. For $r > 1$,

$$\Phi(r)/\Phi'(r) = r - Km(J_r) > 0,$$

so Φ is increasing. Further,

$$\Phi(r)\Phi''(r)/\Phi'(r)^2 = (r - \Phi(r)/\Phi'(r))' = K\chi_J(r), \tag{3.1}$$

where χ_J is the characteristic function of J, except at the boundary of J. This is non-negative, so Φ'' is non-negative except possibly at the boundary of J. It follows that Φ', which is continuous, is non-decreasing and therefore Φ is convex.

We have $m(J_r) = K^{-1}(1 - p_0^{-1}))r + O(1)$ as $r \to \infty$, so that

$$\log \Phi(r) = (p_0 + o(1))\log r,$$

and therefore, from (1.7), $p = P = p_0$. Further, from (3.1),

$$\Phi(r)\Phi''(r)/\Phi'(r)^2 \geq K$$

on J, so E contains J. Now J has density, and therefore from (1.6), logarithmic density, $\alpha(p_0)/K$, from which we conclude that E has lower logarithmic density at least $\alpha(p_0)/K$.

The purpose of the second example is to show that when $K < 1$ and $P < \infty$ in the Theorem it is possible to have $\overline{\text{dens}}\, E = 1$.

Given $P > 1$ and $1 > K > \alpha(P)$, choose $P' > P$ so that $K = \alpha(P')$. The example is made up of parts of $Ar^{P'} + B$, where A and B are constants, interspersed with linear segments.

Suppose that a certain linear segment has slope λ, lies above the axis, and lies below r^P no matter how far it is prolonged in the positive direction. Fix R arbitrarily large and consider $Ar^{P'} + B$ for $r \geq R$, where A is such that

$$P'AR^{P'-1} = \lambda, \tag{3.2}$$

and B is chosen so that the linear segment meets $Ar^{P'} + B$ at R. This latter condition entails $AR^{P'} + B = (\lambda + o(1))R$ as $R \to \infty$, so that, from (3.2), $B = \lambda R(1 - 1/P' + o(1))$. It is thus possible to choose R sufficiently large so that $B > 0$, and we suppose this done.

Let $N > R$ be the largest number such that the tangent to $Ar^{P'} + B$ at N lies under r^P for all $r > N$. (Such an N evidently exists, and the tangent at N will touch the graph of r^P at some point $R_o > N$.) The next two segments making up the example consist of $Ar^{P'} + B$ on $[R, N]$, and the tangent to $Ar^{P'} + B$ at N thereafter (or at least until the next insertion of $A'r^{P'} + B'$).

The function Φ constructed this way is positive, increasing and convex, and $\limsup \log \Phi(r)/\log r = P$. Moreover, recalling that $B > 0$, we have on $[R, N]$,

$$\Phi(r)\Phi''(r)/\Phi'^2 = \alpha(P')\left(1 + Br^{-P'}/A\right) > \alpha(P') = K.$$

Now since the tangent to Φ at N touches the graph of r^P at $R_o > N$, we have $P'AN^{P'-1} = PR_o^{P-1}$. This combined with (3.2) gives

$$\lambda(N/R)^{P'-1} = PR_o^{P-1} > PR^{P-1}.$$

It follows that we can make N/R as large as we please by taking R, which is arbitrary, sufficiently large. It is thus possible to arrange for the upper density of

$$E = \{r : \Phi(r)\Phi''(r)/\Phi'^2 \geq K\}$$

to be 1.

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