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DEFORMATIONS AND THE RATIONAL HOMOTOPY OF THE MONOID OF FIBER HOMOTOPY EQUIVALENCES

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Suppose X is a topological space which has the homotopy type of a CW-complex. Then it is well known that Hurewicz fibrations $X \to E \to B$ are classified up to fiber homotopy equivalence by the homotopy classes of maps $f\colon B \to B\,\mathrm{aut}(X)$ where $\mathrm{aut}(X)$ denotes the topological monoid of self homotopy equivalences of X. It is then highly interesting to calculate certain elementary topological invariants of the classifying space $B\,\mathrm{aut}(X)$ in terms of the invariants of X.

Here we are in particular interested in the rational homotopy of $B\,\mathrm{aut}(X)$. A case which seems to be rather treatable is the case of the so called F_0-spaces.

DEFINITION. A 1-connected space X is said to be of type F_0 if the following conditions are satisfied.

(i) $\dim H^*(X; \mathbf{Q}) < \infty$
(ii) $\dim \pi_*(X) \otimes_{\mathbf{Z}} \mathbf{Q} < \infty$
(iii) $H^{\mathrm{od}}(X; \mathbf{Q}) = 0$.

Under these conditions S. Halperin [2] was able to show that the cohomology $A_0 = H^*(X)$ is a complete intersection, i.e., we have

$$A_0 = P/I_0,$$

where P is a graded polynomial algebra in n generators of even degree and I_0 is an ideal generated by a maximal length prime series $\{f_1, \dots, f_n\}$ of homogeneous elements.

It is exclusively this case which will concern us on the subsequent pages. In the papers [6, 7] W. Meier found formulas for the rational homotopy groups of $B\,\mathrm{aut}(X)$, see e.g., [6], Prop. 2.10 and [7], Prop. 1. In these expressions the evenly graded part $\pi_{\mathrm{ev}}(\mathrm{aut}(X)) \otimes_{\mathbf{Z}} \mathbf{Q}$ of the rational homotopy is interpreted as the negatively graded part of the A_0-module of graded $\mathbf{Q}$-derivations $\mathrm{Der}_{\mathbf{Q}} A_0$. It is a longstanding conjecture that in the case of a complete graded

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intersection A_0 of finite length $(\mathrm{Der}_{\mathbf{Q}} A_0)_- = 0$, see [15]. In its topological form this conjecture stems from S. Halperin and says that every $\mathbf{Q}$-orientable fibration $X \to E \to B$, with X an F_0-space, must have a collapsing Serre-spectral sequence; see [6], (2.3) Conjecture. The equivalence of both conjectures was shown in the above cited paper [6] of Meier. Moreover he proved the conjecture for a series of homogeneous spaces $G/H, H \subset G$ a closed connected subgroup of maximal rank. Newer results stem from Shiga and Tezuka [13] who proved the conjecture for all such homogeneous spaces.

In this note we want to give a deformation theoretic interpretation of the results of Meier's and generalize them to monoids of fiber homotopy equivalences. Suppose we are given a $\mathbf{Q}$-oriented fibration $\xi = \{X \to E \to B\}$ where X is of the F_0 type and B is a formal space. Since we cannot use the Halperin conjecture, we shall assume that the corresponding Serre spectral sequence collapses. Then we will give an algebraic description of the rational homotopy of the monoid $G(\xi)$ of fiber homotopy equivalences of ξ. In the case $B = \{*\}$ a point we have of course $G(\xi) = \mathrm{aut}(X)$ and then our results reduce to those of Meier's. Recall that a space B is called formal if the rational homotopy of B is a formal consequence of its cohomology, i.e., if there exists a map $\rho\colon \mathscr{M}(B) \to H^*(B)$ of differential graded algebras from the minimal model to cohomology inducing an isomorphism in (co)homology. This is for example the case if $H^*(X)$ is a complete intersection; see e.g., [1], Sec. 16. In the following let M_- be the negatively graded part of a $\mathbf{Z}$-graded module M.

THEOREM A. *Let $\xi\colon X \to E \to B$ be a $\mathbf{Q}$-oriented Hurewicz fibration such that X is an F_0-space and B is formal with $H^{\mathrm{od}}(B) = 0$. Then there are canonical isomorphisms*

$$\pi_{\mathrm{ev}}(G_0(\xi)) \otimes_{\mathbf{Z}} \mathbf{Q} \cong \mathrm{Der}_R(H^*(E))_-,$$
$$\pi_{\mathrm{od}}(G_0(\xi)) \otimes_{\mathbf{Z}} \mathbf{Q} \cong T^1_R(H^*(E))_-,$$

where $R = H^*(B)$.

Specializing the above formulas to the case of a fibration $X \to X \to \{*\}$, i.e., $B = \{*\}$, gives the following result.

THEOREM B. *If X is an F_0 space, then there are natural isomorphisms*

$$\pi_{\mathrm{ev}}(\mathrm{aut}_0(X)) \otimes_{\mathbf{Z}} \mathbf{Q} \cong \mathrm{Der}_{\mathbf{Q}}(H^*(X))_-$$
$$\pi_{\mathrm{od}}(\mathrm{aut}_0(X)) \otimes_{\mathbf{Z}} \mathbf{Q} \cong T^1_{\mathbf{Q}}(H^*(X))_-.$$

In all these expressions the index "0" means the 1-connected component of the respective monoids, i.e., the submonoids of those maps which are homotopic to the identity, either because they are homotopic by a homotopy of fiber homotopy equivalences as in the general case or by a homotopy of self homotopy equivalences as in the case $\mathrm{aut}_0 X$. On the algebraic side $\mathrm{Der}_R(A)$ means the A-module of R-derivations of the R-algebra A whereas $T^1_R(A)$ is the A-module of infinitesimal R-deformations of A.

The above formulas show strong connections between the theory of universal fibrations and versal deformations with G_m-action. We therefore consider in this note the relationship between universal fibrations and versal deformations with G_m-action. In the following denote by $\mathrm{aut}^\bullet_0(X) \subset \mathrm{aut}_0(X)$ the submonoid of those self homotopy equivalences which fix a distinguished base point. The classifying space functor on the category of topological monoids gives rise to the universal fibration

$$\xi_u\colon X \to B\,\mathrm{aut}^\bullet_0(X) \to B\,\mathrm{aut}_0(X).$$

Then we show that the application of the cohomology functor to ξ_u produces an algebraic object which in deformation theory is well known under the name of a positively graded versal deformation

THEOREM C. *Let X be a space of type F_0 with $\mathrm{Der}_{\mathbf{Q}}(H^*(X))_- = 0$. Then the $H^*(B\,\mathrm{aut}_0(X))$-algebra $H^*(B\,\mathrm{aut}^\bullet_0(X))$ is the positively graded part of a G_m-equivariant versal deformation of the graded $\mathbf{Q}$-algebra $H^*(X)$.*

We observe that the positively graded part of a graded ($= G_m$-equivariant) deformation is uniquely and canonically determined which is in contrast to the behaviour of a versal deformation which is also unique up to isomorphism but not in a canonical way. The information on the cohomology of the space $B\mathrm{aut}^\bullet_0(X)$ given by Theorem C can now be used to compute the rational homotopy groups of $B\mathrm{aut}^\bullet_0(X)$. Suppose $\partial_f \subset \mathrm{Hom}_{A_0}(I/I^2, A_0)$ is the subspace generated over the ground field Q by the partial derivatives $\partial/\partial f_j$ and let $(\mu(\partial_f) \subset T^1(A_0)$ *be the corresponding image. Then our result can be formulated as follows.*

THEOREM D. *Suppose X is a space of type F_0. Let $A_0 = H^*(x)$ with* $\mathrm{Der}_Q(A_0) - = 0$. *Then $B\mathrm{aut}^*_0 X$ has no odd rational homotopy and we have the formula*

$$\pi_*(B\,\mathrm{aut}^\bullet_0 X) \otimes_{\mathbf{Z}} \mathbf{Q} \cong \left[T^1_{\mathbf{Q}}(A_0)_- / \mu(\partial_f)\right]^* \oplus m_{A_0}/m^2_{A_0},$$

where m_{A_0} is the maximal (augmentation) ideal of A_0.

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1. Derivations and infinitesimal deformations of graded algebras

In the following, all fields, rings and algebras will be commutative. Let k be a field and denote by (R, m_R), $R/m_R \cong k$, a local k-algebra. Denote by A a finitely generated R-algebra. We choose an embedding representation $A = P/I$, $P = R[X_1, \dots, X_n]$, $I \subset P$, the defining ideal. Then the first step in deformation theory is to consider the exact sequence

$$I/I^2 \to \Omega_{P|R} \otimes_P A \to \Omega_{A|R} \to 0.$$

After dualizing with the functor $\mathrm{Hom}_P(-, A)$ we obtain the exact sequence

$$0 \to \mathrm{Hom}_A(\Omega_{A|R}, A) \to \mathrm{Hom}_P(\Omega_{P|R}, A) \to \mathrm{Hom}_A(I/I^2, A).$$

Let $T_R^0(A) = \mathrm{Ker}\,\Delta = \mathrm{Hom}_A(\Omega_{A|R}, A) = \mathrm{Der}_R(A)$ and $T_R^1(A) = \mathrm{Coker}\,\Delta$. Then we have an exact sequence

$$(0)\quad 0 \to \mathrm{Der}_R(A) \to \mathrm{Hom}_P(\Omega_{P|R}, A) \xrightarrow{\Delta} \mathrm{Hom}_P(I, A) \to T_R^1(A) \to 0.$$

We investigate now the special case of a complete intersection A over R. Thus we can assume that I is generated by a regular series of homogeneous elements F_j, $j = 1, \dots, k$, in the indeterminants X_i with coefficients in R. Under these hypotheses the conormal module I/I^2 is freely generated by the derivatives dF_j, $j = 1, \dots, k$; i.e.,

$$I/I^2 \cong \sum_{j=1}^{k} A\, dF_j.$$

The dual object then takes the form

$$\mathrm{Hom}_A(I/I^2, A) \cong \sum_{j=1}^{k} A \frac{\partial}{\partial F_j}.$$

Since the homomorphism δ: $I/I^2 \to \Omega_{P|R}$ is induced by the canonical differential d: $P \to \Omega_P$, we get

$$dF_j = \sum_j \partial F_j/\partial X_i\, dX_i.$$

Obviously $\operatorname{Der}_R(P, A) = \operatorname{Hom}_P(\Omega_{P|R}, A)$ is also free and thus we have

$$\operatorname{Der}_R(P, A) = \sum_{i=1}^{k} A \frac{\partial}{\partial X_i}.$$

On the generators the homomorphism Δ assumes the form

$$\Delta\left(\frac{\partial}{\partial X_i}\right) = \sum_{j=1}^{k} \left(\frac{\partial F_j}{\partial X_i} + I\right) \frac{\partial}{\partial F_j}.$$

Consequently the homomorphism Δ is given by the Jacobian matrix of the polynomials $F_1, \dots, F_k$.

If R is a positively graded local k-algebra with $m_R = m = R^+$ the augmentation ideal, and if A is a graded R-algebra, then all the four terms in the exact sequence (0) have a natural graduation. Let

$$\deg X_i = n_i$$

and

$$\deg F_j = w_j,$$

then $\partial/\partial X_i$ has degree $-n_i$ and $\partial/\partial F_j$ has degree $-w_j$. This implies in particular that $\operatorname{Der}_R(A)$ and $T^1_R(A)$ can have terms of negative degree. Since the data of a graduation are equivalent to the data of an action of the reductive group $G_m = k^*$, we can extract from (0) a "negatively graded" exact sequence

$$\text{(I)} \quad 0 \to \operatorname{Der}_R(A)_- \to \left(\sum_{i=1}^{n} A \frac{\partial}{\partial X_i}\right)_- \xrightarrow{\Delta} \left(\sum_{j=1}^{n} A \frac{\partial}{\partial F_j}\right)_- \to T^1_R(A)_- \to 0.$$

It is in particular this sequence we want to interpret topologically. Suppose $R = k$ is the base field. Then $\operatorname{Hom}_P(P, A)_-$ is a finite k-vector space. The same is true for $\operatorname{Hom}_A(I/I^2, A)_-$. Consequently $\operatorname{Der}_R(A)_-$ and $T^1_R(A)_-$ are also finite k-vector spaces. It might seem that our definition of $T^1_R(A)$ depends on the chosen embedding representation. But this is not the case as follows from the deformation theoretic interpretation which can be given to it; see e.g., [7].

Here we consider an example which will be of a certain interest in the sequel. As usual let R be a positively graded local k-algebra and let $A = R[X]/(F)$, $\deg X = d$, where F is a homogeneous polynomial. Then we have

$$\operatorname{Der}_R(A) = \left\{a \frac{\partial}{\partial X} \,\middle|\, a \frac{\partial F}{\partial X} \in (F),\, a \in A\right\}$$

and

$$T_R^1(A) \cong \frac{A\dfrac{\partial}{\partial F}}{\dfrac{\partial F}{\partial X}A\dfrac{\partial}{\partial F}}.$$

If $F = X^{n+1} - r$, $r \in m_R R[X]$, where R is the graded polynomial ring $R = k[t]$, $\deg t \equiv 0 (\mathrm{mod}\, 2)$, then we obtain

$$T_R^1(A) \cong R[X]/(F, F_X)\frac{\partial}{\partial F},$$

where F_X is the partial derivative of F with respect to X. We now evaluate the R-module $T_R^1(A)$ along the spectrum of $R = k[t]$; i.e., we compute the $k(y)$-vector spaces $T_k^1(A_0) = T_R^1(A) \otimes_R k(y)$ for $y \in \operatorname{Spec} R$.

In the case $y = (t)$ we obtain with $k(y) = k$ and $A_0 = k[X]/(X^{n+1})$ the expression

$$T_k^1(A_0) = k[X]/(X^{n+1}, (n+1)X^n)(-(n+1)),$$

where $(-(n+1))$ indicates a grade shift by $-(n+1)$. If now char $k = 0$ we obtain the version

$$T_k^1(A_0) = k[X]/(X^n)(-(n+1)).$$

At the other hand, if we take a closed point outside the origin, say $(t - y)$, $y \in k - \{0\}$, we obtain with $K = k(y)$ and $A_y = A/(t-y)A$ the expression

$$T_K^1(A_y) \cong K[X]/(f, f'),$$

where f is the polynomial $F(t, X)$ evaluated in $t = y$, i.e., $f(X) = F(y, X)$ and f' means of course the derivative of f with respect to X. Thus we conclude that $T_K^1(A_y)$ is different from zero if and only if the polynomial f has at least one multiple root.

2. Self maps of fiber bundles and rational homotopy

In the proof of Theorem A we proceed now as Meier did in his proof of Prop. 1 in [7]. Let $\xi = \{X \to E \to B\}$ be a fibration satisfying the hypotheses of Theorem A. As shown in [2], the cohomology algebra $A_0 = H^*(X; \mathbf{Q})$ is a complete intersection of finite artinian length; i.e., A_0 can be written as

$$A_0 = P/I_0$$

with $P = \mathbf{Q}[X_1, \dots, X_n]$, $\deg X_i = d_i$, $d_i \equiv 0 (\mathrm{mod}\, 2)$, a graded polynomial algebra and $I_0 = (f_1, \dots, f_n)$, a defining ideal, generated by a maximal length prime series (regular series) of homogeneous elements with $\deg f_j = w_j$. Let $R = H^*(B)$, then by a result analogous to Theorem 1.1 in [3] it follows that the R-algebra $H^*(E)$ has the form

$$H^*(E) \cong P_R/I$$

with $P_R = R[X_1, \dots, X_n]$ and a defining ideal I generated by homogeneous elements of the form $F_j = f_j - r_j$, $r_j \in m_R P_R$. Let $A = H^*(E)$ and $Q = \mathbf{Q}[Y_1, \dots, Y_n]$, $\deg Y_j = w_j$; then consider the commutative diagram

$$\text{(I)} \qquad \begin{tikzcd} P_R \arrow[r] & A \\ P \arrow[u, "F"] \arrow[r] & \mathbf{Q} \arrow[u] \end{tikzcd},$$

where the homomorphism F is given by $F(Y_j) = F_j$, $j = 1, \dots, n$, and the right vertical arrow is simply the structural morphism of the $\mathbf{Q}$-algebra A. Since the polynomials F_j form again a prime series this implies that F is flat. Recall that $H^*(E) = H^{\mathrm{ev}}(E)$, since by the hypothesis B has no odd cohomology. Moreover, by III.3(1), Théorème in [14] the minimal model of E is pure and therefore E is also formal, since B is formal after the hypothesis. Our first goal is to realize the above diagram by a rational fibration

$$E' \to B_\emptyset \times K_1 \to K_2,$$

where the K_i, $i = 1, 2$, are adequate products of rational Eilenberg-McLane spaces, i.e.,

$$K_1 = \prod_{i=1}^{n} K(\mathbf{Q}, d_i)$$

and

$$K_2 = \prod_{j=1}^{n} K(\mathbf{Q}, w_j).$$

Recall that the respective cohomology rings are given by the polynomial rings

$$R_1 = P = H^*(K_1; \mathbf{Q}) \cong \mathbf{Q}[x_1, \dots, x_n], \quad \deg x_i = d_i$$

and

$$R_2 = H^*(K_2, \mathbf{Q}) \cong \mathbf{Q}[y_1, \dots, y_n], \quad \deg y_j = w_j.$$

Now the set of homotopy classes of continuous maps $h\colon K_1 \to K_2$ is given by

$$[K_1, K_2] = \prod_j \left[K_1, K(\mathbf{Q}, w_j)\right] \cong \prod_j H^{w_j}(K_1; \mathbf{Q}),$$

whereas the set of homotopy classes of continuous maps $H\colon B \times K_1 \to K_2$ is given by

$$[B \times K_1, K_2] \cong \prod_j \left[B \times K_1, K(\mathbf{Q}, w_j)\right].$$

Therefore we have

$$[B \times K_1, K_2] \cong \prod_j H^{w_j}(B \times K_1; \mathbf{Q}).$$

Now we choose a map Φ in the homotopy class of the element

$$(F_1, \ldots, F_n) \in \prod_j H^{w_j}(B \times K_1, \mathbf{Q}) \cong \prod_j (R[X_1, \ldots, X_n])^{w_j}.$$

Let $p_1\colon B \times K_1 \to B$ be the trivial fibration where p_1 is the projection onto the first factor. Then we define a map ϕ as the composition of Φ with the inclusion j of the fiber K_1. It is easy to see that ϕ corresponds to a map in the homotopy class of the element

$$(f_1, \ldots, f_n) \in \prod_j H^{w_j}(K_1, \mathbf{Q}).$$

Therefore we have the commutative diagram

$$\begin{tikzcd}
X' \arrow[r, "\nu_0"] \arrow[d, "i"] & K_1 \arrow[r, "\phi"] \arrow[d, "j"] & K_2 \arrow[d, equal] \\
E' \arrow[r, "\nu"] \arrow[d, "\pi"] & B_\emptyset \times K_1 \arrow[r, "\Phi"] \arrow[d, "p_1"] & K_2 \\
B_\phi \arrow[r] & B_\emptyset &
\end{tikzcd}$$

It is clear by construction that the middle horizontal fibration realizes the diagram (I). Moreover it follows from the Eilenberg Moore spectral sequence that the fiber E' of Φ has the cohomology $H^*(E') \cong A$. Consequently the cohomology of the fiber X' of ϕ is given by A_o. Moreover it is clear that the inclusion i of the fiber X' in E' induces precisely the homomorphism

$A \to A_o$. Therefore $\xi' = \{X' \to E' \to B_\emptyset\}$ is a fibration which cohomologically looks like ξ. We have to show that ξ' is fiber homotopy equivalent to

$$\xi_\emptyset = \{X_\emptyset \to E_\emptyset \to B_\emptyset\}.$$

First we recall that B is a formal space. Let $\mathscr{M}(B)$ be the minimal model of B. Then there exists a D.G.A.-morphism ρ_0: $\mathscr{M}(B) \to H^*(B)$ inducing an isomorphism in (co)homology. Moreover A_0 is a complete intersection. It follows from [14], III.3(1), Théorème, that the minimal models of E, resp. E' are pure and thus are also formal, since B is formal; see Appendix, Theorem A.2. Consider $H^*(E)$ and $H^*(E')$ as algebras over $\mathscr{R} = \mathscr{M}(B)$ via the formality map. Then there exist formality maps ρ: $\mathscr{M}(E) \to H^*(E)$ and ρ': $\mathscr{M}(E') \to H^*(E')$ which are also D.G.A.-morphisms over $\mathscr{R}$. Let α: $H^*(E') \to H^*(E)$ be an isomorphism. Consider the diagram

$$\begin{array}{ccc} \mathscr{M}(E') & \xrightarrow{M(\alpha)} & \mathscr{M}(E) \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\rho'} \\ H^*(E') & \xrightarrow{\alpha} & H^*(E) \end{array}$$

Since the differential graded algebras over $\mathscr{R}$ form a closed model category in the sense of Quillen (see e.g., [1][10][11]) there exists a lifting $M(\alpha)$ such that the above diagram becomes homotopy commutative. Let h: $E_\emptyset \to E'$ be the corresponding geometric map. Then h is a rational fiber homotopy equivalence between ξ and ξ'; i.e., from the point of view of rational homotopy theory the fibrations ξ and ξ' can be identified. Thus in the following we put $\xi = \xi'$. The fibration Φ gives now rise to a fibration of mapping spaces:

$$\operatorname{Map}^{l_0}(E, E_\emptyset; l) \to \operatorname{Map}^{l_0}(E, B_\emptyset \times K_1; \nu_l) \to \operatorname{Map}(E, K_2; \Phi \circ \nu_l),$$

Here l: $E \to E_\emptyset$ is the localization map and $\nu_l = \nu \circ l$ is the composition of ν with l, whereas l_0: $B \to B_\emptyset$ is the localization map on the base space B. Recall that by functoriality of the localization we have a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{l} & E_\emptyset \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi_\emptyset} \\ B & \xrightarrow{l_0} & B_\emptyset. \end{array}$$

Then the superscript l_0 distinguishes those maps which lie over l_0. As $H^*(E)$

is evenly graded, we have

$$\pi_{2i-1}(\mathrm{Map}(E, K_2; \Phi \circ \nu_l) = 0$$

for all $i \in \mathbf{N}$. Let p_2: $B_\emptyset \times K_1 \to K_1$ be the projection onto the second factor, then by the same reason we have

$$\pi_{2i-1}(\mathrm{Map}(E, K_1; \phi \circ p_2 \circ \nu_l) = 0.$$

for all $i \in \mathbf{N}$. Therefore by the obvious identification

$$\mathrm{Map}^{l_0}(E, B_\emptyset \times K_1; \nu_l) = \mathrm{Map}(E, K_1; p_2 \circ \nu_l)$$

we obtain as the exact homotopy sequence of this fibration the four term exact sequence

$$\text{(II)} \qquad \begin{aligned} 0 &\to \pi_{\mathrm{ev}} \mathrm{Map}^{l_0}(E, E_\emptyset; l) \to \pi_{\mathrm{ev}} \mathrm{Map}(E, K_1; p_2 \circ \nu_l) \\ &\xrightarrow{\delta} \pi_{\mathrm{ev}} \mathrm{Map}(E, K_2; \Phi \circ \nu_l) \to \pi_{\mathrm{od}} \mathrm{Map}^{l_0}(E, E_\emptyset; l) \to 0. \end{aligned}$$

Recall now that by the canonicity of the localization there is a chain of natural maps

$$\mathrm{Map}^B(E, E) \to \mathrm{Map}^{l_0}(E, E_\emptyset) \to \mathrm{Map}^{B_\emptyset}(E_\emptyset, E_\emptyset) \to \mathrm{Map}^B(E, E)_\emptyset$$

which comes out to be the $\mathbf{Q}$-localization of $\mathrm{Map}^B(E, E)$ and thus is a rational homotopy equivalence. Thus it follows

$$\pi_{\mathrm{ev}} \mathrm{Map}^{l_0}(E, E_\emptyset; l) \cong \pi_{\mathrm{ev}} \mathrm{Map}^B(E, E; \mathrm{id}) \otimes_{\mathbf{Z}} \mathbf{Q}.$$

We consider now the monoid $G(\xi)_0 = \mathrm{Map}^B(E, E; \mathrm{id})$ of B-maps which are homotopic through B-maps to the identity id: $E \to E$. Since the above isomorphisms remain also true in the odd case, the above considered exact sequence take the form:

$$\text{(III)} \qquad \begin{aligned} 0 &\to \pi_{\mathrm{ev}}(G(\xi)_0) \otimes_{\mathbf{Z}} \mathbf{Q} \to \pi_{\mathrm{ev}} \mathrm{Map}(E, K_1; p_2 \circ \nu_l) \to \\ &\xrightarrow{\delta} \pi_{\mathrm{ev}} \mathrm{Map}(E, K_2; \Phi \circ \nu_l) \to \pi_{\mathrm{od}}(G(\xi)_0) \otimes_{\mathbf{Z}} \mathbf{Q} \to 0. \end{aligned}$$

THEOREM 2.1. *The exact sequence* (III) *is canonically isomorphic to the exact sequence* (I) *after reversing the sign of graduation.*

As a corollary of Theorem 2.1 we obtain immediately Theorem A.

Proof of Theorem 2.1. First we show that there are canonical isomorphisms

$$\pi_{\mathrm{ev}}\,\mathrm{Map}(E, K_1; p_2 \circ \nu_l) \cong \mathrm{Hom}_{P_R}(\Omega_{P_R|R}, A)_-$$
$$\pi_{\mathrm{ev}}\,\mathrm{Map}(E, K_2; \Phi \circ \nu_l) \cong \mathrm{Hom}_A(I/I^2; A)_-$$

of graded $\mathbf{Q}$-vector spaces after a graduation reversing. By using Thom theory we observe that there are homotopy equivalences

$$\mathrm{Map}(E, K_1) \simeq \prod_{i=1}^{n} \mathrm{Map}(E, K(\mathbf{Q}, d_i)) \simeq \prod_{i=1}^{n} \prod_{q} K(H^q(E), d_i - q).$$

Since the odd rational cohomology of E vanishes, we have

$$\pi_{\mathrm{od}}\,\mathrm{Map}(E, K_1; p_2 \circ \nu_l) = 0,$$
$$\pi_{\mathrm{ev}}\,\mathrm{Map}(E, K_1; p_2 \circ \nu_l) \cong \sum_{i=1}^{n} \sum_{q=0}^{d_i} H^q(E)$$

and therefore we have a sign reversing isomorphism

$$\pi_{\mathrm{ev}}\,\mathrm{Map}(E, K_1; p_2 \circ \nu_l) \cong \sum_{i=1}^{n} \left(H^*(E) \frac{\partial}{\partial X_i} \right)_- .$$

But the right hand side can of course be interpreted as $\mathrm{Hom}_{P_R}(\Omega_{P_R|R}, A)_-$, which gives the first of the desired isomorphisms. In a quite analogous way we can proceed to prove the second isomorphism.

Thus it remains to show, that the linear map δ in the exact sequence (II) is given by the Jacobian homomorphism $\Delta = \Delta_F$ of the ring homomorphism F. Let S^{2n} be the $2n$-sphere and denote by $[S^{2n} \times E_\emptyset, K_1]_{p_2 \circ \nu}$ the set of homotopy classes of those maps f: $S^{2n} \times E_\emptyset \to K_1$ such that for every $s \in S^{2n}$ the map f_s: $E_\emptyset \to K_1$, $f_s(x) = f(s, x)$, is homotopic to the fiber inclusion ν_l. Thus we have

$$\pi_{2n}\,\mathrm{Map}(E, K_1; p_2 \circ \nu_l) \cong \left[S^{2n} \times E_\emptyset, K_1\right]_{p_2 \circ \nu}.$$

Since $P = R_1 = H^*(K_1)$, the minimal model $M(K_1)$ of K_1 is also given by $M(K_1) \cong R_1$ because the differentials must disappear by graduation reasons. Therefore rational homotopy theory (Sullivan equivalence) gives us the isomorphism

$$\left[S^{2n} \times E_\emptyset, K_1\right] \cong \left[R_1, M(S^{2n} \times E)\right].$$

Substituting $M(S^{2n}) = \Lambda(\xi)/\xi^2, \deg \xi = 2$, for the minimal model of the sphere and using the formality of E we obtain

$$M(S^{2n} \times E) \cong M(E) \otimes \Lambda(\xi)/\xi^2 \cong H^*(E) \otimes \Lambda(\xi)/\xi^2,$$

and therefore

$$\left[S^{2n} \times E_{\emptyset}, K_1\right] \cong \left[R_1, H^*(E) \otimes \Lambda(\xi)/\xi^2\right].$$

It is now a simple exercise to show that two differential graded homomorphisms f and g from R_1 to $H^*(E) \otimes \Lambda(\xi)/\xi^2$ are homotopic in the sense of D.G.A.-morphisms if and only if they coincide. Therefore we obtain the isomorphism

$$\mathrm{Hom}_{\mathrm{D.G.A.}}\left(R_1, H^*(E) \otimes \Lambda(\xi)/\xi^2\right) \cong \left[R_1, H^*(E) \otimes \Lambda(\xi)/\xi^2\right].$$

Now we distinguish a certain subclass of D.G.A.-morphisms: A D.G.A.-morphism h: $R_1 \to H^*(E) \otimes_{\mathbf{Q}} \Lambda(\xi)/\xi^2$ is called *special* if for every $s \in S^{2n}$ the composed homomorphism

$$R_1 \xrightarrow{h} H^*(E) \otimes_{\mathbf{Q}} H^*(S^{2n}) \to H^*(E) \otimes_{\mathbf{Q}} H^*(\{s\})$$

is equivalent to the canonical projection $P \to A = H^*(E)$, i.e., to the homomorphism induced by the fiber inclusion ν. Thus, by composition of the above isomorphisms, we get

$$\pi_{2n} \operatorname{Map}(E, K_1; p_2 \circ \nu_l) \cong \mathrm{Hom}^{\sigma}_{\mathrm{D.G.A.}}\left(R_1, H^*(E) \otimes_{\mathbf{Q}} \Lambda(\xi)/\xi^2\right),$$

where the exponent σ means of course 'special'.

Let $R_2 = H^*(K_2)$; then we consider the set

$$\mathrm{Hom}^{\tau}_{\mathrm{D.G.A.}}\left(R_2, H^*(E) \otimes_{\mathbf{Q}} \Lambda(\xi)\xi^2\right)$$

of homomorphisms g such that the composition

$$R_2 \xrightarrow{g} H^*(E) \otimes_{\mathbf{Q}} H^*(S^{2n}) \to H^*(E) \otimes_{\mathbf{Q}} H^*(\{s\})$$

coincides with the map induced by $\Phi \circ \nu_l$. Then in a similar way we obtain the isomorphism

$$\pi_{2n} \operatorname{Map}(E, K_2; \Phi \circ \nu_l) \cong \mathrm{Hom}^{\tau}_{\mathrm{D.G.A.}}\left(R_2, H^*(E) \otimes_{\mathbf{Q}} \Lambda(\xi)/\xi^2\right)$$

Recall that h takes on the generators X_i, $i = 1, \dots, n$, the value

$$h(X_i) = X_i + I + (q_i + I) \otimes \xi.$$

Therefore we get the series of equalities

$$\begin{aligned}(\delta)(Y_j) &= h\big(F(Y_j)\big)\\ &= h\big(F_j(X_1 + I, \dots, X_n + I)\big),\\ &= F_j\big(h(X_1), \dots, h(X_n)\big) + I,\\ &= F_j\big(X_1 + I + (q_1 + I) \otimes \xi, \dots, X_n + I + (q_n + I) \otimes \xi\big) + I\\ &= F_j(X_1, \dots, X_n) + I + \sum_{k=1}^{n} (q_k + I) \frac{\partial F_j}{\partial X_k} \otimes \xi,\end{aligned}$$

and therefore

$$(\delta)(Y_j) = \sum_{k=1}^{n} (q_k + I) \frac{\partial F_j}{\partial X_k} \otimes \xi.$$

This shows that δ is given by the Jacobian homomorphism $\Delta_F = \Delta$ of F which proves Theorem A.

We conclude this paragraph with an example. We take a fibration of the type

$$S^{2n} \to E \to \mathbf{C}P^k.$$

Then the cohomology of E can be written as

$$H^*(E) = R[X]/(F), F = X^2 - \alpha X + \beta,$$

where α and β are certain homogeneous elements in $R = H^*(\mathbf{C}P^k) = \mathbf{Q}[t]/(t^{k+1})$ such that the defining polynomial F becomes homogeneous. After the formulas in Section 1 we obtain

$$T_R^1(H^*(E)) \cong R[X]/(F, 2X - \alpha) \frac{\partial}{\partial F},$$

and therefore

$$T_R^1(H^*(E)) \cong \mathbf{Q}[t]/\big(t^{k+1}, \beta - \tfrac{1}{4}\alpha^2\big)(-4n).$$

Case 1. $4\beta - \alpha^2 = 0$. Here we obtain

$$T_R^1(H^*(E)) \cong \mathbf{Q}[t]/(t^{k+1})(-4n).$$

Case 2. $4\beta - \alpha^2 \neq 0$. This gives

$$T_R^1(H^*(E)) \cong \mathbf{Q}[t]/(t^{k+1}, t^{2n})(-4n),$$

and therefore

$$T_R^1(H^*(E)) \cong \mathbf{Q}[t]/(t^m)(-4n)$$

with $m = \min(k+1, 2n)$.

It is now a simple exercise to show $\mathrm{Der}_R(H^*(E))_- = 0$. Therefore we obtain for the rational homotopy groups of $G(\xi)_0$:

$$\pi_{2i-1}(G(\xi)_0) \otimes_{\mathbf{Z}} \mathbf{Q} \cong \mathbf{Q} \text{ if } i = 2n - k + j,\ j = 0, \ldots, k,$$
$$\pi_{2i-1}(G(\xi)_0) \otimes_{\mathbf{Z}} \mathbf{Q} \cong \mathbf{Q} \text{ if } i = 2n - m + j,\ j = 0, \ldots, m,$$

and zero otherwise.

3. Versal deformations and universal fibrations

In this paragraph we consider some consequences of Theorem B. Denote by k a field, and let A_0 be a graded local k-algebra; i.e., consider A_0 as a local algebra where the maximal ideal m_{A_0} is given by the augmentation ideal of the positively graded elements in A_0. Let R be a graded ring with a distinguished maximal ideal M, $R/M \cong k$, such that M is homogeneous with respect to the given graduation. In most cases M will be the ideal of positively graded elements. But we don't exclude the more general case of a $\mathbf{Z}$-graduation, i.e., a graduation which allows also negative degrees. As it is well known, the data of a graduation are equivalent to the data of an action of the multiplicative group $G_m = k^*$ on the algebraic object as a group of ring automorphisms. So, we will use both terms simultaneously. Then a G_m-equivariant deformation of A_0 along the ring R is a flat graded R-algebra A such that $A/MA \cong A_0$. We have therefore a commutative diagram

$$\begin{tikzcd} A \arrow[r, "j"] & A_0 \\ R \arrow[u, "f"] \arrow[r] & k, \arrow[u] \end{tikzcd}$$

where f is the flat structural morphism and j denotes a surjection inducing an isomorphism $j\colon A \otimes_R k \to A_0$ of graded k-algebras. In the following, deformations will be denoted by greek letters. A morphism $g\colon \xi \to \eta$ where $\xi = \{R \to A \to A_0\}$ and $\eta = \{S \to B \to A_0\}$ is a commutative diagram

$$\begin{array}{ccc} A_0 & \xrightarrow{\mathrm{id}} & A_0 \\ \uparrow & & \uparrow \\ A & \xrightarrow{g} & B \\ \uparrow & & \uparrow \\ R & \xrightarrow{h} & S \end{array}$$

of graded ring homomorphisms. If $R = S$ and $h = \mathrm{id}$ we speak of an isomorphism between deformations along R. We can subject a deformation ξ to a base change with respect to h. In this way we obtain a new deformation $\xi \otimes_R S = \{S \to A \otimes_R S \to A_0\}$, since flatness is invariant under base change. The deformation ξ is called a *G_m-equivariant versal deformation* if any other graded deformation η is isomorphic to $\xi \otimes_R S$ for some graded ring homomorphism $h\colon R \to S$. If $h = h_\eta$ is uniquely determined by the isomorphism type of η then ξ is called a *universal* deformation. It is a theorem that in the case of an artininian k-algebra A_0 such a versal deformation exists. Its construction will be considered in the special case of a complete intersection in the following. For other information concerning deformation theory we refer to the vast literature on the subject; see e.g. [4] [5] [8] [9] [12].

To explain the construction of a versal deformation in the case of a complete intersection A_0 of finite length, we come back to the exact sequences of Section 1. Consider the exact sequence (I) in the case $R = k$ where I is generated by the polynomials $F_j = f_j$, $j = 1, \dots, n$. Then we have the exact sequence

$$0 \to \mathrm{Der}_k(A_0) \to \sum_{i=1}^{n} A_0 \frac{\partial}{\partial X_i} \xrightarrow{\Delta} \sum_{j=1}^{n} A_0 \frac{\partial}{\partial f_j} \to T_k^1(A_0) \to 0.$$

Since A_0 has finite length as a module over itself, the A_0-module $T_k^1(A_0)$ must also be of finite length and therefore is a finite k-vector space. Let $r = \dim T_k^1(A_0)$, then we choose tuples $g_l = (g_{l1}, \dots, g_{ln})$, $g_{lj} \in P$, homogeneous, $l = 1, \dots, r$, such that the projections of the elements

$$\bar{g}_l = \sum_j (g_{lj} + I) \frac{\partial}{\partial f_j}$$

form a homogeneous basis of $T_k^1(A_0)$. Let now $R = k[t_1, \dots, t_r]$ be the homogeneous polynomial algebra in the indeterminates t_l with $\deg t_l = \deg f_j - \deg g_{lj}$. Then we consider the graded R-algebra A with

$$A = R[X_1, \ldots, X_n]/J,$$

where J is generated by the polynomials $\Phi_j = f_j - \Sigma_l t_l g_{lj}$. It is then easy to see that also the Φ_j form a regular series, which shows that A must be flat over R. The deformation ξ_v we have obtained by this procedure is then a G_m-equivariant versal deformation of the k-algebra A_0 of minimal dimension of the parameter space Spec R. For a proof of this fact we refer to [4] [10] [14].

From the deformation ξ_v we can extract a positively graded versal deformation of A_0 taking simply the positively graded generators $t_1, \ldots, t_s$ of R. Let $S = k[t_1, \ldots, t_s]$ and let h: $R \to S$ be the obvious projection homomorphism. Then ξ_v^+ is defined by $\xi_v^+ = \xi_v \otimes_R S$. Suppose η is a graded deformation along a positively graded local ring Q, then it follows from G_m-equivariance that η is induced from ξ_v^+ by a graded homomorphism h: $S \to Q$. This justifies the notation. If $h = h_\eta$ is uniquely determined by the isomorphism type of η, then we speak of a positively graded universal deformation (= pvd).

Let now S be any local graded k-algebra and let $D_{A_0}(S)$ be the set of isomorphism classes of graded deformations of A_0 along S. Then $D_{A_0}(-)$ gives a functor from the category of graded k-algebras into the category of sets. If $\eta = \{R \to A \to A_0\}$ is a graded deformation, then there is a natural transformation

$$\tau(\eta)\colon \mathrm{Hom}^{G_m}(R, -) \to D_{A_0}(-)$$

of functors, associating to any graded homomorphism f the deformation $\eta \otimes_R (-)$ induced from η by f. Suppose $S = k[\varepsilon], \varepsilon^2 = 0, \deg \varepsilon = 2i$, is the ring of dual numbers of weight $2i$. Then one shows in deformation theory that there is a natural isomorphism between the set $D_{A_0}(-2i) = D_{A_0}(k[\varepsilon])$ of the *infinitesimal deformations* of degree $2i$ and the $2i$-th homogeneous piece $T_k^1(A_0)(-2i)$ of $T_k^1(A_0)$. The map

$$\tau(\eta)(2i)\colon \mathrm{Hom}^{G_m}(R, k[\varepsilon]) \to T_k^1(A_0)(-2i)$$

is then called the $2i$-th homogeneous piece of the Kodaira-Spencer map of η. Now let X be a 1-connected space which is also a space of type F_0. Furthermore we shall assume that the graded $\mathbf{Q}$-algebra $A_0 = H^*(X)$ has no negatively graded derivations. Then we consider the universal fibration ξ_u:

$$X \to B\,\mathrm{aut}_0^\bullet(X) \to B\,\mathrm{aut}_0(X).$$

It follows from Theorem B and rational homotopy theory that $RX = H^*(B\,\mathrm{aut}_0(X))$ is a positively graded polynomial algebra in $r = \dim_{\mathbf{Q}} T^1(A_0)_-$ many generators of even positive degree. Since A_0 has no odd elements, the Serre spectral sequence of ξ_u collapses and $H^*(B\,\mathrm{aut}_0^\bullet(X))$ must be a free RX-module. From the Eilenberg Moore spectral sequence we get an isomorphism

$$H^*(X) \cong H^*\big(B\,\mathrm{aut}_0^\bullet(X)\big) \otimes_R \mathbf{Q}$$

of graded $\mathbf{Q}$-algebras. Consequently the commutative diagram

$$\begin{tikzcd} H^*(B\,\mathrm{aut}_0^\bullet(X)) \arrow[r] & H^*(X) \\ H^*(B\,\mathrm{aut}_0(X)) \arrow[r] \arrow[u] & \mathbf{Q} \arrow[u] \end{tikzcd}$$

represents a positively graded deformation θ_u of A_0.

Denote by the symbol $\mathscr{F}_X(B)$ the set of rational fiber homotopy equivalence classes of oriented fibrations

$$X \to E \to B.$$

Here by fiber homotopy equivalence between two fibrations $\eta = \{X \to E \to B\}$ and $\eta' = \{X \to E' \to B\}$ we mean the existence of a B-map $h\colon E \to E'$ inducing a homotopy equivalence $\bar{h} \in \mathrm{aut}_0 X$ on the fiber. In the case $B = S^{2i}$ an even sphere we write simply $\mathscr{F}_{X_\emptyset}(2i)$.

In the following we put $RX = H^*(B\,\mathrm{aut}_0(X))$. Let us consider the following diagram:

$$\begin{tikzcd} {[B_\emptyset, (B\,\mathrm{aut}_0\,X)_\emptyset]} \arrow[r, "(1)"] \arrow[d, "(2)"'] & \mathscr{F}_{X_\emptyset}(B_\emptyset) \arrow[d, "(4)"] \\ \mathrm{Hom}^{G_m}(RX, H^*(B)) \arrow[r, "(3)"] & D_{A_0}(H^*(B)). \end{tikzcd}$$

Here the vertical maps (2) and (4) are given by passing to cohomology. The first horizontal map (1) is given by $f \mapsto f^*\xi_u$ whereas the second horizontal map (3) is induced by the base change of the deformation θ_u with the given homomorphism $g \in \mathrm{Hom}(RX, H^*(B))$, i.e., from the assignment $g \mapsto \theta_u \otimes_{RX} H^*(B)$.

THEOREM 3.1. *Let X be a space of type F_0 such that $H^{\mathrm{od}}(B\,\mathrm{aut}_0\,X;\mathbf{Q}) = 0$. If B is a formal space then the above diagram is a commutative diagram of isomorphisms.*

It has been shown that in the case of complete intersection A_0 which is generated only by elements of degree 2 the Halperin conjecture is true, i.e., we have $(\operatorname{Der} A_0)_- = 0$; see e.g., [6, 7] and [15]. This implies that $RX^{\mathrm{od}} = 0$ for a space X with $H^*(X) = A_0$. Therefore we have

THEOREM 3.2. *Let X be a space of type F_0 such that $H^*(X)$ is generated by elements of degree 2 and let B be a formal space. Then the above diagram is a commutative diagram of isomorphisms.*

Remark. Let k, $\operatorname{char} k = 0$, be a field and let B be a graded local k-algebra. Then by rational homotopy theory R can be realized by a space B, i.e., $R \cong H^*(B; k)$; see e.g., [1]. Using the isomorphism (3) one has therefore shown that the functor of positively graded deformations of A_0 is representable. In the following let $\mathbf{R}_k$ be the category of the positively graded k-algebras. Denote by **Ens** the category of sets.

THEOREM 3.3. *Let k be a field of characteristic 0 and let $A_0|k$ be a graded complete intersection of finite length with residue field isomorphic to k. If $\operatorname{Der}_k(A_0)_- = 0$ then the functor D: $R_k \to$ **Ens** which associates to any $R \in \mathbf{R}_k$ the set $D_{A_0}(R)$ of graded isomorphism classes of positively graded deformations, is representable.*

In particular this proves Theorem C. In fact Theorem 3.3 says much more, namely that θ_u is a positively graded *universal* deformation and therefore $\theta_u \cong \xi_v^+$ as graded deformations.

Proof of Theorem 3.1. Commutativity is obvious by the definition of the maps. It is not difficult to see that $(B \operatorname{aut}_0 X)_\emptyset$ is a classifying space for oriented fibrations with fiber $X_\emptyset$. That the map (1) is an isomorphism is therefore an immediate consequence of the universality of the localized fibration $\xi_{u\emptyset}$. That the first vertical map (2) is an isomorphism follows from the hypotheses on $B \operatorname{aut}_0 X$, B and rational homotopy theory. Observe that by Sullivan theory

$$\left[B_\emptyset, (B \operatorname{aut}_0 X)_\emptyset\right] \cong [RX, \mathscr{M}(B)],$$

since by the hypothesis on $BX = (B \operatorname{aut}_0 X)_\emptyset$ the minimal model $\mathscr{M}(BX)$ is a polynomial ring and therefore isomorphic to RX. By formality of B we have

$$[RX, \mathscr{M}(B)] \cong [RX, H^*(B)]$$

and consequently

$$[B_\emptyset, (B\,\mathrm{aut}_0\, X)_\emptyset] \cong \mathrm{Hom}(RX, H^*(B)).$$

Now we show that the second vertical map (4) is an isomorphism.

Surjectivity. Suppose we are given a graded deformation $\eta = \{R \to A \to A_0\}$ with $R = H^*(B)$. Since the $\mathbf{Q}$-algebra A_0 is a complete intersection it follows from Theorem 1.1 in [3] that A is also a complete intersection. As usual we write

$$A = R[X_1, \ldots, X_n]/I$$

where the defining ideal I is generated by homogeneous elements $F_1, \ldots, F_n$ of the type $F_j = f_j - r_j$, $r_j \in R_+P_R$. Suppose $d_i = \deg x_i$ and $w_j = \deg f_j = \deg F_j$. Now we proceed as in Section 1.

Let

$$K_1 = \prod_i K(\mathbf{Q}, d_i)$$

and

$$K_2 = \prod_j K(\mathbf{Q}, w_j).$$

Then we choose a map F in the homotopy class of the element

$$(F_1, \ldots, F_n) \in \prod_j H^{w_j}(B \times K_1, \mathbf{Q}) \cong \prod_j (R[X_1, \ldots, X_n])^{w_j}.$$

Define a map f as the composition of F with the inclusion j of the fiber K_1. Then f corresponds to a map in the homotopy class of the element

$$(f_1, \ldots, f_n) \in \prod_j H^{w_j}(K_1, \mathbf{Q}).$$

This gives our commutative diagram

$$\begin{tikzcd}
Y \arrow[r] \arrow[d, "i"] & K_1 \arrow[r, "f"] \arrow[d, "j"] & K_2 \arrow[d, equal] \\
E \arrow[r] \arrow[d, "\pi"] & B \times K_1 \arrow[r, "F"] \arrow[d, "p_1"] & K_2 \\
B \arrow[r] & B. &
\end{tikzcd}$$

Now it follows from the Eilenberg Moore spectral sequence that the fiber E of F has the cohomology $H^*(E) \cong A$. Consequently the cohomology of the fiber Y of f is given by A_0. Moreover it is clear that the inclusion i of the fiber Y in E induces precisely the homomorphism $A \to A_0$. Therefore $\pi_\eta = \{Y \to E \to B\}$ is a fibration which realizes η. This shows the surjectivity of (4).

Injectivity. We consider two fibrations $\eta = \{Y \to E \to B\}$, and $\eta' = \{Y' \to E' \to B\}$. Let γ and γ' be the two deformations induced from the fibrations η and η'. Following the definition of an isomorphism between deformations we have the commutative diagram

$$\begin{tikzcd}
H^*(Y) \arrow[r, "\mathrm{id}"] & H^*(Y') \\
H^*(E) \arrow[r, "\alpha"] \arrow[u] & H^*(E') \arrow[u] \\
H^*(B) \arrow[r, "\mathrm{id}"] \arrow[u] & H^*(B), \arrow[u]
\end{tikzcd}$$

where α is a graded isomorphism. Now we proceed as in Section 2. Let $\mathscr{R}$ be the minimal model of $H^*(B)$. Since B is formal there exists a formality map ρ_0: $\mathscr{R} \to R$. Since A_0 is a complete intersection the minimal models of E, E' respectively are pure; see again [14], III.3(1), Théorème. As in Section 2, one can conclude that E and E' are formal (see also Appendix, Theorem A.2). Let $A = H^*(E)$ and $A' = H^*(E')$ and consider the diagram

$$\begin{tikzcd}
\mathscr{M}(E) \arrow[r, "M(\alpha)"] \arrow[d, "\rho"'] & \mathscr{M}(E') \arrow[d, "\rho'"] \\
A \arrow[r, "\alpha"] & A'.
\end{tikzcd}$$

Here ρ and ρ' are the corresponding formality maps. Consider A and A' as $\mathscr{R}$-algebras via ρ_0 and the structure morphisms. Since C.D.G.A.'s over $\mathscr{R}$ form a closed model category in the sense of Quillen (see e.g., [1], 4.10; also [10], I.5 and [11], p. 234) there exists a C.D.G.A.-morphism $M(\alpha)$ lifting α up to homotopy. It is clear that $M(\alpha)$ induces the identity on A_0 and therefore induces a rational fiber homotopy equivalence f: $E'_\emptyset \to E_\emptyset$.

We conclude that (4) is an isomorphism. Therefore by commutativity also (3) must be an isomorphism. This proves Theorem 3.1.

As a corollary we obtain:

THEOREM 3.4. *Let the hypotheses be as in Theorem* 3.1. *Then there is a commutative diagram of isomorphisms*

$$\begin{array}{ccc} \pi_{2i}(B\,\mathrm{aut}_o\,X)\otimes_{\mathbf{Z}}\mathbf{Q} & \longrightarrow & \mathscr{F}_{X_\emptyset}(2i) \\ \downarrow & & \downarrow \\ \mathrm{Hom}^{G_m}(RX,\mathbf{Q}[\varepsilon]) & \xrightarrow{\tau(\theta_u)(2i)} & \mathscr{D}_{A_o}(-2i). \end{array}$$

Thus we see that in the context of fibrations the Kodaira-Spencer map assumes the simple significance of the map $\pi_{2i}(B\,\mathrm{aut}_o\,X)\otimes_{\mathbf{Z}}\mathbf{Q}\to\mathscr{F}_{X_\emptyset}(2i)$. We can now use Theorem C for to compute the rational homotopy groups of the space $B\,\mathrm{aut}_0^\bullet(X)$ in the case that X is a space of type F_0 with $\mathrm{Der}_{\mathbf{Q}}(H^*(X))_-=0$. Since the ring $H^*(B\,\mathrm{aut}_0^\bullet\,X)$ is isomorphic as a graded RX-algebra to the pvd ξ_v^+ constructed above, we have an explicit expression for $R^\bullet X=H^*(B\,\mathrm{aut}_0^\bullet(X);\mathbf{Q})$. In the following we want to give a rather explicit description of the rational homotopy of the space $B\,\mathrm{aut}_0^\bullet\,X$ using the terminology of the previous pages. Let us consider the following diagram

$$\begin{array}{l} \xrightarrow{\Delta}\mathrm{Hom}_{A_0}(I/I^2,A_0)\xrightarrow{\mu}T^1(A_0)\to 0 \\ \qquad\quad \Vert \\ \quad\ \displaystyle\sum_{j=1}^n A_0\frac{\partial}{\partial f_j} \end{array}$$

where μ denotes the canonical projection modulo the image of the Jacobi homomoprhism. Let $\partial_f\subset\mathrm{Hom}_{A_0}(I/I^2,A_0)$ be the subset generated over the ground field $k\cong A_0^0\subset A_0$ by the partial derivatives $\partial/\partial f_j$.

LEMMA 3.3. *If the embedding dimension* $e(A_0)$ *of* A_0 *is given by* n, *then* $\dim_k\mu(\partial_f)=n$.

Proof. We have the obvious formula

$$\mu(\partial_f)\cong\frac{\displaystyle\sum_{j=1}^n A_0^0\frac{\partial}{\partial f_j}}{\mathrm{Im}\,\Delta\cap\left(\displaystyle\sum_{j=1}^n A_0^0\frac{\partial}{\partial f_j}\right)}.$$

Thus it suffices to prove that

$$\operatorname{Im} \Delta \cap \left(\sum_{j=1}^{n} A_0^0 \frac{\partial}{\partial f_j} \right) = \{0\}.$$

Therefore we have to consider the solvability of the equations

$$\sum_{i=1}^{n} \frac{\partial f_j}{\partial x_i} \lambda_i \equiv c_j \pmod{I_0},\ j = 1, \dots, n,$$

for $c_j \in A^0 \cong k$ where at least one c_j is different from zero. By minimality we can assume that $I_0 \subset (x_1, \dots, x_n)^2$. Suppose there is a non-trivial solution $(\lambda_1, \dots, \lambda_n)$ of the system. Then we consider the system reduced $\operatorname{mod}(x_1, \dots, x_n)$:

$$\sum_{i=1}^{n} \beta_{ij} \bar{\lambda}_i = c_j,\ j = 1, \dots, n,$$

with

$$\beta_{ij} = \frac{\partial f_j}{\partial x_i}(0, \dots, 0).$$

Since $f_j \in (x_1, \dots, x_n)^2$ all the coefficients disappear and thus the reduced system cannot have a solution. Therefore also the original system does not have a solution.

THEOREM D. *Suppose X is a space of type F_0. Let $A_0 = H^*(X)$ with $\operatorname{Der}_{\mathbf{Q}} A_{0-} = 0$. Then $B \operatorname{aut}_0^\bullet X$ has no odd rational homotopy and we have the formula*

$$\pi_*\left(B \operatorname{aut}_0^\bullet X\right) \otimes_{\mathbf{Z}} \mathbf{Q} \cong \left[T_{\mathbf{Q}}^1(A_0)_- / \mu(\partial_f)\right]^* \oplus m_{A_0}/m_{A_0}^2,$$

where m_{A_0} is the maximal (augmentation-) ideal of A_0.

Proof. We can assume that the presentation of A_0 given by $A_0 = P/I_0$ is minimal in the sense that P has the minimal possible number of generators, i.e., $\operatorname{edim} A_0 = \dim P = n$. As a consequence of Lemma 3.3 we have n linearly independent homogeneous elements $t_1, \dots, t_n \in T_k^1(A_0)_-$ which are precisely the images of the $\partial/\partial f_j$ under μ and thus form a homogeneous basis of $\mu(\partial_f)$. We complete now these vectors to a homogeneous basis

$t_1, \ldots, t_n, t_{n+1}, \ldots, t_r$ of $T_k^1(A_0)_-$. After the construction of the versal deformation ξ_v we can write

$$H^*(B\,\mathrm{aut}_0^\bullet X) \cong \frac{RX[X_1, \ldots, X_n]}{(F_1, \ldots, F_n)}$$

with $F_j = F_j' - t_j$, $j = 1, \ldots, n$. Consequently can write $RX = RX'[t_1, \ldots, t_n]$, with $RX' = \mathbf{Q}[t_{n+1}, \ldots, t_r]$. Cancelling the generators t_j, $j = 1, \ldots, n$ using the relations $F_j = 0$ gives

$$H^*(B\,\mathrm{aut}_0^\bullet X) \cong \mathbf{Q}\ \underbrace{[t_{n+1}, \ldots, t_r]}_{[T^1_{\mathbf{Q}}(A_0)_-/\mu(\partial_f)]^*}\ \underbrace{[x_1, \ldots, x_n]}_{m_{A_0}/m^2_{A_0}},$$

which proves the formula. We won't close this paragraph without giving a simple application of the previous result. We take as the space X the complex projective space $\mathbf{C}P^n$. Here we have $A_0 = \mathbf{Q}[x]/I_0$, $\deg x = 2$, where I_0 is generated by $F = x^{n+1}$. Then by the previous exact sequences we obtain

$$T^1_{\mathbf{Q}}(A_0) = \frac{A_0 \frac{\partial}{\partial F}}{\left((n+1)x^n A_0 \frac{\partial}{\partial F}\right)}.$$

Thus we get

$$T^1_{\mathbf{Q}}(A_0) = T^1_{\mathbf{Q}}(A_0)_- = \mathbf{Q}[x]/(x^n)\frac{\partial}{\partial F}.$$

Consequently RX has exactly one generator in the dimensions $4, 6, \ldots, 2n+2$. Now our theorem says that we have to replace the generator which stems from the partial derivative $\partial/\partial F$ by the generator of degree two which comes from x. Thus for the rational homotopy groups of the space $B\,\mathrm{aut}_0^\bullet\, \mathbf{C}P^n$ we get precisely one generator in dimensions $2, 4, \ldots, 2n$ which is precisely the rational homotopy of $BS(U(n) \times U(1))$.

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