ILLINOIS JOURNAL OF MATHEMATICS
Volume 39, Number 2, Summer 1995

SOME REMARKS ON THE CORONA PROBLEM ON STRONGLY PSEUDOCONVEX DOMAINS IN $\mathbf{C}^n$

STEVEN G. KRANTZ[1] AND SONG-YING LI

1. Introduction

Let Ω be a bounded domain in $\mathbf{C}^n$. Let $\mathscr{H}^\infty(\Omega)$ denote the space of all bounded holomorphic functions on Ω with norm $\|\cdot\|_\infty$. It is well known that $\mathscr{H}^\infty(\Omega)$ is a Banach algebra. For each $0 < p < \infty$, we let $\mathscr{H}^p(\Omega)$ denote the usual Hardy spaces over Ω. Let $f_1, f_2, \dots, f_m \in H^\infty(\Omega)$ and $\delta > 0$ be such that

$$\delta^2 \le \sum_{k=1}^{m} |f_k(z)|^2 \le 1, \qquad z \in \Omega. \tag{1.1}$$

Then we can state the corona problem as follows:

Do there exist functions $g_1, g_2, \dots, g_m \in H^\infty(\Omega)$ such that

$$f_1(z)g_1(z) + f_2(z)g_2(z) + \cdots f_m(z)g_m(z) \equiv 1?$$

This problem has been solved by L. Carleson [C] when $n = 1$ and Ω is the unit disk. Carleson's corona theorem has been generalized to a large class of domains in the complex plane. (For example, see Jones [J], Garnett [G] and related references therein). For the case when $n > 1$, the corona problem has been studied by many authors. For examples: In [V1, V2], Varopoulos proved that the corona problem has $\mathscr{H}^p(\Omega)$ solutions when Ω is the unit ball or the unit polydisc in $\mathbf{C}^n$. His theorem for the polydisc has been reproved by S-Y A. Chang [Ch] and by K-C Lin [Lin] for $n \ge 3$ using different methods. Let us consider the operation S_f associated to corona data $f = (f_1, \dots, f_m)$ from the product space $\mathscr{H}^p(\Omega)_m = \mathscr{H}^p(\Omega) \times \mathscr{H}^p(\Omega) \times \cdots \times \mathscr{H}^p(\Omega)$ to $\mathscr{H}^p(\Omega)$ that is defined by letting $S_f(u) = f_1u_1 + \cdots f_mu_m$. Then we may restate the corona problem as follow: Is S_f: $\mathscr{H}^\infty(\Omega)_m \to \mathscr{H}^\infty(\Omega)$ onto? Some substitute results were obtained by Amar [A] and Li [L]; they proved that S_f: $\mathscr{H}^p(\Omega)_m \to \mathscr{H}^p(\Omega)$ is onto when Ω is the unit ball or the unit polydisc for all $1 < p < \infty$, respectively.

Received March 5, 1993.
1991 Mathematics Subject Classification. Primary 32A37; Secondary 32A35, 32M10.
[1]Author partially supported by a grant from the National Science Foundation.

In the general case, counterexamples have been constructed by N. Sibony [S] and J. Fornaess and N. Sibony [FS1], [FS2] to show that the corona problem is not solvable for general pseudoconvex domains, even for a domain in $\mathbf{C}^2$ that is Runge and is strongly pseudoconvex except at a single point. The question of whether the corona problem is solvable for the most standard domains like the unit ball $B = B_n$ and the unit polydisc Δ^n in $\mathbf{C}^n$ is still open. The main purpose of this paper is to provide some remarks on the corona problem when Ω is a strongly pseudoconvex domain in $\mathbf{C}^n$ with smooth boundary. For each $0 < p < \infty$, we shall use $A^p(\Omega)$ to denote the Bergman space over Ω, which is the holomorphic subspace of $L^p(\Omega)$. Let $\Lambda_\gamma(\Omega)$ be the usual Zygmund space over Ω (see [KL2]). Now we are ready to state our main theorems formally:

THEOREM 1.1. *Let $f_1, f_2, \ldots, f_m$ be bounded holomorphic functions in the unit ball B satisfying* (1.1). *Let $0 < p < \infty$. Then S_f: $A^p(B) \to A^p(B)$ is onto. Moreover, for each $h \in A^p(B)$, there are functions $g_1[h], g_2[h], \ldots, g_m[h] \in A^p(B)$ so that*

$$f_1 g_1 + \cdots f_m g_m \equiv h; \qquad \left(\sum_{j=1}^m \|g_j\|_{A^p}^2 \right)^{1/2} \leq \left[C(n,p)/\delta^3 \right] \|h\|_{A^p}.$$

It will be seen that the proof of our Theorem 1.1 actually works for any strictly convex domain in $\mathbf{C}^n$ with C^3 boundary. Next we note that strictly pseudoconvex domains are locally strictly convex (up to local biholomorphism). By applying a decomposition of unity, and then $\bar{\partial}$ theory for smooth functions, we may derive the following corollary:

THEOREM 1.2. *Let Ω be a bounded strictly pseudoconvex domain in $\mathbf{C}^n$ with C^3 boundary. Let $f_1, f_2, \ldots, f_m$ be bounded holomorphic functions in Ω satisfying* (1.1). *Let $1 < p < \infty$. Then S_f: $A^p(\Omega) \to A^p(\Omega)$ is onto. Moreover, for each $h \in A^p(\Omega)$, there are functions $g_1[h], g_2[h], \ldots, g_m[h] \in A^p(\Omega)$ so that*

$$f_1 g_1 + \cdots f_m g_m \equiv h; \qquad \left(\sum_{j=1}^m \|g_j\|_{A^p}^2 \right)^{1/2} \leq \left[C(n,p)/\delta^3 \right] \|h\|_{A^p}.$$

Note here the restriction to $1 < p < \infty$; this is imposed because we must use a factorization theorem from [COU] that is only valid for that range of p.

We shall also generalize the theorem in [A] from $1 < p < \infty$ to the full range $0 < p < \infty$. Notice that the next theorem is formulated in a different topology from the first two:

THEOREM 1.3. *Let Ω be a bounded, strictly pseudoconvex domain in $\mathbf{C}^n$ with C^3 boundary. Let $f_1, f_2, \ldots, f_m$ be bounded holomorphic functions in Ω satisfying* (1.1). *Let $0 < p < \infty$. Then S_f: $\mathscr{H}^p(\Omega) \to \mathscr{H}^p(\Omega)$ is onto. Moreover, for each $h \in \mathscr{H}^p(\Omega)$, there are functions $g_1[h], g_2[h], \ldots, g_m[h] \in \mathscr{H}^p(\Omega)$ so that*

$$f_1 g_1 + \cdots f_m g_m \equiv h; \qquad \left(\sum_{j=1}^{m} \|g_j\|^2_{\mathscr{H}^p} \right)^{1/2} \leq \left[C(n,p)/\delta^3 \right] \|h\|_{\mathscr{H}^p}$$

Furthermore, we shall prove the following theorem:

THEOREM 1.4. *Let Ω be a bounded strongly pseudoconvex domain in $\mathbf{C}^n$ with C^3 boundary. Let $f_1, f_2, \ldots, f_m \in \Lambda_\gamma(\Omega) \cap \mathscr{H}(\Omega)$ with $0 < \gamma < \infty$ satisfy* (1.1). *Then there are $g_1, g_2, \ldots, g_m \in \Lambda_\gamma(\Omega) \cap \mathscr{H}(\Omega)$ such that*

$$f_1 g_1 + f_2 g_2 + \cdots + f_m g_m \equiv 1, \quad \textit{on } \Omega,$$

and

$$\sum_{j=1}^{m} \|g_j\|_{\Lambda_\gamma} \leq C(n,m,\gamma)\delta^{-4-[\gamma]}.$$

Remark 1. From the proof of Theorems 1.1–1.4, one can see that the constants $C(n,p)$ appeared in Theorem 1.1–1.3 and $C(n,m,\gamma)$ appeared in Theorem 1.4 satisfy the following estimates:

$$C(n,p) \leq C(n)(p-1)^{-1/p} p, \qquad C(n,m,\gamma) \leq C(n,m)/\gamma,$$

for all $p > 1$ and $\gamma > 0$. As either $p \to \infty$ or $\gamma \to 0^+$ then the estimate for the constant C blows up. Thus we have not solved the original corona problem.

We note that results related to the material in the present paper appear in [An2]. See also [An1], [AnCa].

2. Preliminaries and lemmas

Let Ω be a bounded domain in $\mathbf{C}^n$. We let $\mathscr{D}_{(0,1)}(\Omega)$ denote the space of all (0, 1) forms whose coefficients are smooth and have compact support in Ω.

We shall use the notation $\mathscr{C}^k_{(0,1)}(\Omega)$ to denote the space of all $(0,1)$ forms whose coefficients lie in $C^k(\bar{\Omega})$, and we let $L^p_{(0,1)}(\Omega)$ be the space of $(0,1)$-forms with coefficients in $L^p(\Omega)$. In order to prove Theorem 1.1, we need only concentrate our attention on the case when $\Omega = B$, the unit ball in $\mathbf{C}^n$. In this section, we shall introduce some notation and collect some results from J. Polking [P] to get the appropriate integral kernel for solutions of the $\bar{\partial}$ equation. Moreover, we shall give several lemmas which will be used in Section 3. Set

$$r(\xi) = 1 - |\xi|^2, \quad \rho(z) = |z|^2 - 1, \quad z, \xi \in B,$$

and

$$\tau(z,\xi) = |\xi - z|^2 + \rho(z)\rho(\xi).$$

Then we define the following $(1,0)$-forms:

$$b(z,\xi) = \frac{1}{2\pi i|\xi - z|^2} \sum_{j=1}^{n} \left(\bar{\xi}_j - \bar{z}_j\right) d\xi_j,$$

$$\beta(z,\xi) = \frac{1}{2\pi i \tau(z,\xi)} \sum_{j=1}^{n} \left(\bar{\xi}_j - \bar{z}_j\right) d\xi_j$$

and

$$\alpha(z,\xi) = \frac{1}{2\pi i(1 - \langle z,\xi\rangle)} \sum_{j=1}^{n} \bar{\xi}_j \, d\xi_j.$$

Here $\langle z, w\rangle \equiv \Sigma_j z_j \bar{w}_j$ is the usual Hermitian inner product on $\mathbf{C}^n$. Moreover, we define the following $(n, n-1)$-forms:

$$B(z,\xi) = b(z,\xi) \wedge \left(\bar{\partial} b(z,\xi)\right)^{n-1},$$

$$\tilde{B}(z,\xi) = \beta(z,\xi) \wedge \left(\bar{\partial}\beta(z,\xi)\right)^{n-1}$$

and

$$S(z,\xi) = \alpha \wedge \left(\bar{\partial}\alpha\right)^{n-1}.$$

Let

$$A(z,w) = [\alpha(z,w) \wedge \beta(z,w)] \wedge \left[\sum_{k=1}^{n-1} \left(\bar{\partial}\alpha\right)^{k-1} \wedge \left(\bar{\partial}\beta\right)^{n-k-1}\right].$$

Then we set

$$E(z,w) = B(z,w) - S(z,\xi) + \bar{\partial}A(z,w).$$

It has been shown in [P] that $E(z,w)$ is a fundamental solution for the $\bar{\partial}$ operator. Moreover, if $p > 1$ and $f \in L^p_{(0,1)}(B)$ is $\bar{\partial}$-closed, then

$$E(f)(z) = \int_\Omega E(z,w) \wedge f(w) \tag{2.1}$$

satisfies the $\bar{\partial}$ equation:

$$\bar{\partial}E(f)(z) = f(z), \qquad z \in B \tag{2.2}$$

in the sense of distributions.

In order to write down $E(z,\xi)$ explicitly, we need to calculate $\bar{\partial}_\xi A(z,\xi)$. For simplicity, we shall write down the details only for the case $n = 2$.

LEMMA 2.1 [P]. *If $n = 2$ then we have*

$$\bar{\partial}A(z,\xi) = S(z,\xi) - \tilde{B}(z,\xi) + (\beta(z,\xi) - \alpha(z,\xi)) \wedge \left(\bar{\partial}\alpha(z,w) + \bar{\partial}\beta(z,\xi)\right).$$

LEMMA 2.2 [P]. *Let $\alpha(z,\xi)$, $\beta(z,\xi)$, $B(z,\xi)$ and $\tilde{B}(z,\xi)$ be defined as above. Then we have*

$$\begin{aligned}(\beta(z,\xi) - \alpha(z,\xi)) &\wedge \left\{\bar{\partial}\alpha(z,w) + \bar{\partial}\beta(z,\xi)\right\} \\ &= r(\xi)E_1(z,\xi) + E_2(z,\xi) \wedge \xi \cdot d\bar{\xi} + E_3(z,\xi) \wedge \xi \cdot d\bar{\xi},\end{aligned}$$

where

$$\begin{aligned} E_1(z,\xi) = {} & \left(\frac{1}{2\pi i}\right)^2 \frac{\left(\dfrac{1}{1 - \langle z,\xi\rangle} + \dfrac{1}{\tau(z,\xi)}\right)}{(1 - \langle z,\xi\rangle)\tau(z,\xi)} d\bar{\xi} \cdot d\xi \wedge \left(\bar{\xi} - \bar{z}\right) \cdot d\xi \\ & - \frac{1}{(2\pi i)^2} \frac{r(z)(1/(1 - \langle z,\xi\rangle)) + (1/\tau(z,\xi))}{(1 - \langle z,\xi\rangle)\tau(z,\xi)} d\bar{\xi} \cdot d\xi \\ & \wedge \bar{\xi} \cdot d\xi, \\ E_2(z,\xi) = {} & \left(\frac{1}{2\pi i}\right)^2 \frac{1}{(1 - \langle z,\xi\rangle)^2 \tau(z,\xi)} \bar{\xi} \cdot d\xi \wedge \left(\bar{\xi} - \bar{z}\right) \cdot d\xi, \\ E_3(z,\xi) = {} & \left(\frac{1}{2\pi i}\right)^2 \frac{r(z)|\xi - z|^2}{(1 - \langle z,\xi\rangle)\tau(z,\xi)^3} \bar{\xi} \cdot d\xi \wedge \left(\bar{\xi} - \bar{z}\right) \cdot d\xi; \end{aligned}$$

here we have used the notation

$$w \cdot dz = \sum_{k=1}^{n} w_k dz_k, \quad dw \cdot dz = \sum_{k=1}^{n} dw_k \wedge dz_k.$$

Combining Lemmas 2.1 and 2.2, we have the following:

LEMMA 2.3. *With notation as above, we have*

$$\begin{aligned} E(z,\xi) &= B(z,\xi) - \tilde{B}(z,\xi) + r(\xi)E_1(z,\xi) \\ &\quad + (E_2(z,\xi) + E_3(z,\xi)) \wedge \xi \cdot d\bar{\xi}. \end{aligned}$$

Now let us compute $B(z,\xi) - \tilde{B}(z,\xi)$. From the definitions of $B(z,\xi)$ and $\tilde{B}(z,\xi)$, one can easily verify that

$$\begin{aligned} B(z,\xi) - \tilde{B}(z,\xi) &= \left(\frac{1}{2\pi i}\right)^2 \sum_{j=1}^{2} \left(\bar{\xi}_j - \bar{z}_j\right) \\ &\quad \times \left(\frac{2r(z)r(\xi)}{|\xi - z|^2 \tau(z,\xi)^2} = \frac{r(z)^2 r(\xi)^2}{|\xi - z|^4 \tau^2} \right) d\xi_j \wedge d\bar{\xi} \cdot d\xi. \end{aligned}$$

First let us give an upper bound for these kernels:

LEMMA 2.4. *With the notation above, we have*

$$\begin{aligned} &\left(\left|B(z,\xi) - \tilde{B}(z,\xi)\right| + r(\xi)|E_1(z,\xi)|\right) r(\xi)^{-1} \\ &\quad \le C(n) \left(\frac{r(z)}{|\xi - z|^3 \tau(z,\xi)} + \frac{1}{|1 - \langle z,\xi\rangle| \tau(z,\xi)^{3/2}} \right). \end{aligned}$$

Proof. Observing that

$$\tau(z,\xi) \le C|1 - \langle z,\xi\rangle|, \quad r(z)r(\xi) \le \tau(z,\xi), \quad \text{and } |\xi - z|^2 \le \tau(z,\xi),$$

one can easily see that the lemma holds. ■

We need to estimate all of the terms that arise in the above calculations. First we show:

LEMMA 2.5. *For* $0 < \varepsilon < 1$, *we have*

$$\int_B \frac{r(z) + r(\xi)}{r(\xi)^{\varepsilon} |\xi - z|^3 \tau(z,\xi)} \, dV(\xi) \le C r(z)^{-\varepsilon}.$$

Proof. We use local coordinates as before. We have

$$\int_B \frac{r(z) + r(\xi)}{r(\xi)^{\varepsilon}|\xi - z|^3 \tau(z, \xi)} \, dV(\xi)$$

$$\leq C\left(\int_0^1 \int_0^1 \int_{|\lambda| \leq 1} \frac{r(z) + t}{t^{\varepsilon}(r(z) + |\lambda| + s)^3} \frac{dA(\lambda) \, ds \, dt}{(r(z) + t + s + |\lambda|)^2} + r(z)^{-\varepsilon}\right)$$

$$\leq C\left(\int_0^1 \int_0^1 \frac{t^{-\varepsilon}}{(r(z) + s)} \frac{1}{r(z) + t + s} \, ds \, dt + r(z)^{-\varepsilon}\right)$$

$$\leq C\left(\int_0^1 \frac{t^{-\varepsilon}}{(r(z) + t)} \, dt + r(z)^{-\varepsilon}\right)$$

$$\leq C_{\varepsilon} r(z)^{-\varepsilon}.$$

Therefore the proof of Lemma 2.5 is complete. ■

The following lemma for controlling non-isotropic integrals will be needed in Section 3. For each $0 < \varepsilon < 1$, and $\alpha, \beta \geq 0$, we define

$$I_{\alpha, \beta, \varepsilon}(z) = \int_{B_2} \frac{r(\xi)^{-\varepsilon}}{|1 - \langle z, \xi \rangle|^{\alpha} \tau(z, \xi)^{\beta}} \, dV(\xi).$$

Now we have:

LEMMA 2.6. (a) *If $\beta > 1$, then there are three cases:*
(i) *If $\alpha + 2\beta + \varepsilon > 4$, then*

$$I_{\alpha, \beta, \varepsilon}(z) \leq C_{\varepsilon, \alpha, \beta} r(z)^{4 - \alpha - 2\beta - \varepsilon};$$

(ii) *If $\alpha + 2\beta + \varepsilon < 4$, then*

$$I_{\alpha, \beta, \varepsilon}(z) \leq C_{\alpha, \beta, \varepsilon};$$

(iii) *If $\alpha + 2\beta + \varepsilon = 4$, then*

$$I_{\alpha, \beta, \varepsilon}(z) \leq C_{\alpha, \beta, \varepsilon}\left(\log \frac{1}{r(z)}\right).$$

(b) *If $\beta = 1/2$, then we have the following two cases:*
(i) *If $\alpha + \varepsilon - 1/2 > 2$, then*

$$I_{\alpha, \beta, \varepsilon}(z) \leq C r(z)^{5/2 - \alpha - \varepsilon};$$

(ii) *If* $\alpha + \varepsilon - 1/2 = 2$, *then*

$$I_{\alpha,\beta,\varepsilon}(z) \le C\left(\log \frac{1}{r(z)}\right).$$

Proof. Let us consider case (i) in part (a) first. Using the usual local coordinates, we have

$$\begin{aligned}
I_{\alpha,\beta,\varepsilon}(z) &= \int_{B_2} \frac{r(\xi)^{-\varepsilon}}{|1-\langle z,\xi\rangle|^{\alpha}\tau(z,\xi)^{\beta}}\,dV(\xi) \\
&\le C\int_0^1\int_0^1\int_{|\lambda|\le 1} \frac{t^{-\varepsilon}}{\left(r(z)+t+s+|\lambda|^2\right)^{\alpha}} \frac{dA(\lambda)\,ds\,dt}{\left(r(z)+t+s+|\lambda|\right)^{2\beta}} \\
&\le C\int_0^1\int_0^1 \frac{t^{-\varepsilon}}{\left(r(z)+t+s\right)^{\alpha+2(\beta-1)}}\,ds\,dt \\
&\le C_{\varepsilon,\alpha,\beta}\, r(z)^{4-\alpha-2\beta-\varepsilon}.
\end{aligned}$$

Applying the same argument as above, we can prove (ii) and (iii) in part (a), and we omit the details here.

Next we consider part (b). We shall prove (i) in (b). We see that

$$\begin{aligned}
I_{\alpha,\beta,\varepsilon}(z) &= \int_{B_2} \frac{r(\xi)^{-\varepsilon}}{|1-\langle z,\xi\rangle|^{\alpha}\tau(z,\xi)^{1/2}}\,dV(\xi) \\
&\le C\int_0^1\int_0^1\int_{|\lambda|\le 1} \frac{t^{-\varepsilon}}{\left(r(z)+t+s+|\lambda|^2\right)^{\alpha}} \frac{dA(\lambda)\,ds\,dt}{\left(r(z)+t+s+|\lambda|\right)} \\
&\le C\int_0^1\int_0^1\int_0^1 \frac{t^{-\varepsilon}}{\left(r(z)+t+s+|\lambda|^2\right)^{\alpha}}\,d|\lambda|\,ds\,dt \\
&\le \int_0^1\int_0^1\int_0^{\infty} \frac{t^{-\varepsilon}}{\left(r(z)+t+s\right)^{\alpha-1/2}\left(1+x^2\right)^{\alpha}}\,dx\,ds\,dt \\
&\le C_{\varepsilon,\alpha,\beta}\, r(z)^{5/2-\alpha-\varepsilon}.
\end{aligned}$$

Similar arguments show that (ii) holds. Therefore the proof of Lemma 2.6 is complete. ∎

3. Proof of Theorem 1.1

In this section, we shall prove Theorem 1.1. In order to avoid unnecessary complications, we restrict our argument to the case when $n = m = 2$. From our proof, it will be clear that the methods work for general n. Then we may apply Koszul complex theory (see [K1]) to extend to general m.

Let $f_1, f_2 \in \mathscr{H}^\infty(B)$ and $\delta > 0$ be such that

$$\delta^2 \leq |f(z)|^2 = |f_1(z)|^2 + |f_2(z)|^2 \leq 1 \quad \text{for all } z \in B. \tag{3.1}$$

Let

$$F = \frac{\bar{f}_1 \overline{\partial f_2} - \bar{f}_2 \overline{\partial f_1}}{|f|^4}$$

where $|f|^4 = (|f_1(z)|^2 + |f_2(z)|^2)^2$. For each $h \in A^p(B)$, it is easy to show that hF is a $\bar{\partial}$-closed $(0,1)$-form, and $\bar{\partial} E(hF) = hF$. Now we set

$$g_1[h] = \varphi_1 h - f_2 E(hF), \quad g_2[h] = \varphi_2 h + f_1 E(hF),$$

where

$$\varphi_j(z) = \frac{\bar{f}_j(z)}{|f(z)|^2}, \qquad j = 1, 2. \tag{3.2}$$

Then one can verify that

$$\bar{\partial} g_j = 0,\ j = 1, 2; \quad \text{and} \quad f_1 g_1[h] + f_2 g_2[h] \equiv h.$$

To prove Theorem 1.1, we therefore need only study the regularity of the solution $E(hF)$. We start with the following lemma.

LEMMA 3.1. *Let F be as above. Then we have*

$$(\xi \cdot d\bar{\xi}) \wedge E_j(z, \xi) \wedge F = \frac{\bar{f}_1 \overline{T_{12} f_2} - \bar{f}_2 \overline{T_{12} f_1}}{|f|^4} \wedge E_j(z, \xi) \wedge d\bar{\xi}_1 \wedge d\bar{\xi}_2,$$

where $T_{12} = \bar{\xi}_1(\partial/\partial \xi_2) - \bar{\xi}_2(\partial/\partial \xi_1)$ and $j = 2, 3$.

Proof. Since

$$\begin{aligned}\xi \cdot d\bar{\xi} \wedge E_j(z,\xi) \wedge F &= \xi \cdot d\bar{\xi} \wedge F \wedge E_j(z,\xi)\\ &= \xi \cdot d\bar{\xi} \wedge \frac{\bar{f}_1\overline{\partial f_2} - \bar{f}_2\overline{\partial f_1}}{|f|^4} \wedge E_j(z,\xi)\\ &= \frac{\bar{f}_1\overline{T_{12}f_2} - \bar{f}_2\overline{T_{12}f_1}(z)}{|f|^4} \wedge E_j(z,\xi),\end{aligned}$$

The proof of Lemma 3.1 is complete. ■

Now we are ready to prove Theorem 1.1.
Since $f_j \in \mathscr{H}^\infty(B)$ and (3.1) holds, we have

$$|\nabla f_1(\xi)| + |\nabla f_2(\xi)| \le \frac{C(n)}{r(\xi)}, \tag{3.3}$$

and

$$|T_{12}f_1(\xi)| + |T_{12}f_2(\xi)| \le \frac{C(n)}{r(\xi)^{1/2}} \tag{3.4}$$

for all $\xi \in B$. Applying Lemmas 2.4 and 3.1, we have

$$\begin{aligned}&|E(hF)(z)|\\ &\quad\le \int_B \left\{ \left|\left(B(z,\xi) - \tilde{B}(z,\xi)\right) \wedge F(\xi)\right| + |r(\xi)E_1(z,\xi) \wedge F(\xi)|\right\} |h(\xi)|\\ &\qquad + \left|\int_B (E_2(z,\xi) + E_3(\xi)) \wedge \xi \cdot d\bar{\xi} \wedge F(\xi)h(\xi)\right|\\ &\quad\le \int_B \left\{ \left|\left(B(z,\xi) - \tilde{B}(z,\xi)\right)\right| r(\xi)^{-1} + |r(\xi)E_1(z,\xi)| r(\xi)^{-1}\right\}\\ &\qquad \times |h(\xi)|\, dV(\xi)\\ &\qquad + \left|\int_B \left(E_2(z,\xi) \wedge \xi \cdot d\bar{\xi} \wedge F(\xi)h(\xi)\right|\right.\\ &\qquad + \left|\int_B E_3(z,\xi) \wedge \xi \cdot d\bar{\xi} \wedge Fh)\right|\\ &\quad = J_1(h)(z) + J_2(h)(z) + J_3(h)(z)\end{aligned}$$

Applying Lemmas 2.4, 2.5 and 2.6 with $\alpha = 1$, $\beta = 3/2$ and $0 < \varepsilon < 1$ and Schur's lemma [Z], we have

$$\| J_1(h) \|_{L^p(B)} \le \frac{C(n,p)}{\delta^3} \|h\|_{A^p}.$$

We study $J_3(h)$ first. Since

$$\begin{aligned}
&\bar{\partial}_\xi E_3(z,\xi)\\
&= \left(\frac{1}{2\pi i}\right)^2 \frac{\left\{(z-\xi)\cdot d\bar{\xi} + \xi \cdot d\bar{\xi}\right\} r(z)|\xi - z|^2}{(1-\langle z,\xi\rangle)^2 \tau(z,\xi)^3}\\
&\qquad \wedge \bar{\xi}\cdot d\xi \wedge \left(\bar{\xi} - \bar{z}\right)\cdot d\xi\\
&\qquad + \left(\frac{1}{2\pi i}\right) \frac{3\left\{(z-\xi)\cdot d\bar{\xi} + r(z)\xi\cdot d\bar{\xi}\right\} r(z)|\xi - z|^2}{(1-\langle z,\xi\rangle)\tau(z,\xi)^4}\\
&\qquad\quad \wedge \bar{\xi}\cdot d\xi \wedge \left(\bar{\xi} - \bar{z}\right)\cdot d\xi\\
&\qquad + \left(\frac{1}{2\pi i}\right) \frac{r(z)(z-\xi)\cdot d\bar{\xi}}{(1-\langle z,\xi\rangle)\tau(z,\xi)^3} \wedge \bar{\xi}\cdot d\xi \wedge \left(\bar{\xi} - \bar{z}\right)\cdot d\xi\\
&\qquad + \left(\frac{1}{2\pi i}\right) \frac{r(z)|\xi - z|^2}{(1-\langle z,\xi\rangle)\tau(z,\xi)^3}\\
&\qquad\quad \times \left\{ d\bar{\xi}\cdot d\xi \wedge \left(\bar{\xi} - \bar{z}\right)\cdot d\xi - \bar{\xi}\cdot d\xi \wedge d\bar{\xi} \wedge d\xi \right\},
\end{aligned}$$

we see that

$$\begin{aligned}
&\left| r(\xi)\bar{\partial}_\xi E_3(z,\xi) \wedge F(\xi) \right|\\
&\quad \le \frac{C}{\delta^3} \left\{ \frac{r(z)}{|1-\langle z,\xi\rangle|^2 \tau(z,\xi)} + \frac{r(z)\left(r(\xi)^{1/2} + |\xi - z|\right)}{|1-\langle z,\xi\rangle|^2 \tau(z,\xi)^2} \right.\\
&\qquad\qquad \left. + \frac{r(z)}{|1-\langle z,\xi\rangle| \tau(z,\xi)^2} + \frac{r(z)^2 r(\xi)^{1/2}|\xi - z|}{|1-\langle z,\xi\rangle| \tau(z,\xi)^3} \right\} dV(\xi)\\
&\quad \le \frac{C}{\delta^3} \left\{ \frac{r(z)}{|1-\langle z,\xi\rangle| \tau(z,\xi)^2} + \frac{r(z) r(\xi)^{1/2}|\xi - z|}{|1-\langle z,\xi\rangle|^2 \tau(z,\xi)^2} \right\} dV(\xi)\\
&\quad \le \frac{C\delta^{-3}}{\tau(z,\xi)^2}\, dV(\xi).
\end{aligned}$$

LEMMA 3.2. *Recall that the dimension n is 2. Let*

$$T(u)(z) = \int_B u(\xi)\tau(z,\xi)^{-2}\, dV(\xi)$$

Then T: $L^p(B) \to L^p(B)$ is bounded for all $1 < p < \infty$.

Proof. This follows from Lemma 2.6 with $\alpha = 0$, $\beta = 2$ and $0 < \varepsilon < 1$ and from Schur's lemma. ■

By using integration by parts, we find that

$$\begin{aligned} |E_3(Fh)(z)| &= \left| \int_B r(\xi)\bar{\partial}_\xi E_3(z,\xi) \wedge F(\xi)h(\xi) \right| \\ &\le \frac{C}{\delta^3}\int_B \tau(z,\xi)^{-2}|h(\xi)|\, dV(\xi) \\ &= T(|h|)(z). \end{aligned}$$

Thus Lemma 3.2 implies that

$$\| J_3(h) \|_{L^p} \le \frac{C(n,p)}{\delta^3} \|h\|_{A^p}$$

for all $1 < p < \infty$.

Next we show that $\|J_2(Fh)\|_{L^p(B)} \le C/\delta^3 \|h\|_{A^p}$.

Since

$$\begin{aligned} &\bar{\partial}_\xi E_2(z,\xi) \\ &= \left(\frac{1}{2\pi i}\right)^2 \frac{2(z-\xi)\cdot d\bar{\xi} + 2\xi \cdot d\bar{\xi}}{(1-\langle z,\xi\rangle)^3 \tau(z,\xi)} \wedge \bar{\xi}\cdot d\xi \wedge \left(\bar{\xi} - \bar{z}\right)\cdot d\xi \\ &+ \left(\frac{1}{2\pi i}\right)^2 \frac{-(\xi - z)\cdot d\bar{\xi} + r(z)\xi\cdot d\bar{\xi}}{(1-\langle z,\xi\rangle)^2 \tau(z,\xi)^2} \wedge \bar{\xi}\cdot d\xi \wedge \left(\bar{\xi} - \bar{z}\right)\cdot d\xi \\ &+ \left(\frac{1}{2\pi i}\right)^2 \frac{1}{(1-\langle z,\xi\rangle)^2 \tau(z,\xi)} d\bar{\xi}\cdot d\xi \wedge \left(\bar{\xi} - \bar{z}\right)\cdot d\xi \\ &- \left(\frac{1}{2\pi i}\right)^2 \frac{1}{(1-\langle z,\xi\rangle)^2 \tau(z,\xi)} \bar{\xi}\cdot d\xi \wedge d\bar{\xi}\cdot d\xi, \end{aligned}$$

we have

$$
\begin{aligned}
&\left|r(\xi)\bar{\partial}_\xi E_2(z,\xi)\wedge F(\xi)\right| \\
&\quad\le \frac{C}{\delta^3}\left(\frac{1}{|1-\langle z,\xi\rangle|^{5/2}\tau(z,\xi)^{1/2}}\right)dV(\xi) \\
&\qquad + Cr(\xi)\left|\left\{\frac{F(\xi)\wedge(\xi-z)\cdot d\bar{\xi}}{(1-\langle z,\xi\rangle)^2\tau(z,\xi)^2}\wedge\bar{\xi}\cdot d\xi\wedge\left(\bar{\xi}-\bar{z}\right)\cdot d\xi\right.\right. \\
&\qquad\qquad \left.\left. + \frac{1}{(1-\langle z,\xi\rangle)^2\tau(z,\xi)}F(\xi)\wedge\bar{\xi}\cdot d\xi\wedge d\bar{\xi}\cdot d\xi\right\}\right| \\
&\quad\le C\left(\frac{1}{\delta^3}G(z,\xi)+G_1(z,\xi)\right);
\end{aligned}
$$

Now let us consider G first; we prove the following lemma.

LEMMA 3.3. *Let $G(z,\xi)$ be defined as above. Set*

$$G(u)(z)=\int_B u(\xi)G(z,\xi)$$

Then G: $L^p(B)\to L^p(B)$ is bounded for all $1<p<\infty$.

Proof. Use Lemma 2.6 with $\alpha=5/2$, $\beta=1/2$ and $0<\varepsilon<1$ and Schur's lemma. ■

In order to estimate $G_1(z,\xi)$, let us do the following calculations:

$$
\begin{aligned}
&\overline{\partial f}(\xi)\wedge\bar{\xi}\cdot d\xi\wedge d\bar{\xi}\cdot d\xi \\
&\quad=\overline{\partial f}(\xi)\wedge\bar{\xi}\cdot d\xi\wedge d\bar{\xi}_1\wedge d\xi_1+\bar{\partial}f(\xi)\wedge\bar{\xi}\cdot d\xi\wedge d\bar{\xi}\wedge d\xi_2 \\
&\quad=\frac{\overline{\partial f}}{\partial\xi_2}\bar{\xi}_2\,d\bar{\xi}_2\wedge d\xi_2\wedge d\bar{\xi}_1\wedge d\xi_1+\frac{\overline{\partial f}}{\partial\xi_1}\bar{\xi}_1\,d\bar{\xi}_1\wedge d\xi_1\wedge d\bar{\xi}_2\wedge d\xi_2 \\
&\quad=\overline{\mathscr{R}f(\xi)}\,d\bar{\xi}_1\wedge d\xi_1\wedge d\bar{\xi}_2\wedge d\xi_2 \\
&\quad=\overline{\mathscr{R}f(\xi)}\,dv(\xi)
\end{aligned}
$$

where

$$\mathscr{R}f(\xi)=\frac{\partial f}{\partial\xi_2}\xi_2+\frac{\partial f}{\partial\xi_1}\xi_1,\quad dv(\xi)=d\bar{\xi}_1\wedge d\xi_1\wedge d\bar{\xi}_2\wedge d\xi_2;$$

and

$$
\begin{aligned}
&\overline{\partial f}(\xi) \wedge (\xi - z) \cdot d\bar{\xi} \wedge \bar{\xi} \cdot d\xi \wedge \left(\bar{\xi} - \bar{z}\right) \cdot d\xi \\
&\quad = \overline{\partial f}(\xi) \wedge (\xi - z) \cdot d\bar{\xi} \\
&\qquad \wedge \left(\bar{\xi}_1\, d\xi_1\left(\bar{\xi}_2 - \bar{z}_2\right) d\xi_2 + \bar{\xi}_2\, d\xi_2 \wedge \left(\bar{\xi}_1 - \bar{z}_1\right) d\xi_1\right) \\
&\quad = \overline{\partial f}(\xi) \wedge (\xi - z) \cdot d\bar{\xi} \wedge \left(\bar{\xi}_1\left(\bar{\xi}_2 - \bar{z}_2\right) - \left(\bar{\xi}_1 - \bar{z}_1\right)\bar{\xi}_2\right) d\xi_1 \wedge d\xi_2 \\
&\quad = \left(\frac{\overline{\partial f}}{\partial \xi_1}(\xi_2 - z_2)\, d\bar{\xi}_1 \wedge d\bar{\xi}_2 - \frac{\overline{\partial f}}{\partial \xi_2}(\xi_1 - z_1)\, d\bar{\xi}_1 \wedge d\bar{\xi}_2\right) \\
&\qquad \wedge \left(\bar{\xi}_1\left(\bar{\xi}_2 - \bar{z}_2\right) - \left(\bar{\xi}_1 - \bar{z}_1\right)\bar{\xi}_2\right) d\xi_1 \wedge d\xi_2 \\
&\quad = -\left(\frac{\overline{\partial f}}{\partial \xi_1}(\xi_2 - z_2) - \frac{\overline{\partial f}}{\partial \xi_2}(\xi_1 - z_1)\right) \\
&\qquad \times \left(\bar{\xi}_1\left(\bar{\xi}_2 - \bar{z}_2\right) - \left(\bar{\xi}_1 - \bar{z}_1\right)\bar{\xi}_2\right) dv(\xi) \\
&\quad = -\left(\bar{\xi}_1 \frac{\overline{\partial f}}{\partial \xi_1}|\xi_2 - z_2|^2 + \frac{\overline{\partial f}}{\partial \xi_2}\bar{\xi}_2|\xi_1 - z_1|^2\right) dv(\xi) \\
&\qquad + \left(\frac{\overline{\partial f}}{\partial \xi_2}\bar{\xi}_1(\xi_1 - z_1)\left(\bar{\xi}_2 - \bar{z}_2\right) + \frac{\overline{\partial f}}{\partial \xi_1}\bar{\xi}_2(\xi_2 - z_2)\left(\bar{\xi}_1 - z_1\right)\right) dv(\xi) \\
&\quad \equiv N(z, \xi)\, d\bar{\xi}_1 \wedge d\xi_1 \wedge d\bar{\xi}_2 \wedge d\xi_2.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&|\xi - z|^2 \overline{\mathscr{R} f(\xi)} + N(z, \xi) \\
&\quad = \bar{\xi}_1 \frac{\overline{\partial f}}{\partial \xi_1}|\xi_1 - z_1|^2 + \bar{\xi}_2 \frac{\overline{\partial f}}{\partial \xi_2}|\xi_2 - z_2|^2 \\
&\qquad + \frac{\overline{\partial f}}{\partial \xi_2}\bar{\xi}_1(\xi_1 - z_1)\left(\bar{\xi}_2 - \bar{z}_2\right) + \frac{\overline{\partial f}}{\partial \xi_1}\bar{\xi}_2(\xi_2 - z_2)\left(\bar{\xi}_1 - \bar{z}_1\right) \\
&\quad = \bar{\xi}_1 \frac{\overline{\partial f}}{\partial \xi_1}|\xi_1 - z_1|^2 + \bar{\xi}_2 \frac{\overline{\partial f}}{\partial \xi_2}|\xi_2 - z_2|^2 \\
&\qquad + \frac{\overline{\partial f}}{\partial \xi_2}\left(\bar{\xi}_1(\xi_1 - z_1) + \bar{\xi}_2(\xi_2 - z_2)\right)\left(\bar{\xi}_2 - \bar{z}_2\right) - \bar{\xi}_2 \frac{\overline{\partial f}}{\partial \xi_2}|\xi_2 - z_2|^2 \\
&\qquad + \frac{\overline{\partial f}}{\partial \xi_1}\left(\bar{\xi}_2(\xi_2 - z_2) + \bar{\xi}_1(\xi_1 - z_1)\right)\left(\bar{\xi}_1 - \bar{z}_1\right) - \bar{\xi}_1 \frac{\overline{\partial f}}{\partial \xi_1}|\xi_1 - z_1|^2 \\
&\quad = \frac{\overline{\partial f}}{\partial \xi_2}\left(\bar{\xi}_1(\xi_1 - z_1) + \bar{\xi}_2(\xi_2 - z_2)\right)\left(\bar{\xi}_2 - \bar{z}_2\right) \\
&\qquad + \frac{\overline{\partial f}}{\partial \xi_1}\left(\bar{\xi}_2(\xi_2 - z_2) + \bar{\xi}_1(\xi_1 - z_1)\right)\left(\bar{\xi}_1 - \bar{z}_1\right) \\
&\quad = Q_1(z, \xi) + Q_2(z, \xi).
\end{aligned}
$$

Since

$$\left|\bar{\xi}_1(\xi_1 - z_1) + \bar{\xi}_2(\xi_2 - z_2)\right| \leq C|1 - \langle z, \xi\rangle|,$$

we have

$$\left| Q_j(z,\xi) \frac{r(\xi)}{(1 - \langle z,\xi\rangle)^2 \tau(z,\xi)^2} \right| \leq C \frac{1}{|1 - \langle z,\xi\rangle| \tau(z,\xi)^{3/2}},$$

for $j = 1, 2$.

Therefore

$$\begin{aligned}|G_1(z,\xi)| &\leq \frac{C}{\delta^3}\left(|Q_1(z,\xi)| + |Q_2(z,\xi)| + \frac{1}{\tau(z,\xi)^2}\right) dv(\xi) \\ &\leq \frac{C}{\delta^3} \frac{1}{|1 - \langle z,\xi\rangle| \tau(z,\xi)^{3/2}} dv(\xi).\end{aligned}$$

Applying Lemma 2.6 and Schur's lemma, we have the following:

LEMMA 3.4. *Let*

$$G_1(u)(z) = \int_B u(\xi) G_1(z,\xi)\, dv$$

Then G_1: $L^p(B) \to L^p(B)$ *is bounded for all* $1 < p < \infty$.

Combining all the above estimates, we have completed the proof of Theorem 1.1 for $1 < p < \infty$.

COROLLARY 3.5. *Let F be as above. Then*

$$|E(F)(z)| \leq \frac{C}{\delta^3}\left(\log \frac{1}{r(z)}\right);$$

and for any $\gamma > 0$, *if* $f \in \Lambda_\gamma(\Omega)$, *then*

$$\|E(F)\|_{L^\infty} \leq \frac{C_\gamma}{\delta^3} \|f\|_{\Lambda_\gamma}.$$

For the case when $0 < p \leq 1$ we cannot prove Theorem 1.1 directly. We need the aid of the following decomposition theorem for Bergman spaces due to R. R. Coifman and R. Rochberg [CR] for $0 < p < 2$:

THEOREM 3.6. *Let* $h \in A^p(B)$. *Then we can write*

$$h(z) = \sum_{k=1}^{\infty} \lambda_k u_k(z) v_k(z).$$

with $u_k \in A^{pq}(B)$ *and* $v_k \in A^{pq'}(B)$ *such that*

$$\sum_{k=1}^{\infty} |\lambda_k|^p \approx \|h\|_{A^p}^p, \qquad \|u_k\|_{pq}\|v_k\|_{pq'} \le 1$$

where $q > 1$, q *and* q' *are conjugate exponents.*

Let $0 < p \le 1$ and $h \in A^p(B)$. Then we choose $q > 1$ so that $pq = 2$ and

$$h = \sum_{k=1}^{\infty} \lambda_k u_k v_k, \qquad u_k \in A^{pq},\, v_k \in A^{pq'}$$

as in Theorem 3.5.

Applying the conclusion of Theorem 1.1 for the case $1 < p < \infty$ to u_k, we see that there exist $g_j[u_k]$ so that

$$\sum_{j=1}^{m} f_j g_j[u_k] \equiv u_k, \quad k = 1, 2, \ldots$$

and

$$\big\| g_j[u_k] \big\|_{L^{pq}} \le \frac{C}{\delta^3} \|u_k\|_{A^{pq}}.$$

Now we let

$$g_j[h] = \sum_{k=1}^{\infty} \lambda_k g_j[u_k] v_k$$

Then we have $g_j[h] \in A^p(B)$ and

$$\begin{aligned}\big\| g_j[h] \big\|_{L^p(B)} &\le \Big\| \sum_{k=1} \lambda_k g_j[u_k] v_k \Big\|_{A^p} \\ &\le \big(C(n,p)/\delta^3\big) \left(\sum_{k=1}^{\infty} |\lambda_k|^p \big\| g_j[u_k] v_k \big\|_p^p \right)^{1/p} \\ &\le \frac{C(n,p)}{\delta^3} \|h\|_{A^p},\end{aligned}$$

for all $j = 1, 2, \ldots, m$. This completes the proof of the case $0 < p \le 1$.

Therefore, the proof of Theorem 1.1 is complete.

The proof of Theorem 1.2 follows similar lines, and has been described in the introduction. We omit the details.

Now we prove Theorem 1.3.

With the help of Theorem 2.2 in [KL1] on Carleson measures and BMOA(Ω) functions, and the decomposition or factorization theorem (Theorem 1.2 in [KL2]) for Hardy spaces in a strongly pseudoconvex domain in $\mathbf{C}^n$ with smooth boundary, we may generalize a theorem in [A] from the unit ball to a strongly pseudoconvex domain with smooth boundary. Thus we have that S_f: $\mathscr{H}^p(\Omega) \to \mathscr{H}^p(\Omega)$ is bounded and onto for all $1 < p < \infty$.

Now we consider the case $0 < p \leq 1$. For each $0 < p \leq 1$ and $h \in \mathscr{H}^p(\Omega)$, we apply the factorization theorem for $\mathscr{H}^p(\Omega)$ (Theorem 1.2 in [KL2]) several times (indeed, k_0 times) with $1 < 2k_0 p \leq 2$. For simplicity, we write down the case $k_0 = 2$; from the proof, one can see how the argument works for any k_0 as above.

Thus, with $k_0 = 2$, we have

$$h(z) = \sum_{k=1}^{\infty} \lambda_k u_k \sum_{l=1}^{\infty} \lambda_{l,k} u_{l,k} v_{l,k}.$$

Here $u_k \in \mathscr{H}^{2p}(\Omega)$, $u_{l,\mathrm{k}}, v_{l,\mathrm{k}} \in \mathscr{H}^{4p}(\Omega)$ and

$$\sum_{k=1}^{\infty} |\lambda_k|^p \approx \|h\|_{\mathscr{H}^p}^p,$$

$$\|u_{l,k}\|_{\mathscr{H}^{4p}} \|v_{l,k}\|_{\mathscr{H}^{4p}} \leq 1,$$

$$\|u_k\|_{\mathscr{H}^{2p}} \Big(\sum |\lambda_{l,k}|^{2p} \Big)^{1/2p} \leq C.$$

Then we apply the previous result to the functions $v_{l,k} \in \mathscr{H}^{2k_0 p}(\Omega)$. We conclude that there are functions $g_j[v_{l,k}] \in \mathscr{H}^{2k_0 p}(\Omega)$ such that

$$\sum_{j=1}^{m} f_j(z) g_j[v_{l,k}](z) = v_{l,k}(z),$$

$$\left(\sum_{j=1}^{m} \| g[v_{l,k}] \|_{2k_0 p}^2 \right)^{1/2} \leq C(n, 2k_0 p) \|v_{l,k}\|_{\mathscr{H}^{2k_0 p}}.$$

Thus if we let

$$g_j[h] = \sum_{k=1} \lambda_k u_k \sum_{l=1}^{\infty} u_{l,k} g_j[v_{l,k}],$$

then $g_j[h]$ is in $\mathscr{H}^p(\Omega)$ and satisfies

$$\sum_{j=1}^{m} f_j(z) g_j[h](z) = h(z),$$

and

$$\left(\sum_{j=1}^{m} \|g_j\|^2_{\mathscr{H}^p}\right)^{1/2} \le \frac{C(n,p)}{\delta^3}\|h\|_{\mathscr{H}^p}.$$

Therefore

$$S_f\colon \mathscr{H}^p(\Omega) \to \mathscr{H}^p(\Omega)$$

is bounded and onto for all $0 < p < \infty$. This completes the proof of Theorem 1.3. ■

4. Proof of Theorem 1.4

In this section, we shall prove Theorem 1.4. For simplicity, we only consider the case when $n = m = 2$ and Ω is the unit ball.

By the argument at the beginning of Section 3, we see that it suffices to prove that $E(F) \in \Lambda_\gamma(B)$. We shall separate our argument into several cases.

Case 1. $0 < \gamma \le 1$. As usual, we shall prove that

$$|\nabla_z E(F)(z)| \le (C(n)/\delta^3) r(z)^{\gamma-1}.$$

Since $\bar\partial E(F) = F$ and $f \in \Lambda_\gamma(B)$, we have

$$|F(z)| \le \frac{C(n)}{\delta^3} r(z)^{\gamma-1}.$$

(Here f denotes the m-tuples $(f_1, \dots, f_m)$; see also the beginning of Section 3.) Thus we need only show that

$$|\partial_z E(F)(z)| \le \frac{C(n)\|f\|_{\Lambda_\gamma}}{\delta^3} r(z)^{\gamma-1}, \quad z \in B.$$

Since $f \in \Lambda_\gamma(B)$, we have

$$\left|T_{ij} f_k(z)\right| \leq C\|f\|_{\Lambda_\gamma} g_\gamma(z),$$

where

$$g_\gamma(z) = \begin{cases} 1 & \text{if } \gamma \geq 1/2 \\ r(z)^{\gamma - 1/2} & \text{if } 0 < \gamma \leq 1/2. \end{cases}$$

Let

$$F_j(z) = \frac{\bar{f}_j(z)}{|f(z)|^4}, \quad z \in B.$$

Then

$$\begin{aligned} \left|\frac{\partial E(F)}{\partial z_i}(z)\right| &= \left|\int_B \frac{\partial}{\partial z_i} E(z, \xi) \wedge F(\xi)\right| \\ &= \left|\int_B \frac{\partial}{\partial z_i} E(z, \xi) \wedge \left(F_1(\xi) - F_1(z)\right)\overline{\partial} \bar{f}_2(\xi)\right. \\ &\qquad\qquad - \left(F_2(\xi) - F_2(z)\right)\overline{\partial} \bar{f}_1(\xi)) \\ &\quad \left. + F_1(z) \int_B \frac{\partial}{\partial z_i} E(z, \xi) \overline{\partial} \bar{f}_2(\xi) - F_2(z) \int_B \frac{\partial}{\partial z_i} E(z, \xi) \overline{\partial} \bar{f}_1(\xi)\right| \\ &\leq \sum_{j=1}^{2} \int_B \left|\frac{\partial}{\partial z_i} E(z, \xi) \wedge \overline{\partial} \bar{f}_j\right| \|f\|_{\Lambda_\gamma} |\xi - z|^\gamma / \delta^4 \\ &\quad + |F_2(z)|\,|\partial_i f_1(z)| + |F_1(z)|\,|\partial_i f_2(z)| \\ &= C\|f\|_{\Lambda_\gamma} \delta^{-4} I(z) + C\|f\|_{\Lambda_\gamma} r(z)^{\gamma - 1}. \end{aligned}$$

By the form of the expression for E_j in Section 2, we have

$$\begin{aligned} &\left|\partial_z\left(r(\xi) E_1(z, \xi) \wedge \overline{\partial} \bar{f}_k(\xi)\right)\right| \\ &\quad \leq C\|f\|_{\Lambda_\gamma} r(\xi)^\gamma \left(\frac{|\xi - z|}{|1 - \langle z, \xi\rangle|^3 \tau(z, \xi)}\right. \\ &\qquad\qquad \left. + \frac{1}{|1 - \langle z, \xi\rangle|^2 \tau(z, \xi)^{3/2}}\right) dV(\xi) \\ &\quad \leq C\|f\|_{\Lambda_\gamma} r(\xi)^\gamma \left(\frac{1}{|1 - \langle z, \xi\rangle|^2 \tau(z, \xi)^{3/2}}\right) dV(\xi). \end{aligned}$$

By Lemma 3.1,

$$
\begin{aligned}
&\left|\partial_z\left(E_2(z,\xi) \wedge \xi \cdot d\bar{\xi} \wedge \overline{\partial f_k}(\xi)\right)\right| \\
&\quad \le C g_\gamma(\xi)\left|\partial_z E_2(z,\xi)\right| \\
&\quad \le C g_\gamma(\xi)\left(\frac{|\xi - z|}{|1 - \langle z,\xi\rangle|^3 \tau(z,\xi)} + \frac{1}{|1 - \langle z,\xi\rangle|^2 \tau(z,\xi)}\right) dV(\xi) \\
&\quad \le C g_\gamma(\xi) \frac{1}{|1 - \langle z,\xi\rangle|^3 \tau(z,\xi)^{1/2}}\, dV(\xi).
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|\partial_z\left(E_3(z,\xi) \wedge \xi \cdot d\bar{\xi} \wedge \overline{\partial f_k}(\xi)\right)\right| \\
&\quad \le C g_\gamma(\xi)\left|\partial_z E_3(z,\xi)\right| \\
&\quad \le C g_\gamma(\xi)\left(\frac{1}{|1 - \langle z,\xi\rangle| \tau(z,\xi)^{3/2}} + \frac{1}{\tau(z,\xi)^2}\right) dV(\xi) \\
&\quad \le C g_\gamma(\xi)\left(\frac{1}{|1 - \langle z,\xi\rangle| \tau(z,\xi)^{3/2}}\right) dV(\xi).
\end{aligned}
$$

Since

$$
\begin{aligned}
&B(z,\xi) - \tilde{B}(z,\xi) \\
&\quad = \left(\frac{1}{2\pi i}\right)^2 \sum_{j=1}^{2} \left(\bar{\xi}_j - \bar{z}_j\right) \frac{2r(z)r(\xi)}{|\xi - z|^2 \tau(z,\xi)^2}\, d\xi_j \wedge d\bar{\xi} \cdot d\xi \\
&\qquad + \left(\frac{1}{2\pi i}\right)^2 \sum_{j=1}^{2} \left(\bar{\xi}_j - \bar{z}_j\right) \frac{r(z)^2 r(\xi)^2}{|\xi - z|^4 \tau(z,\xi)^2}\, d\xi_j \wedge d\bar{\xi} \cdot d\xi,
\end{aligned}
$$

we find that

$$
\begin{aligned}
&\left|\partial_z\left(\left(B(z,\xi) - \tilde{B}(z,\xi)\right) \wedge \overline{\partial f_k}(\xi)\right)\right| \\
&\quad \le C\|f\|_{\Lambda_\gamma} r(\xi)^\gamma \left(\frac{1}{|\xi - z| \tau(z,\xi)^2} + \frac{1}{|\xi - z|^2 \tau(z,\xi)^{3/2}} \right.\\
&\qquad\qquad \left. + \frac{|\xi - z|^{-4}}{\tau(z,\xi)^{1/2}}\right) dV(\xi) \\
&\quad \le C\|f\|_{\Lambda_\gamma} r(\xi)^\gamma \left(\frac{1}{|\xi - z|^4 \tau(z,\xi)^{1/2}}\right) dV(\xi).
\end{aligned}
$$

Combining the above estimates with Lemma 2.6, we have

$$\begin{aligned}
I(z) &= \sum_{k=1}^{2} \int_B \left| \frac{\partial}{\partial z_i} E(z,\xi) \wedge \overline{\partial} f_k \right| |\xi - z|^\gamma \, dV(\xi) \\
&\le C\|f\|_{\Lambda_\gamma} \int_B r(\xi)^\gamma \frac{|\xi - z|^\gamma}{|\xi - z|^4 \tau(z,\xi)^{1/2}} \\
&\quad + C \int_B \frac{g_\gamma(\xi)}{|1 - \langle z,\xi\rangle|^3 \tau(z,\xi)^{1/2}} |\xi - z|^\gamma \\
&\le C\|f\|_{\Lambda_\gamma} r(z)^{\gamma-1} + C\|f\|_{\Lambda_\gamma} r(z)^{\gamma-1} \\
&\le C\|f\|_{\Lambda_\gamma} r(z)^{\gamma-1}.
\end{aligned}$$

Therefore we have

$$\left| \frac{\partial}{\partial z_i} E(F)(z) \right| \le C \frac{\|f\|_{\Lambda_\gamma}^2}{\delta^4} r(z)^{\gamma-1},$$

and the proof of the case $0 < \gamma \le 1$ is complete.

Case 2. $1 < \gamma \le 2$. We calculate that

$$\begin{aligned}
\frac{\partial}{\partial z_i} E(F)(z) &= \int_B \frac{\partial}{\partial z_i} E(z,\xi) \big(F_1(\xi) \overline{\partial} f_2(\xi) - F_2(\xi) \overline{\partial} f_1(\xi) \big) \\
&= \int_B \frac{\partial}{\partial z_i} E(z,\xi) \big((F_1(\xi) - F_1(z)) \overline{\partial} f_2(\xi) \\
&\qquad\qquad - (F_2(\xi) - F_2(z)) \overline{\partial} f_1(\xi) \big) \\
&\quad + \left(F_1(z) \int_B \frac{\partial}{\partial z_i} E(z,\xi) \overline{\partial} f_2(\xi) - F_2(z) \int_B \frac{\partial}{\partial z_i} E(z,\xi) \overline{\partial} f_1(\xi) \right) \\
&= I_1(z) + I_2(z) \\
&= I_{11}(z) + I_{12}(z) + I_2(z).
\end{aligned}$$

Let

$$\begin{aligned}
P_1(F_1)(z,\xi) = \Bigg(F_1(z) - F_1(\xi) - \sum_{k=1}^{2} \big(&\partial_k F_1(\xi)(z_k - \xi_k) \\
&- \overline{\partial}_k F_1(\xi)(\bar{z}_k - \bar{\xi}_k) \big) \Bigg).
\end{aligned}$$

Then

$$
\begin{aligned}
&\frac{\partial}{\partial z_j} I_{11}(z) \\
&\quad = -\frac{\partial}{\partial z_j}\int_B \frac{\partial}{\partial z_i} E(z,\xi) \wedge \overline{\partial f_2}(\xi)\big(F_1(z) - F_1(\xi)\big) \\
&\quad = -\frac{\partial}{\partial z_j}\int_B \frac{\partial E(z,\xi)}{\partial z_i} \wedge \overline{\partial f_2}(\xi) P_1(F_1)(z,\xi) \\
&\qquad - \frac{\partial}{\partial z_j}\int_B \frac{\partial E(z,\xi)}{\partial z_i} \wedge \overline{\partial f_2}(\xi) \sum_{k=1}^{2} \big(\partial_k F_1(\xi)(z_k - \xi_k) \\
&\qquad\qquad - \bar{\partial}_k F(\xi)(\bar{z}_k - \bar{\xi}_k)\big) \\
&\quad = -\int_B \frac{\partial^2 E(z,\xi)}{\partial z_i \partial z_j} \wedge \overline{\partial f_2}(\xi) P_1(F_1)(z,\xi) \\
&\qquad - \int_B \frac{\partial E(z,\xi)}{\partial z_i} \wedge \overline{\partial f_2}(\xi) \frac{\partial}{\partial z_j} P_1(F_1)(z,\xi) \\
&\qquad - \int_B \frac{\partial^2 E(z,\xi)}{\partial z_i \partial z_j} \wedge \overline{\partial f_2}(\xi) \sum_{k=1}^{2} \big(\partial_k F_1(\xi)(z_k - \xi_k) \\
&\qquad\qquad + \bar{\partial}_k F_1(\xi)(\bar{z}_k - \bar{\xi}_k)\big) \\
&\qquad - \int_B \frac{\partial E(z,\xi)}{\partial z_i} \wedge \overline{\partial f_2}(\xi) \frac{\partial}{\partial z_j} \sum_{k=1}^{2} \big(\partial_k F_1(\xi)(z_k - \xi_k) \\
&\qquad\qquad + \bar{\partial}_k F(\xi)(\bar{z}_k - \bar{\xi}_k)\big) \\
&\quad = I_{111}(z) + I_{112}(z) + I_{113}(z) + I_{114}(z).
\end{aligned}
$$

Now it is easy to see that

$$|P_1(F_1)(z,\xi)| \le \frac{C}{\delta^5}\|f\|_{\Lambda_\gamma}\big(r(\xi)^{-2+\gamma} + r(z)^{-2+\gamma}\big)|\xi - z|^2.$$

Thus with some computations, as we did in Case 1, we have

$$\int_B \left|\frac{\partial E(z,\xi)}{\partial z_j}\right| |\xi - z|\, dV(\xi) \le C \tag{4.1}$$

and

$$\int_B \left| \frac{\partial^2 E(z,\xi)}{\partial z_i \partial z_j} \right| |\xi - z|^2 r(\xi)^{\gamma-2} \, dV(\xi) \le C r(z)^{2-\gamma} \tag{4.2}$$

for all $1 < \gamma \le 2$. Therefore, we have

$$\begin{aligned} |I_{111}(z)| &\le \frac{C}{\delta^5} \int_B \left| \frac{\partial^2 E(z,\xi)}{\partial z_i \partial z_j} \wedge \overline{\partial} \bar{f}_2(\xi) \right| \|f\|_{\Lambda_\gamma} r(\xi)^{\gamma-2} |\xi - z|^2 \, dV(\xi) \\ &\quad + \frac{C}{\delta^5} \|f\|_{\Lambda_\gamma} r(z)^{\gamma-2} \int_B \left| \frac{\partial^2 E(z,\xi)}{\partial z_i \partial z_j} \wedge \bar{f}_2(\xi) \right| |\xi - z|^2 \, dV(\xi) \\ &\le \frac{C}{\delta^5} \|f\|^2_{\Lambda_\gamma} r(z)^{\gamma-2}. \end{aligned}$$

Notice that

$$\begin{aligned} \left| \frac{\partial}{\partial z_j} P_1(F_1)(z,\xi) \right| &= \left| \frac{\partial F_1}{\partial z_j}(z) - \frac{\partial F_1}{\partial z_j}(\xi) \right| \\ &\le \frac{C}{\delta^5} \|f\|_{\Lambda_\gamma} \big(r(\xi)^{\gamma-2} + r(z)^{\gamma-2} \big) |\xi - z|. \end{aligned}$$

Arguing as we did in Case 1, we have

$$|I_{112}(z)| \le \frac{C}{\delta^5} \|f\|^2_{\Lambda_\gamma} r(z)^{\gamma-2}.$$

Next,

$$\begin{aligned} &-I_{113}(z) \\ &\quad = \int_B \frac{\partial^2 E(z,\xi)}{\partial z_i \partial z_j} \wedge \overline{\partial} \bar{f}_2(\xi) \sum_{k=1}^{2} \big(\partial_k F_1(\xi)(z_k - \xi_k) + \bar{\partial}_k F_1(\xi)(\bar{z}_k - \bar{\xi}_k) \big) \\ &\quad = \int_B \frac{\partial^2 E(z,\xi)}{\partial z_i \partial z_j} \wedge \overline{\partial} \bar{f}_2(\xi) \\ &\qquad \times \sum_{k=1}^{2} \big((\partial_k F_1(\xi) - \partial_k F(z))(z_k - \xi_k) + \bar{\partial}_k F_1(\xi)(\bar{z}_k - \bar{\xi}_k) \big) \\ &\qquad + \int_B \frac{\partial^2 E(z,\xi)}{\partial z_i \partial z_j} \wedge \overline{\partial} \bar{f}_2(\xi) \sum_{k=1}^{2} \big(\partial_k F_1(z)(z_k - \xi_k) \big) \\ &\quad = I_{1131}(z) + I_{1132}(z). \end{aligned}$$

Thus, by (4.2), we have

$$
\begin{aligned}
&|I_{1131}(z)| \\
&\quad \le C\left|\int_B \frac{\partial^2 E(z,\xi)}{\partial z_i \partial z_j} \wedge \overline{\partial f}_2(\xi) \right.\\
&\qquad \left. \times \sum_{k=1}^{2} \left((\partial_k F_1(\xi) - \partial_k F(z))(z_k - \xi_k) + \bar\partial_k F_1(\xi)(\bar z_k - \bar\xi_k)\right)\right| \\
&\quad \le \frac{C}{\delta^5}\|f\|_{\Lambda_\gamma}\int_B \left|\frac{\partial^2 E(z,\xi)}{\partial z_i \partial z_j} \wedge \overline{\partial f}_2(\xi)\right| \left(r(\xi)^{-2+\gamma} + r(z)^{-2+\gamma}\right) \\
&\quad \times \quad |\xi - z|^2\, dV(\xi) \\
&\qquad + \left| C \frac{\partial^2}{\partial z_i \partial z_j}\int_B E(z,\xi) \wedge \overline{\partial f}_2(\xi) \bar\partial_k F_1(\xi)(\bar z_k - \bar\xi_k)\right| \\
&\quad \le \frac{C}{\delta^5}\|f\|_{\Lambda_\gamma}^2 r(z)^{\gamma-2} \\
&\qquad + C\left| \frac{\partial^2}{\partial z_i \partial z_j}\int_B \bar\partial_\xi E(z,\xi) \wedge \bar f_2(\xi) \bar\partial_k F_1(\xi)(\bar z_k - \bar\xi_k)\right| \\
&\qquad + C\sum_{k=1}^{2}\left| \frac{\partial^2}{\partial z_i \partial z_j}\int_B E(z,\xi) \wedge \bar f_2(\xi) \wedge \bar\partial_\xi\left(\bar\partial_k F_1(\xi)(\bar z_k - \bar\xi_k)\right)\right| \\
&\quad \le \frac{C}{\delta^5}\|f\|_{\Lambda_\gamma}^2 r(z)^{-2+\gamma} + C\sum_{k=1}^{2}\left| \frac{\partial^2}{\partial z_i \partial z_j} \bar f_2(z) \bar\partial_k F_1(z)(\bar z_k - \bar z_k)\right| \\
&\qquad + C\sum_{k=1}^{2}\left| \frac{\partial^2}{\partial z_i \partial z_j}\int_B E(z,\xi)\left(\bar f_2(\xi) - \bar f_2(z)\right) \wedge \bar\partial_\xi \bar\partial_k F_1(\xi)(\bar z_k - \bar\xi_k)\right| \\
&\qquad + C\sum_{k=1}^{2}\left| \frac{\partial^2}{\partial z_i \partial z_j} \bar f_2(z) \int_B E(z,\xi) \bar\partial_\xi\left(\bar\partial_k F_1(\xi)(\bar z_k - \bar\xi_k)\right)\right| \\
&\quad \le \frac{C}{\delta^5}\|f\|_{\Lambda_\gamma}^2 r(z)^{-2+\gamma} + 0 + C\int_B \left|\frac{\partial^2 E(z,\xi)}{\partial z_i \partial z_j}\right| \|f\|_{\Lambda_\gamma}^2 r(\xi)^{\gamma-2} |\xi - z|^2 \\
&\qquad + 0 + 0 \\
&\quad \le \frac{C}{\delta^5}\|f\|_{\Lambda_\gamma}^2 r(z) \quad ,
\end{aligned}
$$

and

$$
\begin{aligned}
&|I_{1132}(z)| \\
&= \left| \int_B \frac{\partial^2 E(z,\xi)}{\partial z_i \partial z_j} \wedge \overline{\partial f_2}(\xi) \sum_{k=1}^{2} \left(-\bar{\partial}_k F_1(\xi)(\bar{z}_k - \bar{\xi}_k) \right) \right| \\
&= \sum_{k=1}^{2} \left| \frac{\partial^2}{\partial z_i \partial z_j} \int_B E(z,\xi) \wedge \overline{\partial f_2}(\xi) \bar{\partial}_k F_1(\xi)(\bar{z}_k - \bar{\xi}_k) \right| \\
&\le \left| \sum_{k=1}^{2} \frac{\partial^2}{\partial z_i \partial z_j} \int_B \bar{\partial}_\xi E(z,\xi) \wedge \bar{f}_2(\xi) \bar{\partial}_k F_1(\xi)(\bar{z}_k - \bar{\xi}_k) \right| \\
&\quad + C \sum_{k=1}^{2} \left| \frac{\partial^2}{\partial z_i \partial z_j} \int_B E(z,\xi) \wedge \bar{f}_2(\xi) \bar{\partial}_\xi \left(\bar{\partial}_k F_1(\xi)(\bar{z}_k - \bar{\xi}_k) \right) \right| \\
&= \sum_{k=1}^{2} \left| \frac{\partial^2}{\partial z_i \partial z_j} \left(\bar{\partial}_k F_1(z)(\bar{z}_k - \bar{z}_k) \bar{f}_2(z) \right) \right| \\
&\quad + C \sum_{k=1}^{2} \left| \frac{\partial^2}{\partial z_i \partial z_j} \int_B E(z,\xi) \left(\bar{f}_2(\xi) - \bar{f}_2(z) \right) \right. \\
&\qquad\qquad \left. \times \bar{\partial}_\xi \left(\bar{\partial}_k F_1(\xi)(\bar{z}_k - \bar{\xi}_k) \right) \right| + 0 \\
&= \sum_{k=1}^{2} \int_B \left| \frac{\partial^2 E(z,\xi)}{\partial z_i \partial z_j} \right| \|f\|^2_{\Lambda_\gamma} |z - \xi|^2 r(\xi)^{\gamma-2} \, dV(\xi) \\
&\quad + C \sum_{k=1}^{2} \int_B \left| \frac{\partial E(z,\xi)}{\partial z_i} \right| \|f\|^2_{\Lambda_\gamma} |\xi - z| r(\xi)^{\gamma-2} \, dV(\xi) \\
&\le C \|f\|_{\Lambda_\gamma} r(z)^{\gamma-2}.
\end{aligned}
$$

Now

$$
\begin{aligned}
I_{114}(z) &= \int_B \frac{\partial E(z,\xi)}{\partial z_i} \wedge \overline{\partial f_2}(\xi) \frac{\partial}{\partial z_j} \sum_{k=1}^{2} \left(\partial_k F_1(\xi)(z_k - \xi_k) \right. \\
&\qquad\qquad \left. - \bar{\partial}_k F(\xi)(\bar{z}_k - \bar{\xi}_k) \right) \\
&= \int_B \frac{\partial E(z,\xi)}{\partial z_i} \wedge \overline{\partial f_2}(\xi) \partial_j F_1(\xi) \\
&= \int_B \frac{\partial E(z,\xi)}{\partial z_i} \wedge \overline{\partial f_2}(\xi) \frac{\partial}{\partial z_j} \left(\partial_j F_1(\xi) - \partial_j F_1(z) \right) \\
&\quad + \partial_j F_1(z) \int_B \frac{\partial E(z,\xi)}{\partial z_i} \wedge \overline{\partial f_2}(\xi) \\
&= \int_B \frac{\partial E(z,\xi)}{\partial z_i} \wedge \overline{\partial f_2}(\xi) \left(\partial_j F_1(\xi) - \partial_j F_1(z) \right) + \partial_j F(z) \partial_i \bar{f}_2(z) \\
&= \int_B \frac{\partial E(z,\xi)}{\partial z_i} \wedge \overline{\partial f_2}(\xi) \left(\partial_j F_1(\xi) - \partial_j F_1(z) \right)
\end{aligned}
$$

Thus

$$\begin{aligned}|I_{114}(z)| &\leq C\int_B \left| \frac{\partial E(z,\xi)}{\partial z_i} \wedge \overline{\partial} \overline{f}_2(\xi) \right| \|f\|_{\Lambda_\gamma} |\xi - z| \, dV(\xi) \\ &\leq C\|f\|^2_{\Lambda_\gamma} \\ &\leq C\|f\|^2_{\Lambda_\gamma} r(z)^{\gamma-2}.\end{aligned}$$

Similarly, we have

$$\left| \frac{\partial}{\partial z_j} I_{12}(z) \right| \leq C\|f\|^2_{\Lambda_\gamma} r(z)^{\gamma-2}.$$

It is easy to check that

$$I_2(z) = 0.$$

Combining the above estimates, we have completed the proof of the case $1 < \gamma \leq 2$.

If we repeat these arguments, we can prove that Theorem 1.4 holds for all $\gamma > 0$. Similar formulations can be found in [K2] and [Siu]; we omit the details here. This completes the proof of Theorem 1.4. ■

References

[A] E. Amar, *On the corona problem*, J. Geometric Analysis **1** (1991), 291–305.

[An] M. Andersson, *Values in the interior of L^2-minimal solutions of the ∂-equation in the unit ball of C^n*, Publ. Mat. **32** (1988), 179–189.

[C] L. Carleson, *Interpolation by bounded analytic functions and the corona problem*, Ann. of Math. **76** (1962), 547–559.

[Ch] S-Y.A. Chang, *Two remarks of $\mathscr{H}^1$ and BMO on the bidisc*, Conference on harmonic analysis in honor of A. Zygmund, Vol II, Chicago, 1981, pp. 373–393.

[CR] R.R. Coifman and R. Rochberg, *Representation theorems for holomorphic and harmonic functions in L^p*, Astérisque **77** (1980), 11–65.

[COU] B. Coupet, *Régularité d'Applications Holomorphes sur des Varietes Totalement Realles. Structure des Espaces de Bergman*, These, l'Université de Provence, 1987.

[FS1] J. Fornæss and N. Sibony, *Counterexamples to the corona and $\bar\partial$ problems*, Proceedings of a Special Year at the Mittag-Leffler Institute, Princeton Lecture Notes, Princeton University Press, Princeton, N.J., 1993.

[FS2] ______, *Smooth pseudoconvex domains in $\mathbf{C}^2$ for which the corona theorem and L^p estimates for $\bar\partial$ fail*, preprint.

[G] J.B. Garnett, *Bounded analytic functions*, Academic Press, San Diego, 1981.

[GP] J.B. Garnett and P. Jones, *The corona theorem for Denjoy domains*, Report 6, Institute Mittag-Leffler, 1984.

[HS] M. Hakim and N. Sibony, *Spectre de $A(\overline{\Omega})$ pour les domaines faiblement pseudoconvexes*, J. Functional Analysis **37** (1980), 127–135.

[J] P. Jones, *L^∞ estimates for $\bar\partial$ problems in a half-plane*, Acta Math. **150** (1983), 137–152.

[K1] S.G. Krantz, *Function theory of several complex variables*, 2nd. ed., Wadsworth, Belmont, 1992.

[K2] ______, *Structure and interpolation theorems for certain Lipschitz spaces and estimates for $\bar{\partial}$ equations*, Duke Math J. **43** (1976), 417–439.

[KL1] S.G. Krantz and S-Y. Li, *A note on Hardy spaces and functions of bounded mean oscillation on domains in $\mathbf{C}^n$*, Michigan J. Math. **41** (1994), 51–72.

[KL2] ______, *On the decomposition theorem for Hardy spaces in a domain in $\mathbf{C}^n$ and applications*, J. Fourier Anal. Appl., to appear.

[L] S-Y. Li, *Corona Problem of several complex variables*, Contemp. Mathematics **137** (1992), 307–328.

[Lin] K.C. Lin, *$\mathscr{H}^p$ solutions for the corona problem on the polydisc in C^n*. Bull Soc. Math. France **110**, (1986), 69–84.

[M] C.M. Moore, *The corona theorem for domains whose boundary lies in a smooth curve*, Proc. Amer. Math. Soc. 100 (1987), 200–204.

[P] J. Polking, *The Cauchy-Riemann equations on convex domains*, Proc. Sympos. Pure Math., vol. 52, part 3, Amer. Math. Soc., Providence, R.I., pp. 309–322.

[Ra] M. Range, *On Hölder and BMO estimates for $\bar{\partial}$ on convex domains in $\mathbf{C}^2$*, J. Geom. Anal., to appear.

[S] N. Sibony, *Problem de la Couronne pour des domaines pseudoconvexes à bord lisse*, Ann. of Math. **126** (1987), 675–682.

[Siu] Y.T. Siu, *The $\bar{\partial}$ problem with uniform bounds on derivatives*, Math. Ann. **207** (1974), 163–176.

[V1] N.Th. Varopoulos, *BMO functions and the $\bar{\partial}$ equation*, Pacific J. Math., **71** (1977), 221–272.

[V2] ______, *Probabilistic approach to some problems in complex analysis*, Bull. Sci. Math. Paris **105** (1981), 181–224.

[Z] K. Zhu, *Operator theory in function spaces*, Marcel Dekker, New York, 1990.

Washington University
St. Louis, Missouri