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A NOTE ABOUT THE VERONESE CONE

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1. Introduction

The area-minimizing property of the cone on the Veronese surface is still unproven. In this note we offer further evidence in support of the conjecture that the cone is indeed area-minimizing. The techniques used here present a combination of twist-calibrations (see [Mh]) and vanishing calibrations (see [Lr]).

The result from [Mh] is that the cone is area-minimizing in a class of surfaces which has two restrictions, namely that the comparison surfaces cannot cannot cross a certain subset of space, and that they satisfy an orientability condition called "L-orientability;" see below for details. The main result of this paper is to remove the first restriction. We show that if S is any surface (integral current mod 2) with the same boundary as the cone, and if the portion of S lying near the cone is L-orientable, then S has more area than the cone.

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2. Twist-calibrating the Veronese cone

The subsections within this section recall relevant material from the articles [Lr] and [Mh].

L-orientability.

To extend the techniques of calibrated geometry to non-orientable submanifolds, it is necessary to define an alternate "orientation." This objective can sometimes be accomplished using lines bundles on a submanifold other than the orientation bundle. For our purposes, we state the following:

DEFINITION. Let L be a smooth Euclidean line bundle (that is, a smooth line bundle with a smooth choice of inner-product on the fibers) on a smooth manifold M^m of dimension m. Suppose N^n is a smooth (embedded)

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n-dimensional submanifold of M. If the orientation bundle of N is isomorphic (as a Euclidean bundle) to the restriction of L to N, then N is said to be an *L-orientable* submanifold, or L-manifold. An *L-orientation* of N is a choice of isomorphism between the restriction of L to N and the orientation bundle of N.

The above definition allows one to prove a version of Stokes's theorem for L-valued forms on L-orientable submanifolds (with boundary). The key idea in the proof of the L-valued Stokes's Theorem is that the L-orientability of an L-manifold N allows one to identify the L-valued forms on N with forms of odd type on the orientation double cover of N; see [Mh] for details. Stokes's theorem is the key ingredient in the one-line proof that calibrated submanifolds minimize mass in homology. See Section 4 below.

The Veronese surface.

The *Veronese surface* is a minimimally embedded real projective plane in the standard 4-sphere. An elementary description is the set of 3 by 3 matrices, thought of as points of $\mathbf{R}^9$, of the form

$$VV^T - \frac{1}{\sqrt{3}}I,$$

where V ranges over all unit length column vectors with 3 entries. This actually sits in a round 4-sphere in a 5-dimensional subspace of $\mathbf{R}^9$ given as a degenerate orbit of the adjoint action of $SO(3)$ on the real vector space of 3 by 3 real symmetric matrices of trace zero. Note that if this vector space is equipped with the inner product

$$\langle m_1, m_2 \rangle = \tfrac{1}{2}\,\mathrm{Trace}(m_1 m_2),$$

then the Veronese surface is the $SO(3)$-orbit of the matrix

$$\mathrm{diag}\left(1/\sqrt{3}\,, 1/\sqrt{3}\,, -2/\sqrt{3}\,\right).$$

The *Veronese cone* is the set of rays from the origin through points on the Veronese surface. We let V_1 denote the set of line segments from the origin to points on the Veronese surface, and also call this set the Veronese cone, relying on context to determine whether we are referring to the infinite cone or the compact cone.

The Veronese cone is a minimal cone with a singularity at the origin. The image of this cone under the antipodal map is called the *opposite Veronese cone*.

The twist-calibration.

The 3-manifold C^3 obtained from cone over the Veronese surface by deleting the origin is twist-calibrated. More precisely, the complement of the opposite Veronese cone in $\mathbf{R}^5$ is an open 5-manifold M^5 that supports a Riemannian double-cover that induces the orientation double-covering on C^3. In [Mh] a twist-calibration on the double-cover of the complement of the opposite Veronese cone is constructed. That is, there exists on the double cover of M^5 a closed 3-form of unit comass that calibrates the orientation double cover of C^3. Moreover, this 3-form pulls-back to its negative by the sheet-interchange involution.

The twist-calibration described above is constructed as a parallel 3-form on the space $SO(3) \times W$, where

$$W = \left\{ \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3) \;\middle|\; \sum_{i=1}^{3} \lambda_i = 0 \text{ and } \lambda_1 \geq \lambda_2 > \lambda_3 \right\}$$

is the cross-section of the $SO(3)$-action on the complement of the opposite cone. The Euclidean metric, when pulled back to $SO(3) \times W$ via the adjoint map, is a sum of squares of 1-forms:

$$\eta^1 = \frac{1}{\sqrt{3}}\, du, \quad \eta^2 = dv, \quad \eta^3 = (u - v)\omega^1, \quad \eta^4 = (u + v)\omega^2, \quad \eta^5 = 2v\omega^3$$

where

$$u = \tfrac{3}{2}(\lambda_1 + \lambda_2), \quad v = \tfrac{1}{2}(\lambda_1 - \lambda_2)$$

and

$$\omega = \begin{pmatrix} 0 & \omega^3 & -\omega^2 \\ -\omega^3 & 0 & \omega^1 \\ \omega^2 & -\omega^1 & 0 \end{pmatrix}$$

is the matrix of left-invariant Maurer-Cartan forms for the standard representation of $SO(3)$. Note that ω satisfies the structure equation $d\omega = -\omega \wedge \omega$.

The twist-calibration of C^3 corresponds to the 3-form

$$\Omega = \eta^1 \wedge \left(\eta^2 \wedge \eta^5 - \eta^3 \wedge \eta^4 \right)$$

on $SO(3) \times W$. For further details of the construction outlined here, the reader is referred to [Mh], especially pp. 247–255.

The method of vanishing calibrations.

As pointed out in the introduction, a drawback of this twist-calibration is that surfaces to which it applies must not cross the opposite Veronese cone. We will resolve this problem by making Ω into a "vanishing calibration."

A vanishing calibration is a calibration which is identically zero on part of the space in which it is defined. For example, there is a calibration of a unit disk in $\mathbf{R}^3$ whose support is contained in a ball of radius $\sqrt{2}$. Such a calibration proves that if S is any surface whose boundary is the unit circle, then the area of the portion of S lying within the ball of radius $\sqrt{2}$ is at least the area of the disk. From this we can deduce that any union of disks in space is area-minimizing if the distance between the centers of any two of them is at least $\sqrt{2}$ times the sum of their radii.

One method of finding a vanishing calibration is to take an existing exact form ω and modify it. If ω is exact, then there is a family of forms whose exterior derivative is ω; the idea is to choose an advantageous representative ψ from this family, multiply ψ by a real-valued function g which vanishes outside some neighborhood, and let $\phi = d(g\psi)$. Then ϕ is a vanishing form which is automatically closed. In order for ϕ to be a calibration, we need $\|\phi\| \leq 1$. This translates to a differential inequality for g. If we can solve the inequality in such a way that g reaches zero, then we have succeeded in finding a vanishing calibration. We can allow ϕ to be discontinuous; a sufficient condition for ϕ to serve as calibration is to require $g\psi$ to be Lipschitz (see [Lr, Lemma A8 in appendix]). For examples of vanishing calibrations, see [Lr].

This idea applies well to the Veronese cone because the twist-calibration in [Mh] is an exact form. (Any twist-calibration is closed, but may or may not be exact). The twist-calibration applies only to comparison surfaces which do not cross the opposite cone. We can remove this restriction if we can modify the form in the manner described above, in such a way that it vanishes around the negative cone.

3. The vanishing calibration for the Veronese cone

The main result of this note is to prove that the 3-form Ω described in the previous section gives rise to a vanishing calibration. To this end we cho0se the d^{-1} representative of Ω to be

$$\psi = \frac{1}{\sqrt{3}}\left(-v^2\,du \wedge \omega^3 - \frac{u^2}{\sqrt{3}}\omega^1 \wedge \omega^2\right).$$

Letting $g = g(t)$ with $t = u/v$, one computes that

$$d(g\psi) = \left(\frac{tg'}{2} - g\right)\eta^{125} + \left(\frac{tg'}{3(1-t^2)} + g\right)\eta^{134} - \frac{g'}{3\sqrt{3}\,(1-t^2)}\eta^{234}.$$

The differential inequalities.

To find the conditions that assure $d(g\psi)$ has comass at most one, we first rotate η^1 and η^2 so as to combine the last two terms. That is, we find a pair of 1-forms $\eta^{1'}$ and $\eta^{2'}$ which are orthonormal and are linear combinations of η^1 and η^2, such that

$$\eta^{1'} \wedge \eta^{2'} = \eta^1 \wedge \eta^2$$

and

$$d(g\psi) = \left(\frac{tg'}{2} - g\right)\eta^{1'2'5} + A\eta^{1'34},$$

where

$$A = \left(\left(\frac{tg'}{3(1-t^2)} + g\right)^2 + \left(\frac{g'}{3\sqrt{3}\,(1-t^2)}\right)^2\right)^{1/2}.$$

Then

$$d(g\psi) = \eta^{1'} \wedge \left(\left(\frac{tg'}{2} - g\right)\eta^{2'5} + A\eta^{34}\right).$$

The comass of this form equals the comass of its second factor; since the four superscripts $2'$, 5, 3, and 4 are all different (and the corresponding 1-forms are orthonormal), the second cousin principle tells us that the comass is the larger of the two coefficients $tg'/2 - g$ and A. Hence we obtain the two differential inequalities

$$\left|\frac{tg'}{2} - g\right| \leq 1,$$

and

$$\left(\frac{tg'}{3(1-t^2)} + g\right)^2 + \left(\frac{g'}{3\sqrt{3}\,(1-t^2)}\right)^2 \leq 1.$$

The initial condition is $g(0) = 1$ (which forces $g'(0) = 0$), and the hope is that some solution reaches $g = 0$ at some positive t.

A solution to the inequalities.

The function $g(t) = 1 - 18t^2 + 27t^3$ for $t \in [0, 1/3]$ and $g(t) = 0$ for $t \geq 1/3$ satisfies these conditions. All of the conditions are easily verifiable for this function except the second differential inequality, which takes a bit more work.

We outline below the process we used to find the polynomial g and to prove that it satisfies the differential inequality. We used an interplay of analysis and computer assistance to get the proof.

(1) Solve the o.d.e. for $g'(t)$. Use a simple numerical method (Euler's method) to find an approximate solution. Near $t = 0$, Euler's method isn't good enough; use $g = 1 - 22.5t^2$ for very small t. The result of this computer analysis is a tentative value of $t = 0.2685$ at which $g = 0$. This gives an idea of what the best solution looks like, but is not a proof without error analysis.

(2) Rather than prove error bounds on the numerical analysis, we chose to look for a polynomial which satisfied both differential inequalities. Something of the form $1 - at^2$ for some positive a might work. But the initial computer solution had an inflection point before g reached zero. To accommodate the gradually increasing second derivative, we tried $1 - at^2 + bt^3$.

(3) We set up a range of integer values of a and rational values of b, and had the computer step through each one with a small increment for t, checking the differential inequality at each t. Among the polynomials which were reported to satisfy the inequality, we chose $g(t) = 1 - 18t^2 + 27t^3$. This reaches zero at $t = 1/3$.

(4) The simpler differential inequality is satisfied by this polynomial. The more complex one reduces (with the help of symbolic computation) to a tenth degree polynomial $f(t)$ which needs to be less than zero for $0 \leq t \leq 1/3$.

(5) A simple estimate showed that $f(t) \leq 0$ for $0 \leq t \leq 1/10$. The computer then reported that $f(t) < -0.1$ for $1/10 \leq t \leq 1/3$. A crude upper bound on $f'(t)$ (namely, 300) showed that if $f(t) < -0.1$ at a set of points separated by a distance of $1/3000$, then $f < 0$ in between as well. This we checked by computer. The roundoff error is provably negligible, and thus the proof is complete.

4. Summary

The support of the vanishing twist-calibration $\phi = d(g\psi)$ is the cone of points (vectors) which make an angle less than or equal to $\tan^{-1}(1/3)$ with the (nearest ray of the) Veronese cone. If we accept the computer evidence of a solution $g(t)$ vanishing at $t = 0.2685$ then this angular radius decreases to $\tan^{-1}(0.2685) \approx 15°$. (Note that the opposite cone makes an angle of $60°$ with the Veronese cone.)

The twist-calibration ϕ applies to surfaces (integral currents mod 2) S such that the part of S within the support of ϕ is L-orientable. If S is such a surface, and has the same boundary modulo 2 as the Veronese cone V_1, then S (in fact, even the part of S within the support of ϕ) has at least as much area as V_1, by the standard one-line calibration proof

$$\operatorname{Area}(V_1) = \int_{V_1} \phi = \int_S \phi \leq \operatorname{Area}(S).$$

This leaves open the possibility that some integral current mod 2 has the same boundary mod 2 as V_1, and has some local unorientability near the cone (such as a small cross-cap), and has less area.

REFERENCES

[Lr] G. LAWLOR, *A sufficient criterion for a cone to be area-minimizing*, Mem. Amer. Math. Soc. **446** (1991).

[Mh] T. Murdoch, *Twisted calibrations*, Trans. Amer. Math. Soc. **328** (1991), 239–257.

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