ILLINOIS JOURNAL OF MATHEMATICS
Volume 39, Number 2, Summer 1995

THE FACTORIZATION OF $A(z) + B(w)$ UNDER COMPOSITION

LEE A. RUBEL[1] AND CHUNG-CHUN YANG

Dedicated to the memory of R.P. Boas, Jr.

1. Introduction

THEOREM. *Let A and B be non-constant entire functions of one complex variable and suppose that*

$$f(g(z,w)) = A(z) + B(w) \tag{1}$$

for all z, w $\in \mathbb{C}$, where f is an entire function of one variable and g is an entire function of two variables. Then f must be affine: $f(\zeta) = \alpha\zeta + \beta$, $\alpha, \beta \in \mathbb{C}$ and g must have the form

$$g(z,w) = a(z) + b(w)$$

for some entire functions a(z) and b(w) of one variable.

This theorem says that the only entire factorizations (under composition of functions) of $A(z) + B(w)$ are the obvious ones. Note that the theorem is a global result dealing with *entire* functions. There is no corresponding local result—witness

$$A(z) + B(w) = \log(\exp(A(z) + B(w)),$$

or indeed

$$A(z) + B(w) = \Phi^{-1}(\Phi(A(z) + (B(w))$$

where Φ and Φ^{-1} are analytic functions suitably defined on regions in $\mathbb{C}$.

Besides some algebraic and analytic manipulating of an elementary kind, the main tool in the proof is Nevanlinna theory based on exhaustions of $\mathbb{C}^2$ by *polydiscs*. In particular, we use a version of the lemma of the logarithmic

---

Received January 6, 1993.

[1]The research of the first author was partially supported by a grant from the National Science Foundation.

1991 Mathematics Subject Classification. Primary 30D05; Secondary 30D35, 32A22.

derivative (LLD) for this context, the proof of which is given in our text. We thank Wilhelm Stoll for his valuable suggestion of how to prove our LLD by using a result in this paper [STO]. An appendix states and proves a several-variables version we need of a theorem of Clunie (see [HAY]) on the growth of the composition of entire functions. At the end of the paper we discuss some open problems, and also discuss a possible alternative approach to proving our theorem, which brings in some interesting considerations on ordinary factorization.

2. Proof of the theorem

Changing notation slightly from

$$f(g(z_1, z_2)) = A_1(z_1) + A_2(z_2) \tag{1}$$

we get

$$f'(g(z_1, z_2))g_1(z_1, z_2) = A_1'(z_1) \tag{2}$$

$$f'(g(z_1, z_2))g_2(z_1, z_2) = A_2'(z_2) \tag{3}$$

$$f''(g(z_1, z_2))g_1(z_1, z_2)g_2(z_1, z_2) + f'(g(z_1, z_2))g_{12}(z_1, z_2) = 0 \tag{4}$$

$$\frac{f''(g(z_1, z_2))}{f'(g(z_1, z_2))} = -\frac{g_{12}(z_1, z_2)}{g_1(z_1, z_2)g_2(z_1, z_2)}, \tag{5}$$

where the subscripts on the g's denote partial differentiation.

At this point we must make a detour into Nevanlinna theory for entire (and meromorphic) functions of n variables using a polydisc exhaustion of $\mathbb{C}^n$, based on the work of Stoll (see [STO]).

The Euclidean space $\mathbb{R}^n$ is partially ordered by its coordinates. Denote by $\|\mathfrak{r}\|$ the length of $\mathfrak{r}$ in $\mathbb{R}^n$. For $\mathfrak{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$, define $|\mathfrak{z}| = (|z_1|, \ldots, |z_n|)$. For $0 < \mathfrak{b} \in \mathbb{R}^n$, define

$$\overset{+}{\mathbb{R}}{}^n(\mathfrak{b}) = \{\mathfrak{r} \in \mathbb{R}^n : 0 < \mathfrak{r} < \mathfrak{b}\} \tag{6}$$

$$\mathbb{D}(\mathfrak{b}) = \{\mathfrak{z} \in \mathbb{C}^n : |\mathfrak{z}| < \mathfrak{b}\} \tag{7}$$

$$\mathbb{D}\langle\mathfrak{b}\rangle = \{\mathfrak{z} \in \mathbb{C}^n : |\mathfrak{z}| = \mathfrak{b}\} \tag{8}$$

Let Ω_n be the rotation-invariant measure on the torus $\mathbb{D}\langle\mathfrak{r}\rangle$ for $\mathfrak{r} \in \overset{+}{\mathbb{R}}{}^n(\mathfrak{b})$ such that $\mathbb{D}\langle\mathfrak{r}\rangle$ has total volume 1. In familiar notation,

$$\Omega_n = \frac{d\theta_1\, d\theta_2 \ldots d\theta_n}{(2\pi)^n}. \tag{9}$$

Denote by

$$\|z, a\| = \frac{|z - a|}{\sqrt{1 + |a|^2}\sqrt{1 + |z|^2}} \tag{10}$$

the chordal metric on $\mathbb{C}$. A meromorphic function f on $\mathbb{C}^n$ is the quotient of two entire functions g and $h \not\equiv 0$, such that $hf = g$ and

$$u = (g, h) : \mathbb{C}^n \to \mathbb{C}^2 \tag{11}$$

defines a meromorphic map from $\mathbb{C}^n$ into $\mathbb{P}_1$ which is identified with f. The representation u is reduced if and only if g and h are coprime at every point of $\mathbb{C}^2$. Finally, the value-distribution functions are defined by

$$T_f(\mathfrak{r}, \mathfrak{q}) = \int_{\mathbb{D}\langle\mathfrak{r}\rangle} \log\sqrt{|g|^2 + |h|^2}\,\Omega_n - \int_{\mathbb{D}\langle\mathfrak{q}\rangle} \log\sqrt{|g|^2 + |h|^2}\,\Omega_n \tag{12}$$

$$m_f(\mathfrak{r}, a) = \int_{\mathbb{D}\langle\mathfrak{r}\rangle} \log\frac{\sqrt{1 + |a|^2}\sqrt{1 + |f|^2}}{|f - a|}\,\Omega_n \tag{13}$$

$$m_f(\mathfrak{r}, \infty) = \int_{\mathbb{D}\langle\mathfrak{r}\rangle} \log\sqrt{1 + |f|^2}\,\Omega_n. \tag{14}$$

For fixed $\mathfrak{q}$,

$$T_f(\mathfrak{r}, \mathfrak{q}) = T_f^{\#}(\mathfrak{r}) + O(1) \tag{15}$$

where

$$T_f^{\#}(\mathfrak{r}) = m_f^{\#}(\mathfrak{r}) + N_f^{\#}(\mathfrak{r}) \tag{16}$$

where

$$m_f^{\#}(\mathfrak{r}) = \int_{\mathbb{D}\langle\mathfrak{r}\rangle} \log^+ |f|\,\Omega_n \tag{17}$$

and where $N_f^{\#}(\mathfrak{r})$ is a certain monotone-increasing function of the pole-set of f (that is, of the zero-set of h in a reduced representation $f = g/h$). This notation is more in line with the analyst's usual notation.

Now in the context of this paper, we take $0 < r < \infty$, take $n = 2$, and take $\mathfrak{r} = (r, 2r)$ and take $\mathfrak{b} = (4r, 4r)$, say. Take $\mathfrak{q} = (s, 2s)$ for s fixed, $0 < s < r$. Now take $0 < q \in \mathbb{R}$ and $\theta > 1$ so that

$$0 < q \cdot \mathfrak{b} \le \mathfrak{q} \le \mathfrak{r} < \theta\mathfrak{r} < \mathfrak{b}.$$

Let us take $s = 1$, $q = (s/8r) = (1/8r)$ so that $4qr < s$. Hence

$$q \cdot \mathfrak{b} = (q4r, q4r) < \mathfrak{q} = (s, 2s) \tag{18}$$

We now state Theorem (6.3) from [STO]:

THEOREM S. *Take* $\mathfrak{b} \in \mathbb{R}_n^+$. *Take* $\mathfrak{r} \in \overset{+}{\mathbb{R}}{}^n(\mathfrak{b})$ *and* $\mathfrak{q} \in \overset{+}{\mathbb{R}}{}^n(\mathfrak{b})$ *with* $0 < \mathfrak{q} \le \mathfrak{r} < \mathfrak{b}$. *Take* $1 < \theta \in \mathbb{R}$ *and* $0 < q \in \mathbb{R}$ *so that* $0 < q \cdot \mathfrak{b} \le \mathfrak{q} \le \mathfrak{r} < \theta\mathfrak{r} < \mathfrak{b}$. *Let* $F \not\equiv 0$ *be a meromorphic function on* $\mathbb{D}(\mathfrak{b})$. *Take* $\tau \in (1, 2, \ldots, n)$ *and define* $F'_\tau = \partial F/\partial z_\tau$. *Then*

$$\begin{aligned}\int_{\mathbb{D}\langle\mathfrak{r}\rangle} \log^+\left|\frac{F'_\tau}{F}\right|\Omega_n \le{}& 8\log^+ T_F(\theta\mathfrak{r}, \mathfrak{q}) + 4\log^+ m_F(\mathfrak{q}, 0)\\ &+ 4\log^+ m_F(\mathfrak{q}, \infty) + 9\log\frac{2\theta}{\theta - 1}\\ &+ 2\log^+\frac{1}{qb_\tau} + 24\log 2\end{aligned} \tag{19}$$

We now apply Stoll's LLD (Theorem S above) to get, for $F : \mathbb{C}^2 \to \mathbb{C}$ an entire function,

$$\begin{aligned}\frac{1}{(2\pi)^2}\int_{\mathbb{D}\langle\mathfrak{r}\rangle} \log^+\left|\frac{1}{F}\frac{\partial F}{\partial z_1}\right| d\theta_1\, d\theta_2 &\le \text{sum of six terms}\\ &= (1) + (2) + (3) + (4) + (5) + (6)\end{aligned} \tag{20}$$

where

$$(6) = 24\log 2 = O(1) \tag{21}$$

$$(5) = 2\log^+\frac{1}{qb_1} = 2\log^+\frac{1}{\frac{1}{8r}4r} = 2\log^+ 2 = O(1) \tag{22}$$

$$(3) = \log^+ m_F(\mathfrak{q}, \infty) = \text{const} = O(1) \tag{23}$$

$$(2) = 4\log^+ m_F(\mathfrak{q}, 0) = \text{const} = O(1) \tag{24}$$

$$(1) = 8\log^+ T_F(\theta\mathfrak{r}, \mathfrak{q}) \le 8\log^+ T_F^{\#}(\theta\mathfrak{r}) \tag{25}$$

$$(4) = 9\log\frac{2\theta}{\theta - 1} \tag{26}$$

Thus, we have

$$\frac{1}{(2\pi)^2}\int_{\mathbb{D}\langle\mathfrak{r}\rangle}\log^+\left|\frac{1}{F}\frac{\partial F}{\partial z_1}\right| d\theta_1\, d\theta_2 \le 8\log^+ T_F^{\#}(\theta\mathfrak{r}) + 9\log\frac{2\theta}{\theta-1} + C, \tag{27}$$

where C is a finite constant, independent of θ. We now apply Borel's lemma that, for μ a positive increasing function,

$$\mu\left(r + \frac{1}{\mu(r)}\right) < 2\mu(r) \tag{28}$$

off an exceptional set of r of finite length. We take

$$\mu(r) = T_F^{\#}(\mathfrak{r}) \tag{29}$$

and

$$r + \frac{1}{\mu(r)} = \theta r \tag{30}$$

so that

$$\theta = 1 + \frac{1}{r\mu(r)}. \tag{31}$$

In other words, this is our choice of θ.

Finally, we have, for any entire function $F:\mathbb{C}^2 \to \mathbb{C}$, with the above choices,

$$\frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\log^+\left|\frac{1}{F}\frac{\partial F}{\partial z_1}\right| d\theta_1\, d\theta_2 \le 17\log^+ T_F^{\#}(\mathfrak{r}) + \log r + C' \tag{32}$$

off a set of finite length, where C' is a finite constant. This is the version of LLD we apply in this paper. A similar result holds for $\partial F/\partial z_2$.

In abbreviated notation, we have in the usual way,

$$\begin{aligned} T^{\#}(\mathfrak{r}, F') &= m(\mathfrak{r}, F') = m\left(\mathfrak{r}, \frac{F'}{F}F\right) \\ &\le m\left(\mathfrak{r}, \frac{F'}{F}\right) + m(\mathfrak{r}, F) \\ &= (1 + o(1))T(\mathfrak{r}, F) \end{aligned} \tag{33}$$

off an exceptional set of finite length. Here, F' denotes either $\partial F/\partial z_1$ or $\partial F/\partial z_2$.

We are freely using here such elementary results as

$$T^{\#}(\mathfrak{r}, F + G) \leq T^{\#}(\mathfrak{r}, F) + T^{\#}(\mathfrak{r}, G) + O(1) \tag{34}$$

$$T^{\#}(\mathfrak{r}, FG) \leq T^{\#}(\mathfrak{r}, F) + T^{\#}(\mathfrak{r}, G) + O(1) \tag{35}$$

and

$$T^{\#}\left(\mathfrak{r}, \frac{1}{F}\right) = T^{\#}(\mathfrak{r}, F) + O(1) \tag{36}$$

Now going way back to (5), we have

$$T^{\#}\left(\mathfrak{r}, \frac{f''(g)}{f'(g)}\right) \leq (3 + o(1))T^{\#}(\mathfrak{r}, g) \tag{37}$$

with the usual allowance for an exception set. This is a key step, and uses (33), (35), and (36). By Clunie's result (see our appendix) f''/f' must be a rational function, so that we have

$$f = \int Pe^{Q} \tag{38}$$

where P and Q are polynomials, because f is entire.

Along a second line of reasoning from (2) and (3) and Clunie's result and LLD, supposing that f' is transcendental, we get

$$T^{\#}(\mathfrak{r}, A_1'(z_1)) \sim T^{\#}(\mathfrak{r}, A_2'(z_2)) \tag{39}$$

off a set E of finite length, where $\mathfrak{r} = (r, 2r)$, $r \to \infty$, and $A_1'(z_1)$ is meant as a function of both z_1 and z_2 and similarly for $A_2'(z_2)$, and $T^{\#}$ is the *polydisc* characteristic. Consequently, off E,

$$T(r, A_1'(z_1)) \sim T(2r, A_1'(z_1)) \tag{40}$$

where the characteristic T is now of functions of one variable.

This is because

$$\begin{aligned} T^{\#}(\mathfrak{r}, A_1'(z_1)) &= m(\mathfrak{r}, A_1'(z_1)) \\ &= \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \log^+ \left|A_1'(re^{i\theta_1})\right| d\theta_1\, d\theta_2 \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ \left|A_1'(re^{i\theta_1})\right| d\theta_1 = m(r, A_1') \\ &= T(r, A_1'), \end{aligned} \tag{41}$$

and $T^{\#}(\mathrm{r}, A_2'(z_2)) = T(2r, A_2')$. Thus

$$T(r, A_1') = (1 + o(1))T(2r, A_2').$$

By symmetry, we get

$$T(r, A_2') = (1 + o(1))T(2r, A_1'),$$

so that $T(r, A_1') = (1 + o(1))T(4r, A_1')$ and (40) follows, on replacing $4r$ above by $2r$.

From (40), as we will show in a minute, $A_1'(z_1)$ must have order zero. (Similarly, $A_2'(z_2)$ must have order 0.) Recall from (2) that

$$f'(g(z_1, z_2))g_1(z_1, z_2) = A_1'(z_1).$$

Hold z_2 fixed, say $z_2 = 0$. By Theorem C of the Appendix, if f is transcendental, then, using LLD (in one variable) we would have $T(r, f'(g(z_1, 0)) = O(r^\alpha)$ for each $\alpha > 0$, off a set E of finite length, and thus by a familiar elementary argument, with *no* exceptions. (For if E is a set of finite length, then for all sufficiently large $r \in E$, the interval $[r, 2r]$ is not contained entirely in E. And if $T(\rho) \leq A\rho^a$ and $\rho \geq r \geq \rho/2$ then $T(r) \leq A2^\alpha r^\alpha$, etc.) Hence $f'(g(z, 0))$ must be of order zero. Another approach to this fact is via Lemma 2 on pp. 751–752 of [HEL]. See also [SOY]. We now quote Theorem 2.9 of [HAY], p. 53 (reversing the notation).

THEOREM H. *Suppose that $f(z)$, $g(z)$ are entire functions and that $\varphi(z) = f(g(z))$ has finite order. Then either $g(z)$ is a polynomial or $f(z)$ has zero order.*

We want to conclude that the $f(z)$ of (1) has zero order. So suppose by way of contradiction, (using Theorem H) that $g(z)$ is a polynomial of degree N. Here, we write $g(z) = g(z, 0)$. Then, given large z, say $|z| > R$, there exists a $w \in \mathbb{C}$ with $g(w) = z$ and $|w| \leq 2|z|^{1/N}$. For fixed $\alpha > 0$ we have

$$|f'(g(w))| \leq A\exp(B|w|^\alpha)$$

so that

$$|f'(z)| \leq A\exp(\overline{B}|z|^{\alpha/N}) \quad |z| > R,$$

where $\overline{B} = 2B$, and thus we see that $f'(z)$ has order zero in any event. But since by (38), $f' = Pe^Q$, we must have $Q = \text{const}$, and $f = P$, say, a polynomial. So in any event, $f = P$.

Now for the promised proof that $A_1'(z_1)$ must have order zero, go back to (40).

Let

$$B_\varepsilon = \{r : T(2r) < (1+\varepsilon)T(r)\} \tag{42}$$

where $T(r) := T(r, A_1')$. If $2^k r \in B_\varepsilon$ for $k = 0, 1, 2, \ldots,$ we have

$$T(2^k r) \leq (1+\varepsilon)^k T(r) \tag{43}$$

$$\frac{T(2^k r)}{(2^k r)^\alpha} \leq \left(\frac{1+\varepsilon}{2^\alpha}\right)^k \frac{T(r)}{r^\alpha} \tag{44}$$

and if $1 + \varepsilon < 2^\alpha$, we have

$$\underline{\lim} \frac{T(\rho)}{\rho^\alpha} = 0.$$

Let J_ε be the complement of B_ε, so that we need $2^k r \notin J_\varepsilon$, $k = 0, 1, 2, \ldots$. That is, $r \notin \bigcup_k J_\varepsilon / 2^k$. But the length of this union is finite, since the length of $J_\varepsilon / 2^k$ is 2^{-k} times the length of J_ε. Again, by the familiar elementary argument, we may dispense with the exceptional set to conclude that $A_1'(z_1)$ has order zero.

Now from (2) and (3),

$$\frac{g_1(z_1, z_2)}{g_2(z_1, z_2)} = \frac{A_1'(z_1)}{A_2'(z_2)}. \tag{45}$$

(This holds off the zeros of $f'(g(z_1, z_2))$, and hence everywhere by continuity, unless $f' \equiv 0$, which is ruled out.)

From (45), it follows that the zeros of $g_1(z_1, z_2)$ are exactly (counting multiplicity) the zeros of $A_1'(z)$ and the zeros of $g_2(z_1, z_2)$ are exactly the zeros of $A_2'(z_2)$. The argument for this is that from (2), every zero of $g_1(z_1, z_2)$ is a zero of $A_1'(z_1)$. In the other direction, suppose there were a $z_1 \in \mathbb{C}$ such that $A_1'(z_1) = 0$ but there were a $z_2 \in \mathbb{C}$ with $g_1(z_1, z_2) \neq 0$. Then for those z_2 we would have $A_2'(z_2) = 0$. Now, for this fixed z_1, $\{z_2 : g_1(z_1, z_2) \neq 0\}$ is an open set in $\mathbb{C}$, and we would have $A_2'(z_2) \equiv 0$, contrary to the hypothesis that $A_2 \neq \text{const}$.

Thus we have

$$A_1'(z_1) = g_1(z_1, z_2) \exp u_1(z_1, z_2) \tag{46}$$

$$A_2'(z_2) = g_2(z_1, z_2) \exp u_2(z_1, z_2) \tag{47}$$

where u_1 and u_2 are entire functions. Going back to (2), we have

$$f'(g(z_1, z_2)) = e^{u(z_1, z_2)}, \tag{48}$$

where $u = u_1 = u_2$. Remember that we know that f is a *polynomial*. If $\deg f \geq 3$ then $\deg f' \geq 2$ and thus f' has at least two zeros. If they are *distinct* zeros, then g cannot omit both of these *two* values, since $g \neq$ const. In the case where $\deg f \leq 2$, $\deg f' \leq 1$, so that

$$f'(w) = \alpha w + \beta, \quad \alpha, \beta \in \mathbb{C}, \tag{49}$$

and we have, from (48),

$$\alpha g(z_1, z_2) + \beta = \exp u(z_1, z_2) \tag{50}$$

$$g(z_1, z_2) = \frac{1}{\alpha}([\exp u(z_1, z_2)] - \beta) \tag{51}$$

$$f(w) = \frac{\alpha w^2}{2} + \beta w + \gamma \tag{52}$$

$$\frac{\alpha}{2}\left[\frac{1}{\alpha}e^{u(z_1, z_2)} - \beta\right]^2 + \beta\left[\frac{1}{\alpha}e^{u(z_1, z_2)} - \beta\right] + \gamma = A_1(z_1) + A_2(z_2). \tag{53}$$

This is of the form

$$\Phi(u(z_1, z_2)) = A_1(z_1) + A_2(z_2) \tag{54}$$

where

$$\Phi(w) = \frac{\alpha}{2}\left[\frac{1}{\alpha}e^w - \beta\right]^2 + \beta\left[\frac{1}{\alpha}e^w - \beta\right] + \gamma. \tag{55}$$

But what we have proved shows that $\Phi(w)$ must be a polynomial, which it obviously is not, unless of course $\alpha = 0$, in which case f is affine and the assertion of the theorem holds. Aside from the untreated case where f' is the *square* of an affine function, we have only the possibility $f'(w) \equiv \beta$, $f(w) = \beta w + \gamma$, and the assertion of the theorem holds.

In the one remaining case,

$$f'(\xi) = (a\xi + b)^n \tag{56}$$

$$f(\xi) = \frac{1}{(n+1)a}(a\xi + b)^{n+1} + C \tag{57}$$

$$g(z_1, z_2) = e^{\frac{1}{2}u(z_1, z_2)} \tag{58}$$

and we would have

$$\Phi(u(z_1, z_2)) = \bar{A}_1(z_1) + \bar{A}_2(z_2) \tag{59}$$

where

$$\Phi(w) = \frac{1}{(n+1)a}\left(ae^{w/2} + b\right)^{n+1} + C. \tag{60}$$

Arguing as above, we know that Φ must be a polynomial, which it obviously is not. This contradiction disposes of the last case, and the theorem is proved.

3. Remarks and open problems

Consider the "easier" problem

$$f(g(z_1, z_2, z_3)) = A_1(z_1) + A_2(z_2) + A_3(z_3). \tag{61}$$

This is easier because setting z_3 = constant gives (1). By a different result (see [RST]), the zero-set of the right-hand side of (61) is connected (actually $A_1(z_1) + A_2(z_2) + A_3(z_3)$, for non-constant entire functions A_1, A_2, A_3, is irreducible as an entire function, in the sense of the usual multiplication of functions). Also f, or $f - \lambda$ for suitable $\lambda \in \mathbb{C}$, mut have infinitely many distinct zeros, and hence at least two, say w_1 and w_2. But then the zero-set of the left-hand side of (61) would have at least two disjoint components, a contradiction. (Notice that replacing f by $f - \lambda$ does not change the form of (61).)

To apply this argument to (1), we would need to know that there are at least two numbers λ_1, λ_2 such that $A_1(z_1) + A_2(z_2) - \lambda_1$, and $A_1(z_1) - A_2(z_2) - \lambda_2$ both have connected zero-sets (which would be implied by these functions being irreducible, for example.) Perhaps $A_1(z_1) + A_2(z_2) - \lambda$ is irreducible (or at least has a connected zero-set for "most" values of λ; perhaps there can be at most one exceptional λ. Notice that the zero set of $\exp z_1 - \exp z_2 - \lambda$ is *disconnected* for $\lambda = 0$, but is connected for all other complex λ. Note that it was proved in [ABR] and [RSTv] that if A_1 and A_2 are non-constant *polynomials*, then $A_1(z_1) - A_2(z_2)$ must have connected zero set. This approach to an alternate proof of our theorem seems promising but difficult. (See [FRI] for some more information about the polynomial case.)

Finally, two open problems.

Problem I. How about

$$f(g(z_1, z_2)) = A_1(z_1)A_2(z_2)? \tag{62}$$

(Notice that the right-hand side of (62) has the *product* instead of the *sum* as in (1).

More generally:

Problem II. How about

$$f(g(z_1, z_2)) = \sum_{j=1}^{n} A_j(z_1) B_j(z_2)? \tag{63}$$

4. Appendix: A theorem of Clunie.

We give a detailed proof of the theorem only outlined in [HAY, p. 54], this time in the context of functions of several complex variables. (We have reversed some of the notation in Hayman's book, interchanging f and g.)

THEOREM C. *Let $f(z)$ be a transcendental meromorphic function of one complex variable, and let $g(z_1, z_2, \ldots, z_n)$ be a non-constant entire function of n complex variables, and let*

$$\varphi(z_1, z_2, \ldots, z_n) = f(g(z_1, z_2, \ldots, z_n)).$$

Then

$$T^{\#}(\mathfrak{r}, \varphi)/T^{\#}(\mathfrak{r}, g) \to \infty$$

as $r_1, r_2, \ldots, r_n \to +\infty$, $\mathfrak{r} = (r_1, r_2, \ldots, r_n)$.

Proof. We may and do assume that $f(w)$ has infinitely many distinct zeros at $w_1, w_2, \cdots \to \infty$. (Otherwise, we could replace f by $f - \lambda$ for some constant λ.) Then

$$N\left(\mathfrak{r}, \frac{1}{\varphi}\right) \geq \sum_{\nu=1}^{p} N\left(\mathfrak{r}, \frac{1}{g(w) - w_\nu}\right) \tag{64}$$

because the averaged counting function N is a monotone increasing function of the pole-set. We also want

$$m^{\#}\left(\mathfrak{r}, \frac{1}{\varphi}\right) \geq \sum_{\nu=1}^{p} m^{\#}\left(\mathfrak{r}, \frac{1}{g(w) - w_\nu}\right) - O(1) \tag{65}$$

Here, we are using the notation

$$m^{\#}(\mathrm{r},\psi) = \left(\frac{1}{2\pi}\right)^n \int_{\theta_1=-\pi}^{\theta_1=\pi} \cdots \int_{\theta_n=-\pi}^{\theta_n=\pi} \log^+ \left|\psi\left(r_1e^{i\theta_1},\dots,r_ne^{i\theta_n}\right)\right| d\theta_1\dots d\theta_n, \tag{66}$$

and observe that $m^{\#} - m = O(1)$ because $\log\sqrt{1+x^2} - \log^+ x$ is bounded.
Now fix n and let

$$0 < \delta < \tfrac{1}{10}\min\{|w_i - w_j| : i \neq j, i, j = 1,\dots,n\}. \tag{67}$$

Write

$$f(w) = (w - w_1)^{m_1}\dots(w - w_p)^{m_p}\Phi(w) \tag{68}$$

where $\Phi(w)$ is regular at each w_i, and where we also choose δ so small that $\Phi(w) \neq 0$ for $0 < |w - w_i| \leq \delta$ for all $i = 1,2,\dots,p$; say $|\Phi(w)| \geq \varepsilon > 0$. Now let

$$E = \bigcup_{i=1}^{p} \{(z_1,\dots,z_n) : |g(z_1,\dots,z_n) - w_i| \leq \delta\}. \tag{69}$$

Now for $(z_1,\dots,z_n) \in E$, we have

$$\log^+\left|\frac{1}{f(g(z_1,\dots,z_n))}\right| \geq \sum_{\nu=1}^{p} \log^+\left|\frac{1}{g(z_1,\dots,z_n) - w_\nu}\right| - M \tag{70}$$

for a suitable constant M depending on p, δ, and ε. But

$$\left(\frac{1}{2\pi}\right)^n \int_{\theta_1=-\pi}^{\theta_1=\pi} \cdots \int_{\theta_n=-\pi}^{\theta_n=\pi} \log^+\left|\frac{1}{g\left(r_1e^{i\theta_1-w_i},\dots,r_ne^{i\theta_n}\right) - w_i}\right| d\theta_n\dots d\theta_1 \tag{71}$$

is asymptotic, as $r_1,\dots,r_n \to \infty$ to

$$\left(\frac{1}{2\pi}\right)^n \int\cdots\int_{E_i} \log\left|\frac{1}{g\left(r_1e^{i\theta_1},\dots,r_ne^{i_n}\right)}\right| d\theta_n\dots d\theta_1 \tag{72}$$

where

$$E_i = \left\{(\theta_1,\dots,\theta_n) : \left|g\left(r_1e^{i\theta_1},\dots,r_ne^{i\theta_n}\right) - w_i\right| \leq \delta\right\} \tag{73}$$

because the integral over the remaining part is less than $\log^+(1/\delta)$.

We conclude, using (70), that

$$T^{\#}\left(\mathrm{r}, \frac{1}{\varphi}\right) \geq pT^{\#}(\mathrm{r}, g) + O(1) \tag{67}$$

for any integer p, and the result follows. This completes the text of this paper.

REFERENCES

[ABR] SHREERAM S. ABHYANKAR and LEE A. RUBEL, *Every difference polynomial has a connected zero-set*, J. Indian Math. Soc. **43** (1979), 69–78.

[FRI] MICHAEL FRIED, *Irreducibility results for separated variables equations*, J. Pure Appl. Algebra **48** (1987), 9–22.

[HAY] WALTER K. HAYMAN, *Meromorphic functions*, Oxford at the Clarendon Press, 1964.

[HEL] S. HELLERSTEIN, *The distribution of values of a meromorphic function and a theorem of H.S. Wilf*, Duke Math. J. **32** (1965), 749–764.

[RST] LEE A. RUBEL, WILLIAM SQUIRES and B.A. TAYLOR, *Irreducibility of certain entire functions, with applications to harmonic analysis*, Ann. of Math. **108** (1978), 553–567.

[RSTv] LEE A. RUBEL, A. SCHINZEL and H. TVERBERG, *On difference polynomials and hereditarily irreducible polynomials*, J. Number Theory **12** (1980), 230–235.

[SOY] G.D. SONG and CHUNG-CHUN YANG, *Further growth properties of compositions of entire and meromorphic functions*, Indian J. Pure Appl. Math. **15** (1984), 67–82.

[STO] WILHELM STOLL, *Value distribution and the lemma of the logarithmic derivative on polydiscs*, Internat. J. Math. Math. Sci. **6** (1983), 617–669.

UNIVERSITY OF ILLINOIS
URBANA, ILLINOIS

HONG KONG UNIVERSITY OF SCIENCE AND TECHNOLOGY
CLEAR WATER BAY, KOWLOON, HONG KONG