ILLINOIS JOURNAL OF MATHEMATICS
Volume 39, Number 2, Summer 1995

ON A SPECIAL SCHAUDER BASIS FOR THE SOBOLEV SPACES $W_0^{1,p}(\Omega)$

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1. Introduction

We are interested in Schauder bases for the Sobolev spaces $W_0^{1,p}(\Omega)$ for Ω a general smooth subdomain of R^n. In the particular case where Ω is a cube it has been proved by Z. Ciesielski and J. Domsta in [3] that $W^{1,p}(\Omega)$ has a Schauder basis made of functions that are mutually orthogonal in $L^2(\Omega)$. Also, for the particular case $p = 2$ it is well known that the eigenfunctions of the Laplace operator constitute a basis of $W_0^{1,2}$ with the property that the elements of the basis are mutually orthogonal in L^2. For a general domain Ω and general p, the existence of a Schauder basis for $W_0^{1,p}(\Omega)$ was proved by S. Fucik, O. John and J. Necas in [4]. However, it is not known whether the elements of this basis are mutually orthogonal in L^2. The existence of a Schauder basis for $W_0^{1,p}(\Omega)$, for general p and Ω, made of elements that are mutually orthogonal in $L^2(\Omega)$ seem to be an open question. It should be mentioned that the Gram-Schmidt orthonormalization of a Schauder basis may fail to be a Schauder basis (see [11], [9]).

In this paper we prove the existence of a Schauder basis of $W_0^{1,p}(\Omega)$ with a property that is weaker than that of having elements that are mutually orthogonal in $L^2(\Omega)$ but that can usefully be substituted for it in some contexts. To wit, let $\{w^i\}_{i\geq 1}$ be a Schauder basis of $W_0^{1,p}(\Omega)$ and V_n the closure in L^2 of the subspace of $W_0^{1,p}(\Omega)$ spanned by the subsequence $\{w^i\}_{i>n}$. We show the existence of a basis with the property that

$$\forall n, \quad V_n \cap \operatorname{Span}\{w^1, \dots, w^n\} = 0. \tag{1}$$

Note that whenever the elements of a basis are mutually orthogonal in $L^2(\Omega)$, (1) is trivially satisfied.

Our interest in this problem came from needs encountered while dealing with the question of existence of a solution to a partial differential equation (see [2]). We elucidate the connection between the two: Schauder bases of Banach spaces are used in the theory of partial differential equations in connection with the Galerkin method. This is one of the methods commonly used to establish the existence of a solution to a partial differential equation.

Received November 20, 1992.
1991 Mathematics Subject Classification. Primary 46; Secondary 35.

The blueprint for applying this method is fairly simple and well known. However, in applying it to specific equations one may encounter difficulties in carrying out some of the required tasks. The use of Schauder bases that have special properties has been successful in some cases (see J.L. Lions, [7], in particular, §6.3 *Base speciale* and §1.7. *Un autre resultat de regularite. Bases speciales*). Among such special properties, the property that the elements of the basis are mutually orthogonal in L^2 is frequently used, and often is a key element in the proof. The property is used essentially in the following way: In implementing the Galerkin method one seeks at first a solution $u_n \in \operatorname{Span}\{w^1, \dots, w^n\}$. If $v = \sum_{i=1}^{\infty} \alpha_i w^i$ and the elements of the basis are mutually orthogonal in L^2 then

$$\int u_n v \, dx = \int u_n \left(\sum_{i=1}^{n} \alpha_i w^i \right) dx \tag{2}$$

and this then can be used very fruitfully (it amounts to enlarging the class of test functions).

Denote by $1 - Q_n$ the L^2 projection onto V_n. Then, obviously, we always have, without any orthogonality assumption on the w^i,

$$\int Q_n(u_n) v \, dx = \int Q_n(u_n) \left(\sum_{i=1}^{\infty} \alpha_i w^i \right) dx. \tag{3}$$

One is then tempted to substitute $Q_n(u_n)$ for u_n in the partial differential equation. Property (1) (when satisfied) allows to do this and still carry out the rest of the steps as called for by the Galerkin method to a conclusive end (see [2], for example). It should be pointed out that in the case where the elements of the basis are mutually orthogonal $Q_n(u_n) = u_n$.

Finally, we mention that our results can be extended to spaces other than $W_0^{1,p}(\Omega)$ (see Remark 2).

2. Some auxiliary results

We begin by recalling an elementary result of $L^p(\Omega)$ spaces:

THEOREM 2.1. *Let Ω be a bounded domain of R^m. There exists a sequence $\{v_i\}$ which is a Schauder basis in $L^p(\Omega)$ for $p \in (1, \infty)$. Furthermore, such $\{v_i\}$ can be found with*

(a)
$$\int_\Omega v_i v_j \, dx = \delta_{ij},$$

(b)
$$P_n(f) = \sum_{i=1}^{n} \alpha_i v_i; \quad \alpha_i = \int_\Omega f \cdot v_i \, dx$$

and the norm of P_n in $\mathscr{L}(L^p(\Omega), L^p(\Omega))$ uniformly bounded for p in a compact subset of $(1, \infty)$.

Proof. In the particular case where $\Omega \subset R$, it is well known that the Haar system has all of the properties listed in the theorem. For the case $m > 1$ the sequence $\{v_i\}$ can be constructed as the image of the Haar system via an isomorphism from $L^p(\Omega)$ to $L^p(0,1)$, such as the isomorphism in [5], [6]. It should be noted that it is important that the isomorphism used be independent of p and isometric when $p = 2$. □

The sequence mentioned above is known to be made of functions which are not continuous. We need a sequence of smooth functions with all of the properties listed in the above theorem. Our intention is to establish the following theorem, whose proof will be delayed until we have introduced some notation and proved some lemmas.

THEOREM 2.2. *There exists a sequence $\{u_i\}$ such that $u_i \in C^\infty(\Omega)$ $\forall i$ and $\{u_i\}$ has all of the properties listed in Theorem* 2.1.

It is relatively easy to verify that by an appropriate smoothing of the elements of the sequence $\{v_i\}$ of Theorem 2.1 one again obtains a Schauder basis. The major difficulty lies in showing that the elements of such a sequence are mutually orthogonal in L^2. Attempting to achieve this by orthogonalizing the smoothed elements of the sequence $\{v_i\}$, by using the Gram-Schmidt process, for example, is not guaranteed to yield a Schauder basis. Indeed, it is well known that the Gram-Schmidt orthonormalization of a Schauder basis may fail to be a Schauder basis (see [11], [9]). Our main difficulty will be to show that in our context, the desired smoothing can be done successfully without losing orthogonality.

Let $p_0 \in (2, \infty)$ and p_0' be its conjugate. Throughout the rest of the paper we will assume that $p \in [p_0', p_0]$. By $\|u\|_p$ we mean the norm of u in $L^p(\Omega)$. Similarly, for an operator T we denote by $\|T\|_p$ its norm in $\mathscr{L}(L^p(\Omega), L^p(\Omega))$. The subscript p is omitted whenever no confusion may ensue.

Let $\rho_\varepsilon(x)$ be the usual mollifier function, and $\{\varepsilon_i\}$ be a decreasing sequence of positive numbers converging to 0. Define

$$T: L^p(\Omega) \to L^p(\Omega)$$

$$T(v_i) = v_i * \rho_{\varepsilon_i}.$$

LEMMA 2.3. $\forall c > 0$, $\forall p_0 \in [2, \infty)$ *there exists a sequence $\{\varepsilon_i^0\}$ such that for any sequence $\{\varepsilon_i\}$ satisfying* $0 \le \varepsilon_i \le \varepsilon_i^0$,

$$\|I - T\|_p \le c \quad \forall p \in (p_0', p_0).$$

Proof. We will prove the above inequality for $p = p_0$. For fixed i, $\rho_\varepsilon * v_i$ converges to v_i in $L^p(\Omega)$ as ε tends to 0. We choose ε_i^0 such that

$$\|\rho_{\varepsilon_i} * v_i - v_i\| \le c \cdot 2^{-i} \cdot \min\left(1, \|v_i\|^{-1}\right)$$

whence $\forall u \in L^p(\Omega)$, $u = \sum \alpha_i v_i$,

$$\begin{aligned}\|(I - T)(u)\|_{L^p(\Omega)} &\leq \sum_{i=1}^{\infty} |\alpha_i| \cdot \|(I - T)v_i\| \\ &\leq \|u\|_{L^p(\Omega)} \cdot c \sum_{i=1}^{\infty} 2^{-i} \\ &\leq \|u\|_{L^p(\Omega)} \cdot c.\end{aligned}$$

The proof is similar for $p = p_0'$. The lemma then follows from the Riesz-Thorin Theorem. □

PROPOSITION 2.1. $\forall p \in [p_0', p_0]$,
(a) *T is an automorphism on $L^p(\Omega)$,*
(b) *$w_i = Tv_i$ is a Schauder basis of $L^p(\Omega)$ $\forall p \in [p_0', p_0]$,*
(c) *$w_i \in C^\infty$ $\forall i$.*

Proof. (a) Direct consequence of Lemma 2.3 (using Newman series).
(b) Direct consequence of (a) and Theorem 2.1.
(c) Follows from the fact that ρ_{ε_i} is a smooth function. □

Remark 1. Assuming that the sequence $\{\varepsilon_i\}$ converges to 0 fast enough we easily deduce from Lemma 2.3 that

$$\|T\| \leq 2,\ \|T^*\| \leq 2,\ \|T^{-1}\| \leq 2,\ \left\|(T^*)^{-1}\right\| \leq 2 \quad \forall p \in (p_0', p_0). \tag{4}$$

Let $E_n = \operatorname{span}\{v_1, \ldots, v_n\}$, $E_{n,\varepsilon} = \operatorname{span}\{w_1, \ldots, w_n\}$. We denote by P_n the L^2 orthogonal projection onto E_n and by $P_{\varepsilon,n}$ the L^2 orthogonal projection onto $E_{n,\varepsilon}$.

LEMMA 2.4. *If $g \perp E_{n,\varepsilon}$ then $T^*g \perp E_n$.*

Proof. $\langle T^*g, v_i \rangle = \langle g, Tv_i \rangle = 0$ for $i < n$ because $Tv_i = w_i \in E_{n,\varepsilon}$. □

LEMMA 2.5. $\forall f \in L^p(\Omega)$,

$$P_n T^* f = P_n T^* P_{\varepsilon,n}(f).$$

Proof. This follows form the previous lemma and the fact that $(f - P_{\varepsilon,n} f) \perp E_{n,\varepsilon}$. □

LEMMA 2.6. *$\forall \mathscr{E} > 0$, there exists $\varepsilon^0 = \{\varepsilon_i^0\}$ such that for any $\varepsilon = \{\varepsilon_i\}$, $0 \leq \varepsilon_i \leq \varepsilon_i^0$, and $\forall f \in L^p(\Omega)$,*

$$\|(I - P_n) T^* P_{\varepsilon,n}(f)\| \leq \mathscr{E} \|P_n T^* P_{\varepsilon,n}(f)\| \quad \forall p \in [p_0', p_0].$$

Proof. As in the proof of Lemma (2.3) we will prove the estimate above for $p = p_0$, $p = p_0'$ and derive the lemma via the Riesz-Thorin Theorem. Since the proofs for the case $p = p_0$ and the case $p = p_0'$ are identical we will show the proof for only one of the cases. For the remainder of this proof $p = p_0$.

We are interested in the restriction of $(I - P_n)T^*$ to $E_{n,\varepsilon}$. But since

$$E_{n,\varepsilon} = TE_n,$$

we will, equivalently, study the restriction of

$$(I - P_n)T^*T$$

to E_n. Since $T^*Tu = u + (T^*T - I)u$, for $u \in E_n$ we have

$$(I - P_n)T^*T(u) = (I - P_n)(T^*T - I)u.$$

Since $\|I - P_n\| \leq M$ we only need to estimate the norm of $(T^*T - I)$ on E_n. We have

$$(I - T^*T) = (I - T^*) + T^*(I - T)$$

whence

$$\|I - T^*T\|_p \leq \|I - T\|_{p'} + \|T\|_{p'}\|I - T\|_p,$$

where we used that $\|A^*\|_p = \|A\|_{p'}$. To summarize, we then have, for $u \in E_n$,

$$\begin{aligned}
&\|(I - P_n)T^*T(u)\| \\
&\quad = \|(I - P_n)[T^*T - I]u\| \\
&\quad = \|(I - P_n)[(T^* - I) - T^*(I - T)]u\| \\
&\quad \leq M(\|I - T\|_{p'} + \|T\|_{p'}\|I - T\|_p)\|u\|_p.
\end{aligned}$$

By Lemma (2.3) and (4) the factor $M(\|I - T\|_{p'} + \|T\|_{p'}\|I - T\|_p)$ can be made arbitrarily small (independently of p) for an appropriate ε.

Given $\delta > 0$ we choose ε such that $M(\|I - T\|_{p'} + \|T\|_{p'}\|I - T\|_p) \leq \delta$. Hence $\forall u \in E_n$,

$$\|(I - P_n)T^*T(u)\| \leq \delta\|u\|.$$

Since $T^{-1}P_{\varepsilon,n}f \in E_n$ for any f, we then have for $u = T^{-1}P_{\varepsilon,n}f$,

$$\|(I - P_n)T^*P_{\varepsilon,n}(f)\| \leq \delta\|T^{-1}P_{\varepsilon,n}(f)\| \quad \forall f.$$

Hence,

$$\begin{aligned}\|(I-P_n)T^*P_{\varepsilon,n}(f)\| &\\ &\le \delta\left\|T^{-1}(T^*)^{-1}T^*(P_{\varepsilon,n}(f))\right\|\\ &\le \delta 4\left\|T^*(P_{\varepsilon,n}(f))\right\| \quad \forall f,\end{aligned}$$

using (4).

We then have (using $\|v\| \le \|P_n v\| + \|I - P_n v\|$) that

$$\|(I-P_n)T^*P_{\varepsilon,n}(f)\| \le 4\delta(\|(I-P_n)T^*P_{\varepsilon,n}(f)\| + \|P_nT^*P_{\varepsilon,n}(f)\|) \quad \forall f.$$

Hence

$$\|(I-P_n)T^*P_{\varepsilon,n}(f)\| \le \frac{4\delta}{(1-4\delta)}\|P_nT^*P_{\varepsilon,n}(f)\|$$

for δ small enough, i.e., $4\delta/(1-4\delta) > 0$. □

LEMMA 2.7. *There exists a sequence $\{\varepsilon_i^0\}$ and a constant $C > 0$ independent of p, n such that for any sequence $\{\epsilon_i\}$, $0 \le \varepsilon_i \le \varepsilon_i^0$ $\forall i$ and for any f*

$$\|P_{\varepsilon,n}f\| \le C\|f\|.$$

Proof. We have that

$$\begin{aligned}\|T^*P_{\varepsilon,n}f\| &\le \|(I-P_n)T^*P_{\varepsilon,n}f\| + \|P_nT^*P_{\varepsilon,n}f\|\\ &\le \underbrace{(1+\mathscr{C})}_{C_1}\|P_nT^*P_{\varepsilon,n}f\| \quad \text{by Lemma 2.6}\\ &\equiv C_1\|P_nT^*f\| \quad \text{by Lemma 2.5.}\end{aligned}$$

Since the norms of $(T^*)^{-1}$ and P_n are bounded independently of p it then follows that

$$\|P_{\varepsilon,n}f\| \le C\|f\|. \quad \square \tag{5}$$

Proof of Theorem 2.2. Let u_i be the sequence constructed by applying the Gram-Schmidt process to the sequence Tv_i. By construction the sequence u_i is orthonormal in L^2. Since the u_i are finite linear combinations of the Tv_i it immediately follows from part (c) of the proposition that $u_i \in C^\infty$. Set $\alpha_i = \int_\Omega f \cdot u_i\, dx$ and it then follows that $f_n(x) = \sum_{i=1}^n \alpha_i u_i = P_{\varepsilon,n}f$.

By a Theorem of Nikolskii (see T. Marti, p. 57, Theorem 5, for example), we then deduce from the estimate (5) that u_i is a Schauder basis of $L^p(\Omega)$ $\forall p \in [p_0', p_0]$. Also, from an examination of the proof of the above theorem it can easily be seen that $\forall f \in L^P(\Omega)$,

$$\lim_{n \to \infty} \|f - P_{\varepsilon, n} f\|_p = 0.$$

3. The main result

$$W^{m,p}(\Omega) = \left\{ u \in L^p(\Omega) \text{ such that } \|u\|_{m,p}^p \equiv \|u\|_{L^p}^p + \sum_{|\alpha| \le m} \|D^\alpha u\|_{L^p}^p < \infty \right\}.$$

Also,

$$W_0^{1,p}(\Omega) = \left\{ u \in W^{1,p}(\Omega) \text{ such that } u = 0 \text{ on } \partial\Omega \right\}.$$

Let Δ denote the Laplace operator. We recall that if $(-\Delta + 1)$ is taken to be the isomorphism from $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ onto $L^p(\Omega)$, then $L \equiv (-\Delta + 1)^{1/2}$ is an isomorphism from $W_0^{1,p}(\Omega)$ onto $L^p(\Omega)$ and is self-adjoint in L^2 (see [10], p. 334, for example). We are now in a position to state the following proposition.

PROPOSITION 3.1. *There exists a sequence of functions w^k, $k = 1, \dots, \infty$, which forms a Schauder basis for $W_0^{1,p}(\Omega, R^m)$ $\forall p \in (p_0', p_0)$ and satisfies*:

1. $w^k \in C^\infty \cap W_0^{1,2}(\Omega)$.
2. $\int_\Omega Lw^j \cdot Lw^i \, dx = \delta_{i,j}$.
3. *If $f \in W_0^{1,p}(\Omega)$, $\alpha_i = \int_\Omega Lf \cdot Lw^i \, dx$ and $P_n(f) = \sum_{k=1}^n \alpha_k w^k$ then $P_n(f)$ converges to f and the norm of P_n in $\mathscr{L}(W_0^{1,p}(\Omega); W_0^{1,p}(\Omega))$ is bounded uniformly in n and p.*

Proof. Take $w^i = L^{-1} u_i$ where u_i is the sequence of Theorem 2. □

In order to state our next result we need the following notation. Denote by V_n the closure in L^2 of the space spanned by the subsequence $\{w^i\}_{i > n}$.

THEOREM 3.1.

$$\forall n \quad V_n \cap \operatorname{Span}\left\{w^1, \dots, w^n\right\} = 0. \tag{6}$$

Proof. Suppose not. Then there exists a nonzero element in the intersection. Without loss of generality assume that the element in question is w^n. Then there exist a_i^k such that $f_k = \sum_{i > n} a_i^k w^i$ and f_k converges to w^n in L^2 as k goes to ∞. Therefore (using the fact that $L \cdot Lw^n \in L^2$) we have

$$\lim_{k \to \infty} (f_k, L \cdot L(w^n)) = (w^n, L \cdot Lw^n) = (Lw^n, Lw^n) = 1.$$

On the other hand

$$(f_k, L \cdot L(w^n)) = \sum_{i>n} a_i^k \left(Lw^i, Lw^n\right) \equiv 0. \quad \square$$

Denoting by Q_n the L^2 projection onto the space $V_n^{\perp}$ (in L^2) we then have the following corollary:

COROLLARY 3.2. *If* $a_{i,j} = \int_\Omega Q_n(w^i) \cdot Q_n(w^j)\,dx$ *for* $i \leq n$ *and* $j \leq n$ *then the matrix* $(a_{i,j})_{i,j}$ *is definite positive and invertible.*

Proof. The statement of the corollary is equivalent to saying that the vectors $(Q_n(w^i))_{i=1,\,n}$ are linearly independent. Suppose they are not. Then there exist α_i, $i = 1, \ldots, n$ not all 0 such that $\sum_{i=1}^n \alpha_i Q_n(w^i) = 0$. Therefore

$$\begin{aligned}\operatorname{Span}\left\{w^1, \ldots, w^n\right\} \ni \sum_{i=1}^n \alpha_i w^i &= \sum_{i=1}^n \alpha_i (1 - Q_n)\left(w^i\right) + \sum_{i=1}^n \alpha_i Q_n\left(w^i\right)\\ &= \sum_{i=1}^n \alpha_i (1 - Q_n)\left(w^i\right) \in V_n.\end{aligned}$$

From the previous theorem it then follows that $\sum_{i=1}^n \alpha_i w^i = 0$. Since the vectors w^i are linearly independent, $\alpha_i = 0$ for each i, which contradicts the assumption. $\square$

Remark 2. It is possible to derive such results for spaces other than $W_0^{1,p}(\Omega)$ without a significant change in the proof. Indeed the specificity of $W_0^{1,p}(\Omega)$ entered the proof only via the operator L of Section 3. Therefore by selecting another suitable L it would be possible to derive the results for other spaces.

Acknowledgment. The author thanks Professor R.F. Wheeler for bringing to his attention the references]11] and [9]. Thanks are also due to an anonymous referee who suggested Remark 2.

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