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RATIONAL PERIOD FUNCTIONS ON $G(\sqrt{2})$ AND $G(\sqrt{3})$ WITH HYPERBOLIC POLES ARE NOT HECKE EIGENFUNCTIONS

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1. Introduction

The theory of automorphic forms lends itself naturally to a variety of generalizations. One such, which was initiated by M. Knopp in [Kn1], [Kn2], [Kn4], is the theory of automorphic integrals and their associated rational period functions. This generalization of the notion of automorphic forms has provoked much activity in recent years: see for example [As], [Ch], [CP1], [CP2], [CZ], [Ge1], [Ge2], [Ha], [HK], [Kn3], [MR], [Pa1], [Pa2], [PR], and [Sc].

The object of this paper is to obtain an analogue to a theorem which appears in [Ge1], [Ge2]. In particular, in [Ge1], [Ge2] it is shown that a rational period function defined on the modular group with at least one quadratic irrational pole cannot be an eigenfunction of the induced Hecke operator $\hat{T}(n)$. (This problem was originally posed by M. Knopp in [Kn3].) In fact, there are exactly two settings in which an analogue may take place, namely for rational period functions defined on the two Hecke groups $G(\sqrt{2})$ and $G(\sqrt{3})$. In this paper we show that a rational period function defined on $G(\lambda)$ for $\lambda = \sqrt{2}$ or $\sqrt{3}$ with a pole that is the fixed point of a hyperbolic element of $G(\lambda)$ cannot be an eigenfunction of the induced Hecke operator $\hat{T}_\lambda(n)$. This will be accomplished by using results from [Ge1, Ge2] as well as a linear map [BK, PR] between the space of rational period functions defined on $G(\lambda)$ and the space of rational period functions defined on $\Gamma(1)$, the modular group.

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2. Definitions

DEFINITION 2.1. For each positive integer $n \geq 3$, the Hecke group $G(\lambda_n)$ is the group of linear fractional transformations generated by

$$S_{\lambda_n} = \begin{pmatrix} 1 & \lambda_n \\ 0 & 1 \end{pmatrix} \quad \textit{and} \quad T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

where $\lambda_n = 2\cos(\pi/n)$.

Note that when $n = 3$ the group $G(\lambda_3)$ is $\Gamma(1)$, the modular group.

DEFINITION 2.2. The slash operator is given by

$$(F \mid_r M)(z) = (cz + d)^{-r} F(Mz), \tag{1}$$

where $M = \begin{pmatrix} * & * \\ c & d \end{pmatrix}$ and $r \in R$.

Rational period functions can be defined on the Hecke groups $G(\lambda_n)$ for $n \geq 3 \in Z$. To make this more precise, we give a definition.

DEFINITION 2.3. Suppose f is meromorphic in $\mathscr{H}$ for $n \geq 3$ in Z, and satisfies

$$(f \mid_{2k} S_{\lambda_n})(z) = f(z)$$

and

$$(f \mid_{2k} T)(z) = f(z) + q(z),$$

where k is an integer and $q(z)$ is a rational function. If, in addition, f is meromorphic at $i\infty$, then f is an automorphic integral of weight $2k$ with associated rational period function (abbreviated as RPF) $q(z)$.

Under such circumstances, we say that $q(z)$ is an *RPF of weight* $2k$ *on* $G(\lambda_n)$. If $q \equiv 0$, then f is an *automorphic form* of weight $2k$. In fact, a rational function q is an RPF of weight $2k$ on $G(\lambda_n)$ if and only if the following two functional equations are satisfied [Kn4]:

$$q \mid_{2k} T + q = 0 \tag{2}$$

and

$$\sum_{i=1}^{n-1} q \mid_{2k} (S_{\lambda_n} T)^i + q = 0. \tag{3}$$

Moreover, in the special case that $n = 3$, since $G(\lambda_3) = \Gamma(1)$, f is a modular integral, and if, in addition, $q \equiv 0$, then f is a modular form.

To give substance to the notion of RPFs defined on $G(\lambda_n)$, we note that Parson and Rosen in [PR] gave an infinite family of (non-trivial) RPFs for each group $G(\lambda_n)$ as follows:

$$q_n(z) = \frac{1}{(z^2 - bz - 1)^k} + \frac{1}{(z^2 + bz - 1)^k}, \tag{4}$$

where $k \geq 1$ is an odd integer, and

$$b = \frac{\lambda_n + \sqrt{\lambda_n^2 + 4}}{2} - \frac{2}{\lambda_n + \sqrt{\lambda_n^2 + 4}}.$$

In fact, when $n = 3$ the function $q_3(z)$ is precisely Knopp's original example of RPFs on $\Gamma(1)$ with quadratic irrational poles [Kn1]. Specifically,

$$q_3(z) = \frac{1}{(z^2 - z - 1)^k} + \frac{1}{(z^2 + z - 1)^k}, \tag{5}$$

where k is a positive odd integer.

In [Ge1] and [Ge2] the following theorem, a conjecture of Knopp [Kn3], was proved.

THEOREM 2.4. *Let $q(z)$ be a rational period function with at least one real quadratic irrational pole. Then $q(z)$ is not an eigenfunction of the induced Hecke operator $\hat{T}(n)$ for any $n > 1$.*

In this paper, we obtain an analogue to Theorem 2.4 for RPFs defined on $G(\lambda_4) = G(\sqrt{2})$ and $G(\lambda_6) = G(\sqrt{3})$. Before stating the analogue, we provide some of the necessary machinery.

3. Background

Unless otherwise specified, for the remainder of this paper, let $\lambda = 1, \sqrt{2}$, or $\sqrt{3}$.

DEFINITION 3.1. Suppose z_0 is fixed by $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G(\lambda)$. Then M is hyperbolic if $|\mathrm{Trace}(M)| = |\alpha + \delta| > 2$.

Remark 3.1. If $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G(\lambda)$ is hyperbolic, then $\beta, \gamma \neq 0$. Otherwise, if, say, $\beta = 0$, then $\alpha\delta - \beta\gamma = 1$ implies that $\alpha = \delta = \pm 1$, in which case $|\mathrm{Trace}(M)| = 2$, a contradiction. The same argument holds if $\gamma = 0$.

THEOREM 3.2. *If z_0 is a finite non-zero pole of an RPF q_λ defined on $G(\lambda)$, for $\lambda = \sqrt{2}$ or $\sqrt{3}$, then z_0 is fixed by a hyperbolic element of $G(\lambda)$.*

We omit the proof of Theorem 3.2 because it is almost verbatim that of the proof that Knopp gave in [Kn2] for the case $\lambda = 1$. Also, in [Sc], T. Schmidt has an alternative proof of Theorem 3.2 using λ-continued fractions.

Remark 3.2. It is convenient to note that for $\lambda = \sqrt{2}$ or $\sqrt{3}$, the elements of $G(\lambda)$ fall into two categories, the *even* elements $\begin{pmatrix} a & b\lambda \\ c\lambda & d \end{pmatrix}$ and the *odd* elements $\begin{pmatrix} a\lambda & b \\ c & d\lambda \end{pmatrix}$, where $a, b, c, d \in Z$ and $ad - bc\lambda^2 = 1$ and $ad\lambda^2 - bc = 1$, respectively. In fact, this description also holds for the elements of $\Gamma(1)$ simply by observing that the two categories of elements actually coincide since $\lambda = 1$. For details, see [Hu1] and [Yo].

In some sense, Theorem 3.2 describes *what* the finite non-zero poles of rational period functions are, namely, the fixed points of hyperbolic elements of $G(\lambda)$. The following corollary of Theorem 3.2 sheds more light on the nature of these poles by describing *where* they are. But first, Theorem 3.2 inspires the following definition.

DEFINITION 3.3. If z_0 is a finite non-zero pole of an RPF q_λ defined on $G(\lambda)$ for $\lambda = 1, \sqrt{2}$, or $\sqrt{3}$, then z_0 is said to be a hyperbolic pole of q_λ.

COROLLARY 3.4. *If z_0 is a hyperbolic pole of an RPF q_λ on $G(\lambda)$ for $\lambda = 1$, $\sqrt{2}$ or $\sqrt{3}$, then*

(a) *z_0 is a root of a quadratic polynomial of the form $P(z) = \lambda a z^2 + bz + \lambda c$, where $a, b, c \in Z$ such that $a, c \neq 0$, $\gcd(a, b, c) = 1$, and $b^2 - 4\lambda^2 ac > 0$.*

Consequently,

(b) *$z_0 \in Q(\lambda, \sqrt{N}) \setminus \lambda Q$, for some positive integer N, where if N is a square or if $N = \lambda^2 (N')^2$ for some positive integer N', then $z_0 \in Q(\lambda) \setminus \lambda Q$.*

The proof of Corollary 3.4 is elementary and is left to the reader.

Remark 3.3. (i) When $\lambda = 1$, Corollary 3.4(b) simply restates a theorem of Knopp [Kn2] regarding the non-zero poles of RPFs defined on $\Gamma(1)$. That is, such poles must belong to $Q(\sqrt{N}) \setminus Q$, where, in this case, N is a positive non-square integer, and therefore poles are real quadratic irrational numbers.

(ii) It also follows from [MR, Theorem 2] that the finite poles of RPFs defined on $G(\sqrt{2})$ and $G(\sqrt{3})$ must be in $Q(\lambda, \sqrt{N}) \setminus \lambda Q$, for some positive integer N.

(iii) The condition that finite non-zero poles of RPFs are fixed points of hyperbolic elements of $G(\lambda)$ is necessary, but not sufficient.

Corollary 3.4 motivates the following definition.

DEFINITION 3.5. Suppose $0 \neq z_0 \in Q(\lambda, \sqrt{N}) \setminus \lambda Q$ for some positive integer N. If z_0 is the root of a quadratic polynomial of the form $P(z) = a\lambda z^2 + bz + c\lambda$ where $a, b, c \in Z$, $ac \neq 0$ and $\gcd(a, b, c) = 1$, then $P(z)$ is said to be an associated quadratic for z_0.

Remarks. (i) Since $z_0 \neq 0$ and $z_0 \notin \lambda Q$, a priori, an associated quadratic $P(z) = a\lambda z^2 + bz + c\lambda$ must satisfy $ac \neq 0$. Moreover, since z_0 is real (and $z_0 \notin \lambda Q$) we must have $b^2 - 4\lambda^2 ac > 0$.

(ii) When $\lambda = 1$, an associated quadratic for z_0 is (up to multiplication by -1) the minimal polynomial for z_0, and hence is uniquely determined.

(iii) When $\lambda = \sqrt{2}$ or $\sqrt{3}$, associated quadratics are not necessarily minimal polynomials. For example, if $z_0 = p/q$ is a rational number in lowest terms, then $q^2\lambda z^2 - p^2\lambda$ is an associated quadratic for z_0, but is not the minimal polynomial for z_0. More importantly, though, associated quadratics are unique up to multiplication by -1, as will be verified in the following proposition.

PROPOSITION 3.6. *Suppose $0 \neq z_0 \in Q(\lambda, \sqrt{N}) \setminus \lambda Q$ for some positive integer N. Then an associated quadratic for z_0 (if it exists) is unique up to multiplication by -1.*

The proof of Proposition 3.6 is a straightforward use of the Euclidean Algorithm for polynomials, and may be found in [Ge2].

By Corollary 3.4, all hyperbolic poles of RPFs defined on $G(\lambda)$ have associated quadratics. To prove the main theorem in this paper, quite often we will study real numbers that are potential poles of RPFs because they are roots of polynomials of the form $Q(z) = a\lambda z^2 + bz + c\lambda$. However, these quadratic polynomials will not necessarily be associated quadratics. That is, it may be the case that $\gcd(a, b, c) \neq 1$. These quadratic polynomials provide a means by which we can associate a potential pole of an RPF with a triple of integers, namely $[a, b, c]$. For convenience, then, we write $Q(z) = [a, b, c]$ in place of $Q(z) = a\lambda z^2 + bz + c\lambda$ wherever appropriate. Also, we say that a is the *lead coefficient* of $Q(z)$ (as opposed to $a\lambda$), b is the *second coefficient* of $Q(z)$, and c is the *constant term* (as opposed to $c\lambda$).

DEFINITION 3.7. Suppose z_0 is in $Q(\lambda, \sqrt{N}) \setminus \lambda Q$, for some positive integer N, and has associated quadratic given by $P(z) = [a, b, c]$. Define $\operatorname{disc}(z_0)$ to be the discriminant of the polynomial $P(z)$. That is, $\operatorname{disc}(z_0) = b^2 - 4\lambda^2 ac$.

Note that $\operatorname{disc}(P(z)) = \operatorname{disc}(-P(z))$, so that by Proposition 3.6, $\operatorname{disc}(z_0)$ is well defined.

In order to make a distinction between the definition of $\operatorname{disc}(z_0)$ and the discriminant of any quadratic polynomial of which z_0 is a root, we give the following definition.

DEFINITION 3.8. Suppose $P(z) = rz^2 + sz + t$ is in $R[z]$. Then $D_{P(z)} = s^2 - 4rt$. In other words, $D_{P(z)}$ is the discriminant of the quadratic polynomial $P(z)$.

If in fact $P(z)$ is an associated quadratic for z_0, then $\operatorname{disc}(z_0) = D_{P(z)}$.

LEMMA 3.9. *Suppose z_0 is a root of the polynomial $P(z) = rz^2 + sz + t$ in $R[z]$ and let $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ be a linear fractional transformation such that* $\det(M) = d \neq 0$. *Then*

(a) *Mz_0 is a root of the (at most quadratic) polynomial $Q(z) = (P|_{-2} M')(z)$, where $M' \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}$, and*

(b) $D_{Q(z)} = d^2 D_{P(z)}$.

Proof. First note that

$$\begin{aligned}
(P|_{-2} M')(Mz_0) &= (-\gamma(Mz_0) + \alpha)^2 P(M'(Mz_0)) \\
&= (-\gamma(Mz_0) + \alpha)^2 P\left(\begin{pmatrix} \alpha\delta - \beta\gamma & 0 \\ 0 & \alpha\delta - \beta\gamma \end{pmatrix} z_0\right) \\
&= (-\gamma(Mz_0) + \alpha)^2 P\left(\frac{dz_0}{d}\right) \\
&= (-\gamma(Mz_0) + \alpha)^2 P(z_0) \\
&= 0,
\end{aligned}$$

because z_0 is a root of $P(z)$.

Moreover, $Q(z)$ is a polynomial of at most degree 2 because

$$\begin{aligned}
Q(z) &= (-\gamma z + \alpha)^2 \left(r\left(\frac{\delta z - \beta}{-\gamma z + \alpha}\right)^2 + s\left(\frac{\delta z - \beta}{-\gamma z + \alpha}\right) + t \right) \\
&= r(\delta z - \beta)^2 + s(\delta z - \beta)(-\gamma z + \alpha) + t(-\gamma z + \alpha)^2 \\
&= (r\delta^2 - s\gamma\delta + t\gamma^2)z^2 + (s(\alpha\delta + \beta\gamma) - 2(r\beta\delta + t\alpha\gamma))z \\
&\quad + (r\beta^2 - s\alpha\beta + t\alpha^2).
\end{aligned}$$

Finally, the second statement follows from the first since

$$\begin{aligned} D_{Q(z)} &= \big(s(\alpha\delta+\beta\gamma)-2(r\beta\delta+t\alpha\gamma)\big)^2 \\ &\quad -4(r\beta^2-s\beta\alpha+t\alpha^2)(r\delta^2-s\gamma\delta+t\gamma^2) \\ &= d^2(s^2-4rt) \\ &= d^2 D_{P(z)}, \end{aligned}$$

as desired. □

COROLLARY 3.10. *Suppose z_0 is a hyperbolic pole of an RPF q_λ on $G(\lambda)$ for $\lambda = 1, \sqrt{2}$, or $\sqrt{3}$. If an associated quadratic for z_0 is $P(z) = [r, s, t]$, then for any $M \in G(\lambda)$,*

(a) *an associated quadratic for Mz_0 exists, and is given by $Q(z) = (P|_{-2} M^{-1})(z)$, and consequently,*

(b) $\operatorname{disc}(z_0) = \operatorname{disc}(Mz_0)$.

The proof of Corollary 3.10 is a straightforward and elementary computation and may be found in [Ge2]. Further, we note that for $G(\lambda) = \Gamma(1)$, with the same hypotheses as in Corollary 3.10, statement (b) follows directly from the fact that we may view $P(z)$ and $Q(z)$ as binary quadratic forms. That is, if we let $Q_1(x, y) = ax^2 + bxy + cy^2$ and $Q_2(x, y)$ be the binary quadratic form with the same coefficients as $(P|_{-2} M^{-1})(z)$ (so that $P(z) = Q_1(z, 1)$ and $(P|_{-2} M^{-1})(z) = Q_2(z, 1)$), then since $Q_1(x, y)$ and $Q_2(x, y)$ are equivalent in the narrow sense, we have that the discriminant of $P(z)$ is the same as the discriminant of $(P|_{-2} M^{-1})(z)$. In other words, $\operatorname{disc}(Mz_0) = \operatorname{disc}(z_0)$. (For more information on binary quadratic forms, see [Bu] and [Za].)

LEMMA 3.11. *Let q be a rational period function on $G(\lambda)$ for $\lambda = 1, \sqrt{2}$ or $\sqrt{3}$. If z_0 is a hyperbolic pole of q, then given a fixed prime p, there is a hyperbolic pole z_1 of q satisfying* $\operatorname{disc}(z_1) = \operatorname{disc}(z_0)$ *and with associated quadratic $[r, s, t]$ such that* $\gcd(r, p) = 1$. *In other words, q has a hyperbolic pole with associated quadratic whose lead coefficient is relatively prime to p.*

The proofs of Lemma 3.11 for $\lambda = 1, \sqrt{2}$, and $\sqrt{3}$ are analogous, and the case $\lambda = 1$ appears as Lemma 2.2 in [Ge1] and as Lemma 2.1.13 in [Ge2].

4. More Background

The main result of this paper is restricted to RPFs defined on the two groups $G(\sqrt{2})$ and $G(\sqrt{3})$ essentially because of the existence of Hecke operators on the space of automorphic integrals of a given weight, which, in turn, induce operators on the corresponding space of RPFs. To better understand why this is so, we give a definition.

DEFINITION 4.1. Suppose G_1 and G_2 are subgroups of a group G such that for some $g, h \in G$,

$$\left[G_1 : g(G_1 \cap G_2)g^{-1}\right] < \infty \text{ and } \left[G_2 : h(G_1 \cap G_2)h^{-1}\right] < \infty$$

(i.e., $g(G_1 \cap G_2)g^{-1}$ and $h(G_1 \cap G_2)h^{-1}$ are of finite index in G_1 and G_2 respectively). Then G_1 is said to be commensurable with G_2.

The Hecke groups, $G(\lambda_n)$, are subgroups of $SL(2, R)$. Leutbecher, in [Le], showed that the only Hecke groups which are pairwise commensurable are $\Gamma(1)$, $G(\sqrt{2})$, and $G(\sqrt{3})$. In [BK], J. Bogo and W. Kuyk used the pairwise commensurability of $\Gamma(1)$, $G(\sqrt{2})$, and $G(\sqrt{3})$ to show the existence of, and subsequently define Hecke operators on the space of automorphic forms on $G(\sqrt{2})$ and $G(\sqrt{3})$. Implicit in their construction of Hecke operators was the use of the map ψ_λ, defined by Hecke, which maps the space of automorphic forms on $G(\lambda)$, of weight $2k$ for $\lambda = \sqrt{2}$ or $\sqrt{3}$, to the space of modular forms of the same weight.

In [PR], A. Parson and K. Rosen applied results of [BK] to the space of automorphic integrals and the corresponding space of associated rational period functions. By doing so, they created new modular integrals and rational period functions defined on the modular group, from automorphic integrals and RPFs defined on $G(\sqrt{2})$ and $G(\sqrt{3})$.

Moreover, it is straightforward to see that the Hecke operators defined in [BK] also act as operators on the space of automorphic integrals of weight $2k$, and Parson and Rosen showed that these Hecke operators induce operators on the corresponding space of RPFs. Therefore, in view of Theorem 3.2, that the finite non-zero poles of RPFs on $G(\sqrt{2})$ and $G(\sqrt{3})$ are fixed points of hyperbolic elements of $G(\sqrt{2})$ and $G(\sqrt{3})$, and in light of the fact that induced Hecke operators exist on these spaces, we are irresistibly drawn to an analogue of Theorem 2.4.

To this end, the results mentioned above which are relevant to the results of this paper are summarized in the following theorem and the next four definitions, which can be found, collectively, in [PR, BK, Kn1, Ap].

THEOREM 4.2. *Let $\lambda = \sqrt{2}$ or $\sqrt{3}$.*

(a) *If f_λ is an automorphic integral of weight $2k$ on $G(\lambda)$, then $\psi_\lambda(f_\lambda)$ is a modular integral of weight $2k$, where*

$$\psi_\lambda(f_\lambda) = f_\lambda(\lambda z) + \lambda^{-2k} \sum_{t=0}^{\lambda^2 - 1} f_\lambda\left(\frac{z+t}{\lambda}\right),$$

(b) *If q_λ is the RPF associated with f_λ, then $\hat{\psi}(q_\lambda)$ is an RPF on* $\Gamma(1)$, *where*

$$\begin{aligned}\hat{\psi}(q_\lambda) &= (\psi_\lambda(f_\lambda))|_{2k}T - \psi_\lambda(f_\lambda)\\ &= q_\lambda(\sqrt{2}z) + (\sqrt{2})^{-2k} q_\lambda\left(\frac{z}{\sqrt{2}}\right)\\ &\quad + (\sqrt{2})^{-2k} q_\lambda\left(\frac{z-1}{\sqrt{2}}\right) + (1-z)^{-2k} q_\lambda\left(\frac{\sqrt{2}z}{1-z}\right)\end{aligned}$$

if $\lambda = \sqrt{2}$, *and*

$$\begin{aligned}\hat{\psi}(q_\lambda) &= q_\lambda(\sqrt{3}z) + (\sqrt{3})^{-2k} q_\lambda\left(\frac{z}{\sqrt{3}}\right) + (\sqrt{3})^{-2k} q_\lambda\left(\frac{z-1}{\sqrt{3}}\right)\\ &\quad + (\sqrt{3})^{-2k} q_\lambda\left(\frac{z+1}{\sqrt{3}}\right)\\ &\quad + (z+1)^{-2k} q_\lambda\left(\frac{\sqrt{3}z}{z+1}\right) + (z-1)^{-2k} q_\lambda\left(\frac{\sqrt{3}z}{z-1}\right),\end{aligned}$$

if $\lambda = \sqrt{3}$.

The Hecke operators defined on the space of modular integrals of weight $2k$ are given as follows.

DEFINITION 4.3. For $\lambda = 1$ and f a modular integral of weight $2k$ on $\Gamma(1)$, the Hecke operators $T(n)$ are defined as follows.

$$\begin{aligned}T(n)f &= n^{2k-1} \sum_{\substack{ad=n\\ d>0\\ 0\le b<d}} d^{-2k} f\left(\frac{az+b}{d}\right)\\ &= n^{2k-1} \sum_{\substack{ad=n\\ d>0\\ 0\le b<d}} f|_{2k}\begin{pmatrix} a & b\\ 0 & d\end{pmatrix}.\end{aligned}$$

The Hecke operators defined above induce operators on the space of RPFs of weight $2k$ defined on $\Gamma(1)$ as follows.

DEFINITION 4.4. If f is a modular integral of weight $2k$ with associated RPF $q(z)$, then the induced Hecke operator, $\hat{T}_{2k}(n)$ is given by $\hat{T}(n)q = (T(n)f)|_{2k}T - T(n)f$.

The Hecke operators defined on the space of automorphic integrals of weight $2k$ are defined as follows.

DEFINITION 4.5. For $\lambda = \sqrt{2}$ or $\sqrt{3}$ and f_λ an automorphic integral of weight $2k$ on $G(\lambda)$, the Hecke operators $T_\lambda(n)$ are defined as follows.

(a) *If* $\lambda^2 \nmid n$, *then*

$$T_\lambda(n) f_\lambda = n^{2k-1} \sum_{\substack{ad=n \\ d>0 \\ 0\leq b<d}} d^{-2k} f_\lambda\left(\frac{az + b\lambda}{d}\right)$$

$$= n^{2k-1} \sum_{\substack{ad=n \\ d>0 \\ 0\leq b<d}} f_\lambda|_{2k}\begin{pmatrix} a & b\lambda \\ 0 & d \end{pmatrix}.$$

(b) *If* $n = \lambda^2$,

$$T_\lambda(\lambda^2) f_\lambda = (\lambda^2)^{2k-1} \sum_{\substack{ad=\lambda^2 \\ d>0 \\ 0\leq b<d}} f_\lambda|_{2k}\begin{pmatrix} a & b\lambda \\ 0 & d \end{pmatrix} + \sum_{t=1}^{\lambda^2-1} f_\lambda|_{2k}\begin{pmatrix} \lambda z & t \\ 0 & \lambda \end{pmatrix}.$$

(c) *If* $n = (\lambda^2)^r$ *for some integer* $r > 1$,

$$T_\lambda\left((\lambda^2)^{r+1}\right) f_\lambda = T_\lambda(\lambda^2) T_\lambda\left((\lambda^2)^r\right) f_\lambda - (\lambda^2)^k T_\lambda\left((\lambda^2)^r\right) f_\lambda$$
$$- (\lambda^2)^{2k-1} T_\lambda\left((\lambda^2)^{r-1}\right) f_\lambda.$$

As is the case for the spaces of modular forms and modular integrals, the Hecke operators defined on the space of automorphic integrals on $G(\lambda)$ are multiplicative (see [PR, BK]). Moreover, in analogy to Definition 4.4, the Hecke operators in Definition 4.5 induce operators on the corresponding space of RPFs on $G(\lambda)$, which are given as follows.

DEFINITION 4.6. For $\lambda = \sqrt{2}$ or $\sqrt{3}$, if f_λ is an automorphic integral of weight $2k$ on $G(\lambda)$ with associated RPF q_λ, then $\hat{T}_\lambda(n) q_\lambda = (T_\lambda(n) f_\lambda)|_{2k} T - T_\lambda(n) f_\lambda$.

It is worth noting that in addition to multiplicativity, the induced Hecke operators inherit the recursion formula in Definition 4.5 (c).

We now have the vocabulary with which to restate Theorem 2.4 as well as a useful corollary.

THEOREM 2.4. *Let $q(z)$ be a rational period function of weight $2k$ defined on $\Gamma(1)$ with at least one hyperbolic pole. Then $q(z)$ is not an eigenfunction of the induced Hecke operator $\hat{T}_{2k}(n)$ for any $n > 1$.*

The following corollary to Theorem 2.4 captures the essence of the proof of Theorem 2.4 in [Ge1], [Ge2] and will be relevant in establishing the main result of this paper.

COROLLARY 4.7. *Suppose $q(z)$ is a rational period function of weight $2k$ defined on $\Gamma(1)$ with at least one hyperbolic pole. Then the* RPF $\hat{T}_{2k}(n)q$ *has a hyperbolic pole X_0 with the property that for every hyperbolic pole z_0 of q,* $\operatorname{disc}(X_0) > \operatorname{disc}(z_0)$.

5. Statement of the theorem

THEOREM 5.1. *Let $\lambda = \sqrt{2}$ or $\sqrt{3}$. If q_λ is a rational period function with at least one hyperbolic pole, then q_λ is not an eigenfunction of $\hat{T}_\lambda(n)$ for any $n > 1$.*

The proof of Theorem 5.1 will be a proof by contradiction, in which we will use Theorem 2.4 and the fact that 'essentially' $\hat{\psi}(\hat{T}_\lambda(n)q_\lambda) = \hat{T}(n)\hat{\psi}(q_\lambda)$. However, in order to apply Theorem 2.4 in such a proof, we must guarantee that if q_λ has a hyperbolic pole, then so does $\hat{\psi}(q_\lambda)$. This is enitrely the purpose of the next section.

6. The Poles of $\hat{\psi}(q_\lambda)$

From now on, assume that $\lambda = 1$, $\sqrt{2}$, or $\sqrt{3}$, and that all automorphic integrals and RPFs are of weight $2k$, k is a positive integer, unless otherwise specified.

The goal of this section is to prove the following proposition.

PROPOSITION 6.1. *For $\lambda = \sqrt{2}$ or $\sqrt{3}$, suppose q_λ is an* RPF *defined on $G(\lambda)$ with a hyperbolic pole. If $\hat{\psi}_\lambda$ is the map defined in Theorem* 4.2 (b), *then $\hat{\psi}_\lambda(q_\lambda)$, an* RPF *defined on* $\Gamma(1)$, *has a hyperbolic pole.*

Before proving Proposition 6.1 we give one lemma.

LEMMA 6.2. *Suppose q_λ is an* RPF *on $G(\sqrt{2})$ with a hyperbolic pole z_0 such that* $\operatorname{disc}(z_0) \equiv 1 \pmod 2$. *Then there is a hyperbolic pole z_1 of q_λ satisfying* $\operatorname{disc}(z_1) = \operatorname{disc}(z_0)$ *with associated quadratic $[r, s, t]$ such that $2|r$. In other words, q_λ has a hyperbolic pole with associated quadratic whose lead coefficient is divisible by* 2.

Lemma 6.2 is quite useful in proving Proposition 6.1 for the case $\lambda = \sqrt{2}$. Unfortunately, the analogous lemma for $\lambda = \sqrt{3}$ is false, as evidenced by the following example. For k a positive odd integer, let

$$q(z) = \frac{1}{(\sqrt{3}z^2 - z - \sqrt{3})^k} + \frac{1}{(\sqrt{3}z^2 + z - \sqrt{3})^k} \tag{6}$$

First, we verify that q is an RPF of weight $2k$ on $G(\sqrt{3})$ by checking functional equations (2) and (3).

For convenience, we use the notation introduced in Section 3 after Proposition 3.6 to write $q(z) = [1, -1, -1]^{-k} + [1, 1, -1]^{-k}$. Note that since k is odd, we have $-[a, b, c]^{-k} = [-a, -b, -c]^{-k}$. Hence, it is straightforward to see that q satisfies (2), and therefore, we check only (3). In that case we need only show

$$\sum_{i=1}^{5} [1, -1, -1]^{-k}|_{2k}(ST)^i + \sum_{i=1}^{5} [1, 1, -1]^{-k}|_{2k}(ST)^i \tag{7}$$

$$= -\left([1, -1, -1]^{-k} + [1, 1, -1]^{-k}\right). \tag{8}$$

To this end,

$$\begin{aligned} &\sum_{i=1}^{5} [1, -1, -1]^{-k}|_{2k}(ST)^i + \sum_{i=1}^{5} [1, 1, -1]^{-k}|_{2k}(ST)^i \\ &\quad = [1, -5, 1]^{-k} + [-1, -1, 1]^{-k} + [-3, 7, -1]^{-k} + [-3, 11, -3]^{-k} \\ &\qquad + [-1, 7, -3]^{-k} + [3, -7, 1]^{-k} + [3, -11, 3]^{-k} \\ &\qquad + [1, -7, 3]^{-k} + [-1, 1, 1]^{-k} + [-1, 5, -1]^{-k} \\ &\quad = -\left([1, -1, -1]^{-k} + [1, 1, -1]^{-k}\right), \end{aligned}$$

as desired.

Next, note that the poles of q are $(1 \pm \sqrt{13})/2\sqrt{3}$ and $(-1 \pm \sqrt{13})/2\sqrt{3}$ which have associated quadratics $[1, -1, -1]$ and $[1, 1, -1]$, respectively, neither of which have lead coefficient divisible by 3, and both of which have discriminant equal to 13 ($\equiv 1 (\bmod 3)$).

Therefore, we must resort to another technique, an algorithm, in the proof of Proposition 6.1 for the case $\lambda = \sqrt{3}$.

Proof of Lemma 6.2. Suppose an associated quadratic for z_0 is $[a, b, c]$. Without loss of generality, assume $2 \nmid a$. By functional equation (2), $Tz_0 = -1/z_0$ is a pole of q_λ which, by Corollary 3.10 (a) has associated quadratic $[c, -b, a]$. If $2|c$, we are done. Otherwise, $\gcd(c, 2) = 1$. Now by functional equation (3) and the fact that $z_0 \notin \sqrt{2}Q$ (Corollary 3.4 (b)), at least one of $(S_{\sqrt{2}}T)^3 z_0$, $(S_{\sqrt{2}}T)^2 z_0$, or $(S_{\sqrt{2}}T) z_0$ is a pole of q_λ, and by Corollary 3.10 (a), these potential poles have associated quadratics

$$[2a + b + c, -4a - b, a],$$
$$[a + b + 2c, -4a - 3b - 4c, 2a + b + c],$$

and

$$[c, -b - 4c, a + b + 2c],$$

respectively. By hypothesis, $\operatorname{disc}(z_0) = b^2 - 8ac$ is odd, so that b is odd. Thus, since a and c are odd, the first two associated quadratics have even lead coefficients, and therefore if either $(S_{\sqrt{2}}T)^3 z_0$ or $(S_{\sqrt{2}}T)^2 z_0$ is a pole, we are done. Otherwise, $(S_{\sqrt{2}}T) z_0$ is a pole whose associated quadratic has even constant term, and in that case we need only use $-1/(S_{\sqrt{2}}T) z_0$ which has associated quadratic $[a + b + 2c, b + 4c, c]$ with even lead coefficient. □

The proofs of Proposition 6.1 for $\lambda = \sqrt{2}$ and $\lambda = \sqrt{3}$ are analogous only up to a point. Therefore, we present the proof of Proposition 6.1 for $\lambda = \sqrt{2}$ for as long as the analogy holds, which will, in fact, completely take care of the case $\lambda = \sqrt{2}$. To finish the proof, we then address what remains of the case $\lambda = \sqrt{3}$.

Proof of Proposition 6.1. We wish to show that if q_λ, an RPF on $G(\lambda)$ has a hyperbolic pole, then $\hat{\psi}_\lambda(q_\lambda)$, an RPF on $\Gamma(1)$, has a hyperbolic pole. To this end, recall from Theorem 4.2 that if $\lambda = \sqrt{2}$,

(9)
$$q_2(z) := \hat{\psi}_\lambda(q_\lambda(z)) = q_\lambda(\sqrt{2}z) + 2^{-k} q_\lambda\left(\frac{z}{\sqrt{2}}\right) + 2^{-k} q_\lambda\left(\frac{z-1}{\sqrt{2}}\right) + (1-z)^{-2k} q_\lambda\left(\frac{\sqrt{2}z}{1-z}\right),$$

and if $\lambda = \sqrt{3}$,

(10)
$$\begin{aligned} q_3(z) &:= \hat{\psi}_\lambda(q_\lambda(z)) \\ &= q_\lambda(\sqrt{3}z) + 3^{-k} q_\lambda\left(\frac{z}{\sqrt{3}}\right) + 3^{-k} q_\lambda\left(\frac{z-1}{\sqrt{3}}\right) + 3^{-k} q_\lambda\left(\frac{z+1}{\sqrt{3}}\right) \\ &\quad + (z+1)^{-2k} q_\lambda\left(\frac{\sqrt{3}z}{z+1}\right) + (1-z)^{-2k} q_\lambda\left(\frac{\sqrt{3}z}{1-z}\right). \end{aligned}$$

For $i = 2, 3$ we wish to show that q_i, a priori an RPF on $\Gamma(1)$, has a hyperbolic pole. Since q_i is a sum of terms of the form $q_\lambda|_{2k}M$, where M is a linear fractional transformation of determinant λ, we look to the poles of q_λ in order to search for *potential* poles of q_i.

For example, if z_2 is a hyperbolic pole of q_λ, for $\lambda = \sqrt{2}$, then $\sqrt{2}z_2$ is potentially a pole of q_2 because the second term in equation (9) is $2^{-k}q_\lambda(z/\sqrt{2})$. On the other hand, $\sqrt{2}z_2$ may be a removable singularity of q_2 if any of $2z_2$, $(\sqrt{2}z_2 - 1)/\sqrt{2}$, and $2z_2/(1 - \sqrt{2}z_2)$ are poles of q_λ. Note here that $(1 - z)^{-2k}q_\lambda(\sqrt{2}z/(1 - z))$ is the fourth term in equation (9). By Corollary 3.4 (b), since $z_2 \notin \sqrt{2}Q$, we have $1 - \sqrt{2}z_2 \neq 0$ and so $(1 - z)^{-2k}$ cannot provide a zero-denominator when $z = \sqrt{2}z_2$.

Similarly, if z_3 is a hyperbolic pole of q_λ for $\lambda = \sqrt{3}$, then by examining the second term in equation (10), we see that $\sqrt{3}z_3$ is potentially a pole of q_λ, but we have no guarantee that $\sqrt{3}z_3$ is not a removable singularity of q_3.

Therefore, we proceed as follows. First, for the sake of clarity, we temporarily restrict the discussion to the case $\lambda = \sqrt{2}$. We wish to find a hyperbolic pole z_2 of q_λ such that $2z_2$, $(\sqrt{2}z_2 - 1)/\sqrt{2}$ and $2z_2/(1 - \sqrt{2}z_2)$ are not poles of q_λ, and in doing so, will guarantee that $\sqrt{2}z_2$ is a pole of q_2. This pole will necessarily be hyperbolic because since z_2 is non-zero and finite, then $\sqrt{2}z_2$ is also non-zero and finite. By Theorem 3.2, then, we will have that $\sqrt{2}z_2$ is a hyperbolic pole of q_2.

To this end, let z_2 be a hyperbolic pole of q_λ with associated quadratic $[a_2, b_2, c_2]$ such that $D_2 = b_2^2 - 8a_2c_2$ is maximal with respect to all hyperbolic poles of q_λ. For convenience, let

$$N_1 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad N_2 = \begin{pmatrix} \sqrt{2} & -1 \\ 0 & \sqrt{2} \end{pmatrix}, \quad \text{and} \quad N_3 = \begin{pmatrix} 2 & 0 \\ -\sqrt{2} & 1 \end{pmatrix},$$

so that

$$N_1z_2 = 2z_2, \quad N_2z_2 = \frac{\sqrt{2}z_2 - 1}{\sqrt{2}}, \quad \text{and} \quad N_3z_2 = \frac{2z_2}{1 - \sqrt{2}z_2}.$$

Next, we apply Lemma 3.9 (a) to N_1z_2, N_2z_2, and N_3z_2 to find that they satisfy the (not necessarily associated) quadratic polynomials

$$P_1(z) = a_2\sqrt{2}z^2 + 2b_2z + 4c_2\sqrt{2},$$
$$P_2(z) = 2a_2\sqrt{2}z^2 + (4a_2 + 2b_2)z + (a_2 + b_2 + 2c_2)\sqrt{2},$$

and

$$P_3(z) = (a_2 + b_2 + 2c_2)\sqrt{2}z^2 + (2b_2 + 8c_2)z + 4c_2\sqrt{2},$$

respectively. First, note that we must show that $P_1(z)$, $P_2(z)$ and $P_3(z)$, or else normalizations of $P_1(z)$, $P_2(z)$ and $P_3(z)$, are of the necessary form with respect to Definition 3.5 so that $N_1 z_2$, $N_2 z_2$, and $N_3 z_2$ may legitimately be hyperbolic poles of q_λ. To this end, we need only show that the lead coefficients and constant terms of $P_1(z)$, $P_2(z)$ and $P_3(z)$ are all non-zero. This is clearly true of $P_1(z)$ because $a_2 c_2 \neq 0$ since $[a_2, b_2, c_2]$ is an associated quadratic for z_2. Therefore, it remains to show that $a_2 + b_2 + 2c_2 \neq 0$. But, if $a_2 + b_2 + 2c_2 = 0$, then $N_2 z_2 = (\sqrt{2} z_2 - 1)/\sqrt{2}$ is a root of $P_2(z) = 2a_2\sqrt{2}z^2 + (4a_2 + 2b_2)z$. That is,

$$2a_2\sqrt{2}\,(N_2 z_2)^2 + (4a_2 + 2b_2) N_2 z_2 = 0,$$

which means that either $N_2 z_2 = 0$ or else $2a_2\sqrt{2} N_2 z_2 + (4a_2 + 2b_2) = 0$. This is a contradiction because in either case, we have $N_2 z_2$ and hence z_2 is a rational multiple of $\sqrt{2}$. Thus, $N_1 z_2$, $N_2 z_2$, and $N_3 z_2$ all have associated quadratics, and hence, we return to the usual notation for the $P_i(z)$, namely

$$P_1(z) = [a_2, 2b_2, 4c_2],$$

$$P_2(z) = [2a_2, 4a_2 + 2b_2, a_2 + b_2 + 2c_2]$$

and

$$P_3(z) = [a_2 + b_2 + 2c_2, 2b_2 + 8c_2, 4c_2].$$

Since for $j = 1, 2, 3$ the determinant of N_j is 2, by Lemma 3.9 (b), we have $D_{P_j(z)} = 4D_{P(z)}$, or, alternatively, $D_{P_j(z)} = 4D_2$ because $P(z)$ is an associated quadratic for z_2. If it turns out that for $j = 1, 2, 3$, $P_j(z)$ is an *associated* quadratic for $N_j z_2$, then $N_j z_2$ will not be a pole of q_λ by the maximality of D_2, and then we are done. That is, if $P_j(z)$, when viewed as a triple of integers, does not have 2 as a common factor, then $P_j(z)$ is an associated quadratic for $N_j z_2$, since the remaining condition is to check that the coefficients are relatively prime as a triple.

A similar discussion regarding potential hyperbolic poles of q_3 provides the following information. First, choose z_3 with associated quadratic $[a_3, b_3, c_3]$ so that $D_3 := \mathrm{disc}(z_3)$ is maximal. Then in order to guarantee that $\sqrt{3} z_3$ is a (necessarily hyperbolic) pole of q_3, we must ensure that

$$3z_3, \quad \frac{\sqrt{3} z_3 - 1}{\sqrt{3}}, \quad \frac{\sqrt{3} z_3 + 1}{\sqrt{3}}, \quad \frac{3z_3}{\sqrt{3} z_3 + 1} \quad \text{and} \quad \frac{3z_3}{1 - \sqrt{3} z_3}$$

are not poles of q_λ for $\lambda = \sqrt{3}$. We apply Lemma 3.9 (a) to each of the above numbers, and discover that they satisfy, respectively, the quadratic polynomials

$$Q_1(z) = [a_3, 3b_3, 9c_3],$$

$$Q_2(z) = [3a_3, 6a_3 + 3b_3, a_3 + b_3 + 3c_3],$$

$$Q_3(z) = [3a_3, -6a_3 + 3b_3, a_3 - b_3 + 3c_3],$$

$$Q_4(z) = [a_3 - b_3 + 3c_3, 3b_3 - 18c_3, 9c_3],$$

and

$$Q_5(z) = [a_3 + b_3 + 3c_3, 3b_3 + 18c_3, 9c_3].$$

Note also that by Lemma 3.9 (b), $D_{Q_j(z)} = 9D_3$ for $j = 1, 2, 3, 4, 5$; if it then turns out that each $Q_j(z)$ is an associated quadratic, we are done by the maximality of D_3. That is, if each $Q_j(z)$, when viewed as a triple of integers, does not have 3 as a common factor, then $Q_j(z)$ is an associated quadratic. We condense the information obtained thus far in Figure 1.

The remainder of the proof is given in two cases, the second of which includes an algorithm. Recall that D_2 and D_3 are maximal and fixed for the rest of the proof.

$\lambda = \sqrt{2}$		$\lambda = \sqrt{3}$	
Pole of q_λ?	Corresponding Quadratic Polynomial	Pole of q_λ?	Corresponding Quadratic Polynomial
$2z_2$	$P_1(z) =$ $[a_2, 2b_2, 4c_2]$	$3z_3$	$Q_1(z) =$ $[a_3, 3b_3, 9c_3]$
$\frac{\sqrt{2}z_2 - 1}{\sqrt{2}}$	$P_2(z) =$ $[2a_2, 4a_2 + 2b_2, a_2 + b_2 + 2c_2]$	$\frac{\sqrt{3}z_3 - 1}{\sqrt{3}}$	$Q_2(z) =$ $[3a_3, 6a_3 + 3b_3, a_3 + b_3 + 3c_3]$
$\frac{2z_2}{1 - \sqrt{2}z_2}$	$P_3(z) =$ $[a_2 + b_2 + 2c_2, 2b_2 + 8c_2, 4c_2]$	$\frac{\sqrt{3}z_3 + 1}{\sqrt{3}}$	$Q_3(z) =$ $[3a_3, -6a_3 + 3b_3, a_3 - b_3 + 3c_3]$
		$\frac{3z_3}{\sqrt{3}z_3 + 1}$	$Q_4(z) =$ $[a_3 - b_3 + 3c_3, 3b_3 - 18c_3, 9c_3]$
		$\frac{3z_3}{1 - \sqrt{3}z_3}$	$Q_5(z) =$ $[a_3 + b_3 + 3c_3, 3b_3 + 18c_3, 9c_3]$

FIG. 1 Potential poles of q_λ related to z_i

Case 1. $D_i \equiv 0 (\mathrm{mod}\ \lambda^2)$.

Since $D_i = b_i^2 - 4\lambda^2 a_i c_i$, we have $b_i \equiv 0 (\mathrm{mod}\ \lambda^2)$. By Lemma 3.11, we may assume without loss of generality that $\gcd(a_i, \lambda^2) = 1$. Then every potential pole of q_λ in Figure 1 is eliminated because each corresponding quadratic polynomial has either lead coefficient or constant term relatively prime to λ^2. In other words, all polynomials in Figure 1 are associated quadratics. Therefore, λz_i is a hyperbolic pole of q_i.

Case 2. $D_i \not\equiv 0 (\mathrm{mod}\ \lambda^2)$

Since $D_i = b_i^2 - 4\lambda^2 a_i c_i$, we have $b_i \not\equiv 0 (\mathrm{mod}\ \lambda^2)$.

The remainder of Case 2 is given in two steps, the first of which gives a procedure for producing a hyperbolic pole of $\hat{\psi}_\lambda(q_\lambda)$, although under somewhat restrictive circumstances (due, in part, to the failure of Lemma 6.2 for $\lambda = \sqrt{3}$). The purpose of the second step is to show that even in the worst case, we may always return to the first step.

Step 1. If there exists a pole z_{m_i} with $\mathrm{disc}(z_{m_i}) = D_i$, and with associated quadratic $[r_i, s_i, t_i]$ such that $\lambda^2 | r_i$, and if among all such poles, we choose z_{m_i} so that $|z_{m_i}|$, is maximal, then all potential poles of q_λ, shown in Figure 2 are eliminated. That is, $2z_{m_2}$ and $3z_{m_3}$ are eliminated by virtue of the maximality

$\lambda = \sqrt{2}$		$\lambda = \sqrt{3}$	
Pole of q_λ?	Corresponding Quadratic Polynomial	Pole of q_λ?	Corresponding Quadratic Polynomial
$2z_{m_2}$	$[r_2, 2s_2, 4t_2]$	$3z_{m_3}$	$[r_3, 3s_3, 9t_3]$
$\dfrac{\sqrt{2}z_{m_2} - 1}{\sqrt{2}}$	$[2r_2, 4r_2 + 2s_2, r_2 + s_2 + 2t_2]$	$\dfrac{\sqrt{3}z_{m_3} - 1}{\sqrt{3}}$	$[3r_3, 6r_3 + 3s_3, r_3 + s_3 + 3t_3]$
$\dfrac{2z_{m_2}}{1 - \sqrt{2}z_{m_2}}$	$[r_2 + s_2 + 2t_2, 2s_2 + 8t_2, 4t_2]$	$\dfrac{\sqrt{3}z_3 + 1}{\sqrt{3}}$	$[3r_3, -6r_3 + 3s_3, r_3 - s_3 + 3t_3]$
		$\dfrac{3z_{m_3}}{\sqrt{3}z_{m_3} + 1}$	$[r_3 - s_3 + 3t_3, 3s_3 - 18t_3, 9t_3]$
		$\dfrac{3z_{m_3}}{1 - \sqrt{3}z_{m_3}}$	$[r_3 + s_3 + 3t_3, 3s_3 + 18t_3, 9t_3]$

FIG. 2 Potential poles of q_λ related to z_{m_i}

of $|z_{m_i}|$, and all of the remaining potential poles of q_λ are eliminated because each has corresponding quadratic polynomial with either lead coefficient or constant term relatively prime to λ^2.

Remark. By Lemma 6.2, when $\lambda = \sqrt{2}$, we can always find a hyperbolic pole of q_λ satisfying the hypotheses given in Step 1. Therefore, when $\lambda = \sqrt{2}$, if q_λ has a hyperbolic pole, so does $\hat{\psi}_\lambda(q_\lambda)$, and we are done. On the other hand, when $\lambda = \sqrt{3}$, as promised, the situation is more complicated, and we deal with it in the next step.

Step 2. $\lambda = \sqrt{3}$, $D_3 \not\equiv 0 (\mathrm{mod}\, 3)$. Recall that z_3 is a hyperbolic pole of q_λ with associated quadratic $[a_3, b_3, c_3]$ such that $D_3 = \mathrm{disc}(z_3)$ is maximal. Note that since $D_3 \not\equiv 0 (\mathrm{mod}\, 3)$, we must have $\gcd(b_3, 3) = 1$.

If $3|a_3$, then go to Step 1. Otherwise, $\gcd(a_3, 3) = 1$. In that case, $3z_3$ is eliminated as a possible pole of q_3 because the lead coefficient of Q_1 (see Figure 1) is not divisible by 3.

It remains either to eliminate as potential poles of q_3, or else use to our advantage, the final four potential poles of q_λ in the second column of Figure 1. To this end, suppose $X_4 = 3z_3/(\sqrt{3}z_3 + 1)$ is a pole of q_λ. By the maximality of D_3, and because $\gcd(3, b_3) = 1$, we must have that $[(a_3 - b_3 + 3c_3)/3, b_3 - 6c_3, 3c_3]$ is an *associated* quadratic for X_4. In that case, $-1/X_4$ is a pole of q_λ with associated quadratic whose lead coefficient is divisible by 3. Go to Step 1. Similarly, if $X_5 = 3z_3/(1 - \sqrt{3}z_3)$ is a pole of q_λ, then $-1/X_5$ is a pole of q_λ with associated quadratic whose lead coefficient divisible by 3. Go to Step 1.

Now, without loss of generality, assume that neither X_4 nor X_5 are poles of q_λ. We will eliminate both $X_2 = (\sqrt{3}z_3 - 1)/\sqrt{3}$ and $X_3 = (\sqrt{3}z_3 + 1)/\sqrt{3}$ as potential poles of q_λ as follows. Observe that the constant terms of Q_2 and Q_3 (the corresponding quadratics for X_2 and X_3) are $(a_3 + b_3 + 3c_3)$ and $(a_3 - b_3 + 3c_3)$, respectively. Therefore, we may eliminate one of X_2 or X_3 depending on whether or not $a_3 \equiv b_3 (\mathrm{mod}\, 3)$. By a solicitous choice of z_3, we may eliminate X_2 and X_3 simultaneously.

In particular, if $a_3 \equiv b_3 (\mathrm{mod}\, 3)$, the among all such poles of q_λ with maximal discriminant and $a_3 \equiv b_3 (\mathrm{mod}\, 3)$, choose z_3 to be the largest (furthest to the right on the real axis). Then X_2 is eliminated because the constant term of Q_2 is not divisible by 3, and X_3 is eliminated because

$$\frac{\sqrt{3}z_3 + 1}{\sqrt{3}} = z_3 + \frac{1}{\sqrt{3}} > z_3.$$

Similarly, if $a_3 \not\equiv b_3 (\mathrm{mod}\, 3)$, then among all such poles of maximal discriminant, choose z_3 to be the smallest. Then X_3 is eliminated because the constant term of Q_3 is not divisible by 3, and X_2 is eliminated because

$$\frac{\sqrt{3}z_3 - 1}{\sqrt{3}} = z_3 - \frac{1}{\sqrt{3}} < z_3.$$

Therefore, $\sqrt{3}z_3$ is a hyperbolic pole of q_3. This concludes Step 2 and Case 2.

In all cases, we have shown that if q_λ is an RPF on $G(\lambda)$ with a hyperbolic pole, then $\hat{\psi}_\lambda(q_\lambda)$, an RPF on $\Gamma(1)$, has a hyperbolic pole. □

7. Relationship among $\hat{T}(n)$, $\hat{T}_\lambda(n)$, and $\hat{\psi}_\lambda$

In this section we find a formula for the relationship among $\hat{\psi}_\lambda$, $\hat{T}(n)$, and $\hat{T}_\lambda(n)$, where $\hat{T}(n)$ and $\hat{T}_\lambda(n)$ are the induced Hecke operators on the space of RPFs on $\Gamma(1)$ and $G(\lambda)$ respectively. See Theorem 4.2 (b), Definition 4.4, and Definition 4.6.

In the next lemma, we begin by giving a formula for the relationship among $T(n)$, $T_\lambda(n)$, and ψ_λ, the usual Hecke operators, and the map from the space of automorphic integrals on $G(\lambda)$ to the space of modular integrals on $\Gamma(1)$, respectively. See Definition 4.3, 4.5 and Theorem 4.2 (a). Then we will show that the corresponding formulas hold for the induced map and operators $\hat{\psi}_\lambda$, $\hat{T}(n)$, and $\hat{T}_\lambda(n)$.

LEMMA 7.1. *For $\lambda = \sqrt{2}$ or $\sqrt{3}$, if f_λ is an automorphic integral of weight $2k$ on $G(\lambda)$, then*

(a) $\psi_\lambda(T_\lambda(n)f_\lambda) = T(n)\psi_\lambda(f_\lambda)$ *if* $\lambda^2 \nmid n$

and

(b) $\psi_\lambda(T_\lambda(n)f_\lambda) = T(n)\psi_\lambda(f_\lambda) + (\lambda^2 - 1)\lambda^{-2k}\psi_\lambda(f_\lambda)$ *if* $n = \lambda^2$.

Proof. The proofs for $\lambda = \sqrt{2}$ and $\sqrt{3}$ are analogous, and so we present only the case $\lambda = \sqrt{2}$.

(a) First, for convenience we write f and ψ in place of $f_{\sqrt{2}}$ and $\psi_{\sqrt{2}}$ respectively. Next, by Definition 4.5 (a), since $2 \nmid n$, we have

$$T_{\sqrt{2}}(n)f = n^{2k-1} \sum_{\substack{ad=n \\ 0 \le b < d}} f|_{2k}\begin{pmatrix} a & b\sqrt{2} \\ 0 & d \end{pmatrix}, \tag{11}$$

and by Theorem 4.2 (a),

$$\psi(f) = f|_{2k}\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{2} \end{pmatrix} + f|_{2k}\begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{pmatrix} + f|_{2k}\begin{pmatrix} 1 & 1 \\ 0 & \sqrt{2} \end{pmatrix}, \tag{12}$$

so that

$$\begin{aligned}(13)\quad \psi(T_{\sqrt{2}}(n)f) &= \left(n^{2k-1}\sum_{\substack{ad=n\\0\le b<d}} f|_{2k}\begin{pmatrix}a & b\sqrt{2}\\0 & d\end{pmatrix}\right)\Bigg|_{2k}\begin{pmatrix}1 & 0\\0 & \sqrt{2}\end{pmatrix}\\
&+\left(n^{2k-1}\sum_{\substack{ad=n\\0\le b<d}} f|_{2k}\begin{pmatrix}a & b\sqrt{2}\\0 & d\end{pmatrix}\right)\Bigg|_{2k}\begin{pmatrix}\sqrt{2} & 0\\0 & 1\end{pmatrix}\\
&+\left(n^{2k-1}\sum_{\substack{ad=n\\0\le b<d}} f|_{2k}\begin{pmatrix}a & b\sqrt{2}\\0 & d\end{pmatrix}\right)\Bigg|_{2k}\begin{pmatrix}1 & 1\\0 & \sqrt{2}\end{pmatrix}\\
&= n^{2k-1}\Bigg[\sum_{\substack{ad=n\\0\le b<d}} f|_{2k}\begin{pmatrix}a & 2b\\0 & d\sqrt{2}\end{pmatrix} + \sum_{\substack{ad=n\\0\le b<d}} f|_{2k}\begin{pmatrix}a\sqrt{2} & b\sqrt{2}\\0 & d\end{pmatrix}\\
&\qquad + \sum_{\substack{ad=n\\0\le b<d}} f|_{2k}\begin{pmatrix}a & a+2b\\0 & d\sqrt{2}\end{pmatrix}\Bigg].\end{aligned}$$

On the other hand, by Definition 4.3,

$$\begin{aligned}(14)\quad T(n)\psi(f) &= n^{2k-1}\sum_{\substack{ad=n\\0\le b<d}} f|_{2k}\begin{pmatrix}1 & 0\\0 & \sqrt{2}\end{pmatrix}\Bigg|_{2k}\begin{pmatrix}a & b\\0 & d\end{pmatrix}\\
&+ n^{2k-1}\sum_{\substack{ad=n\\0\le b<d}} f|_{2k}\begin{pmatrix}\sqrt{2} & 0\\0 & 1\end{pmatrix}\Bigg|_{2k}\begin{pmatrix}a & b\\0 & d\end{pmatrix}\\
&+ n^{2k-1}\sum_{\substack{ad=n\\0\le b<d}} f|_{2k}\begin{pmatrix}1 & 1\\0 & \sqrt{2}\end{pmatrix}\Bigg|_{2k}\begin{pmatrix}a & b\\0 & d\end{pmatrix}\\
(15)\quad &= n^{2k-1}\Bigg[\sum_{\substack{ad=n\\0\le b<d}} f|_{2k}\begin{pmatrix}a & b\\0 & d\sqrt{2}\end{pmatrix} + \sum_{\substack{ad=n\\0\le b<d}} f|_{2k}\begin{pmatrix}a\sqrt{2} & b\sqrt{2}\\0 & d\end{pmatrix}\\
&\qquad + \sum_{\substack{ad=n\\0\le b<d}} f|_{2k}\begin{pmatrix}a & b+d\\0 & d\sqrt{2}\end{pmatrix}\Bigg].\end{aligned}$$

In order to see that $\psi(T_{\sqrt{2}}(n))f = T(n)\psi(f)$, it suffices to show that the summation of the first and last terms of equation (13) equals the summation of the first and last terms of equation (15), because the second terms in both equations are identical. Specifically, it suffices to show that

$$\sum_{\substack{ad=n\\0\le b<d}} \left[f|_{2k}\begin{pmatrix} a & 2b \\ 0 & d\sqrt{2} \end{pmatrix} + f|_{2k}\begin{pmatrix} a & a+2b \\ 0 & d\sqrt{2} \end{pmatrix} \right] \tag{16}$$

is the same as

$$\sum_{\substack{ad=n\\0\le \tilde{b}<d}} \left[f|_{2k}\begin{pmatrix} a & \tilde{b} \\ 0 & d\sqrt{2} \end{pmatrix} + f|_{2k}\begin{pmatrix} a & \tilde{b}+d \\ 0 & d\sqrt{2} \end{pmatrix} \right]. \tag{17}$$

For convenience in computations to follow, we write n/d in place of a.

First, observe that we need only examine the upper right-hand entries of the matrices in (16) and (17), since all other corresponding entries are identical. By doing so, we may rewrite (16) and (17) as follows. Recall that $2 \nmid n$ so that d and n/d are odd. Let

$$\begin{aligned} E1 &= \{0, 2, \dots 2d-2\}, \\ O1 &= \left\{\frac{n}{d}, \frac{n}{d}+2, \dots, \frac{n}{d}+2d-2\right\}, \\ O2 &= \{1, 3, \dots 2d-1\}. \end{aligned}$$

In other words, $E1$ is the list of consecutive even integers from 0 to $2d-2$, $O1$ is the list of consecutive odd integers from n/d to $n/d+2d-1$, and $O2$ is the list of consecutive odd integers from 1 to $2d-1$. In that case, (16) may be written as

$$\begin{aligned} &\sum_{\substack{ad=n\\b\in E1}} f|_{2k}\begin{pmatrix} a & b \\ 0 & d\sqrt{2} \end{pmatrix} + \sum_{\substack{ad=n\\b\in O1\cap O2}} f|_{2k}\begin{pmatrix} a & b \\ 0 & d\sqrt{2} \end{pmatrix} \\ &\quad + \sum_{\substack{ad=n\\b\in O1\setminus(O1\cap O2)}} f|_{2k}\begin{pmatrix} a & b \\ 0 & d\sqrt{2} \end{pmatrix}, \end{aligned} \tag{18}$$

and (17) may be written as

$$\begin{aligned} &\sum_{\substack{ad=n\\\tilde{b}\in E1}} f|_{2k}\begin{pmatrix} a & \tilde{b} \\ 0 & d\sqrt{2} \end{pmatrix} + \sum_{\substack{ad=n\\\tilde{b}\in O1\cap O2}} f|_{2k}\begin{pmatrix} a & \tilde{b} \\ 0 & d\sqrt{2} \end{pmatrix} \\ &\quad + \sum_{\substack{ad=n\\\tilde{b}\in O_2\setminus(O1\cap O2)}} f|_{2k}\begin{pmatrix} a & \tilde{b} \\ 0 & d\sqrt{2} \end{pmatrix}. \end{aligned} \tag{19}$$

Therefore, it suffices to show that the last terms in (18) and (19) are the same. That is, we must show that

$$\sum_{\substack{ad=n \\ b\in O1\backslash(O1\cap O2)}} f|_{2k}\begin{pmatrix} a & b \\ 0 & d\sqrt{2} \end{pmatrix}. \tag{20}$$

is the same as

$$\sum_{\substack{ad=n \\ \tilde{b}\in O2\backslash(O1\cap O2)}} f|_{2k}\begin{pmatrix} a & \tilde{b} \\ 0 & d\sqrt{2} \end{pmatrix}. \tag{21}$$

To this end, note that for $d \neq n$,

$$O1 \setminus (O1 \cap O2) = \left\{2d+1, 2d+3, \ldots, \frac{n}{d} + 2d - 2\right\},$$

and

$$O2 \setminus (O1 \cap O2) = \left\{1, 3, \ldots, \frac{n}{d} - 2\right\},$$

and we see that the difference between every pair of corresponding elements in the above two lists is $2d$. When $d = n$, $O1 \setminus (O1 \cap O2)$ and $O2 \setminus (O1 \cap O2)$ are empty. Therefore, we may rewrite (21) as

$$\sum_{\substack{ad=n \\ b\in O1\backslash O1\cap O2}} f|_{2k}\begin{pmatrix} a & b-2d \\ 0 & d\sqrt{2} \end{pmatrix}, \tag{22}$$

which means that (20) and (21) are identical because

$$\begin{aligned} f|_{2k}\begin{pmatrix} a & b-2d \\ 0 & d\sqrt{2} \end{pmatrix} &= (d\sqrt{2})^{-2k} f\left(\frac{az+b-2d}{d\sqrt{2}}\right) \\ &= (d\sqrt{2})^{-2k} f\left(\frac{az+b}{d\sqrt{2}} - \sqrt{2}\right) \\ &= f|_{2k}\begin{pmatrix} a & b \\ 0 & d\sqrt{2} \end{pmatrix} \end{aligned}$$

since f is periodic with period $\sqrt{2}$. This proves (a).

(b) We will compute $\psi(T_{\sqrt{2}}(2)f)$ and $T(2)\psi(f)$, and then compare the results. To this end, by Definition 4.5 (b), we have

$$(23)\qquad T_{\sqrt{2}}(2)(f) = f|_{2k}\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix} + f|_{2k}\begin{pmatrix}1 & \sqrt{2}\\ 0 & 2\end{pmatrix} + f|_{2k}\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix} + f|_{2k}\begin{pmatrix}\sqrt{2} & 1\\ 0 & \sqrt{2}\end{pmatrix},$$

and by Definition 4.3,

$$(24)\quad T(2)\psi(f) = \psi(f)|_{2k}\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix} + \psi(f)|_{2k}\begin{pmatrix}1 & 1\\ 0 & 2\end{pmatrix} + \psi(f)|_{2k}\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}.$$

Then combining (12) and (24) yields

$$(25)\, T(2)\psi(f) = f|_{2k}\begin{pmatrix}\sqrt{2} & 0\\ 0 & 2\end{pmatrix} + f|_{2k}\begin{pmatrix}1 & 0\\ 0 & 2\sqrt{2}\end{pmatrix} + f|_{2k}\begin{pmatrix}1 & 2\\ 0 & 2\sqrt{2}\end{pmatrix}$$
$$+ f|_{2k}\begin{pmatrix}\sqrt{2} & \sqrt{2}\\ 0 & 2\end{pmatrix} + f|_{2k}\begin{pmatrix}1 & 1\\ 0 & 2\sqrt{2}\end{pmatrix} + f|_{2k}\begin{pmatrix}1 & 3\\ 0 & 2\sqrt{2}\end{pmatrix}$$
$$+ f|_{2k}\begin{pmatrix}2\sqrt{2} & 0\\ 0 & 1\end{pmatrix} + f|_{2k}\begin{pmatrix}2 & 0\\ 0 & \sqrt{2}\end{pmatrix} + f|_{2k}\begin{pmatrix}2 & 1\\ 0 & \sqrt{2}\end{pmatrix}.$$

On the other hand, combining (12) and (23) yields

$$\psi\big(T_{\sqrt{2}}(2)f\big) = T(2)\psi(f) + 2^{-k}f\left(\frac{z+1}{\sqrt{2}}\right) + (\sqrt{2})^{-k}f(\sqrt{2}z+\sqrt{2}) + 2^{-k}f\left(\frac{z}{\sqrt{2}}+\sqrt{2}\right).$$

Since f is periodic with period $\sqrt{2}$, we conclude that

$$\psi\big(T_{\sqrt{2}}(2)f\big) = T(2)\psi(f) + (\sqrt{2})^{-2k}\psi(f),$$

as desired. □

The corresponding formulas hold for q, the RPF associated with f, as is shown in the next corollary.

COROLLARY 7.2. *For $\lambda = \sqrt{2}$ or $\sqrt{3}$, if q_λ is an* RPF *on $G(\lambda)$, then*
(a) $\hat{\psi}_\lambda(\hat{T}_\lambda(n)q_\lambda) = \hat{T}(n)\hat{\psi}_\lambda(q_\lambda)$ *if* $\lambda^2 \nmid n$
and
(b) $\hat{\psi}_\lambda(\hat{T}_\lambda(n)q_\lambda) = \hat{T}(n)\hat{\psi}_\lambda(q_\lambda) + (\lambda^2 - 1)\lambda^{-2k}\hat{\psi}_\lambda(q_\lambda)$ *if* $n = \lambda^2$.

The proof of Corollary 7.2 is a straightforward computation using Theorem 4.2, Definition 4.4, and Lemma 7.1.

We need one final lemma in order to prove Theorem 5.1.

LEMMA 7.3. *Suppose q_λ is an* RPF *of weight $2k$ on $G(\lambda)$ for $\lambda = \sqrt{2}$ or $\sqrt{3}$, and suppose q_λ has a hyperbolic pole. If $s \geq 1$ is an integer, then the* RPF *$\hat{\psi}(\hat{T}_\lambda((\lambda^2)^s)q_\lambda)$ (defined on $\Gamma(1)$) has a hyperbolic pole z_s with the following property: if z_0 is any hyperbolic pole of $\hat{\psi}(q_\lambda)$, the* disc$(z_s) >$ disc(z_0).

Proof. We proceed by induction on r. Specifically, we will show that for all integers $r \geq 1$, $\hat{\psi}(\hat{T}_\lambda((\lambda^2)^r)q_\lambda)$ has a hyperbolic pole z_r such that for any hyperbolic pole z'_{r-1} of $\hat{\psi}(\hat{T}_\lambda((\lambda^2)^{r-1})q_\lambda)$ we have disc$(z_r) >$ disc(z'_{r-1}). To finish the proof, we apply this result to the case $r = 1$, or equivalently, to the hyperbolic pole of $\hat{\psi}(q_\lambda)$.

Let $r = 1$. Then

$$\hat{\psi}\left(\hat{T}_\lambda((\lambda^2)^r)q_\lambda\right) = \hat{\psi}\left(\hat{T}_\lambda(\lambda^2)q_\lambda\right) \tag{26}$$
$$= \hat{T}(\lambda^2)\hat{\psi}(q_\lambda) + (\lambda^2 - 1)\lambda^{-2k}\hat{\psi}(q_\lambda)$$

by Corollary 7.2 (b). Moreover, by Corollary 4.7, $\hat{T}(\lambda^2)\hat{\psi}(q_\lambda)$ has a hyperbolic pole z_1 such that disc$(z_1) >$ disc(z_0) for any hyperbolic pole z_0 of $\hat{\psi}(q_\lambda)$, and therefore, by equation (26), so does $\hat{\psi}(\hat{T}_\lambda(\lambda^2)q_\lambda)$.

Now suppose the induction hypothesis holds for all positive integers j such that $1 \leq j \leq r$. It remains to show that $\hat{\psi}(\hat{T}_\lambda((\lambda^2)^{r+1})q_\lambda)$ has a hyperbolic pole z_{r+1} such that disc$(z_{r+1}) >$ disc(z'_r) for any hyperbolic pole z'_r of $\hat{\psi}(\hat{T}_\lambda((\lambda^2)^r)q_\lambda)$. To accomplish this, we use the recursive definition of $\hat{T}_\lambda((\lambda^2))^{r+1})$ as given by Definition 4.6 (c) in Section 4. Specifically,

$$\hat{\psi}\left(\hat{T}_\lambda\left((\lambda^2)^{r+1}\right)q_\lambda\right) \tag{27}$$
$$= \hat{\psi}\left(\hat{T}_\lambda(\lambda^2)\hat{T}_\lambda\left((\lambda^2)^r\right)q_\lambda - (\lambda^2)^k\hat{T}_\lambda\left((\lambda^2)^r\right)q_\lambda - (\lambda^2)^{2k-1}\hat{T}_\lambda\left((\lambda^2)^{r-1}\right)q_\lambda\right).$$

Since $\hat{\psi}$ is linear, and by applying Corollary 7.2 (b), we can rewrite equation (27) as

$$\begin{aligned}\hat{\psi}\Big(\hat{T}_\lambda\big((\lambda^2)^r\big)q_\lambda\Big) &= \hat{T}(\lambda^2)\hat{\psi}\Big(\hat{T}_\lambda\big((\lambda^2)^r\big)q_\lambda\Big) + (\lambda^2-1)\lambda^{-2k}\hat{\psi}\Big(\hat{T}_\lambda\big((\lambda^2)^r\big)q_\lambda\Big)\\ &\quad -(\lambda^2)^k\hat{\psi}\Big(\hat{T}_\lambda\big((\lambda^2)^r\big)q_\lambda\Big) - (\lambda^2)^{2k-1}\hat{\psi}\Big(\hat{T}_\lambda\big((\lambda^2)^{r-1}\big)q_\lambda\Big)\\ &= \hat{T}(\lambda^2)\hat{\psi}\Big(\big(\hat{T}_\lambda\big((\lambda^2)^r\big)q_\lambda\Big)\\ &\quad +\Big[(\lambda^2-1)\lambda^{-2k} - (\lambda^2)^k\Big]\hat{\psi}\Big(\hat{T}_\lambda\big((\lambda^2)^r\big)q_\lambda\Big)\\ &\quad -(\lambda^2)^{2k-1}\hat{\psi}\Big(\hat{T}_\lambda\big((\lambda^2)^{r-1}\big)q_\lambda\Big).\end{aligned}$$

In total,

$$\begin{aligned}(28)\quad &\hat{\psi}\Big(\hat{T}_\lambda\big((\lambda^2)^{r+1}\big)q_\lambda\Big)\\ &= \hat{T}(\lambda^2)\big(\hat{\psi}\big(\hat{T}_\lambda\big((\lambda^2)^r\big)q_\lambda\big)\big) + C_1\hat{\psi}\big(\hat{T}_\lambda\big((\lambda^2)^r\big)q_\lambda\big) + C_2\hat{\psi}\big(\hat{T}_\lambda\big((\lambda^2)^{r-1}\big)q_\lambda\big),\end{aligned}$$

where $C_1 = (\lambda^2-1)\lambda^{-2k} - (\lambda^2)^k$ and $C_2 = -(\lambda^2)^{2k-1}$.

By the induction hypothesis, $\hat{\psi}(\hat{T}_\lambda((\lambda^2)^r)q_\lambda)$ has a hyperbolic pole, z_r such that for any hyperbolic pole z'_{r-1} of $\hat{\psi}(\hat{T}_\lambda((\lambda^2)^{r-1})q_\lambda)$, we have $\mathrm{disc}(z_r) > \mathrm{disc}(z'_{r-1})$. Moreover, by Corollary 4.7, $\hat{T}(\lambda^2)\hat{\psi}(\hat{T}_\lambda((\lambda^2)^r)q_\lambda)$ has a hyperbolic pole z_{r+1} such that for any hyperbolic pole z'_r of $\hat{\psi}(\hat{T}_\lambda((\lambda^2)^r)q_\lambda)$, we have $\mathrm{disc}(z_{r+1}) > \mathrm{disc}(z'_r)$. Therefore, by equation (28), the same can be said of $\hat{\psi}(\hat{T}_\lambda((\lambda^2)^{r+1})q_\lambda)$. This completes the induction on r.

To finish the proof, simply note that for $1 \le j \le s$, we know that $\hat{\psi}(\hat{T}_\lambda((\lambda^2)^j)q_\lambda)$ has a hyperbolic pole z_j such that $\mathrm{disc}(z_j) > \mathrm{disc}(z'_{j-1})$ for any hyperbolic pole z'_{j-1} of $\hat{\psi}(\hat{T}_\lambda((\lambda^2)^{j-1})q_\lambda)$. In particular, if z_0 is any hyperbolic pole of $\hat{\psi}(q_\lambda)$, then $\mathrm{disc}(z_s) > \mathrm{disc}(z_0)$, as desired. □

8. Proof of Theorem 5.1

We are now ready to prove Theorem 5.1, which we restate.

THEOREM 5.1. *For $\lambda = \sqrt{2}$ or $\sqrt{3}$, if q_λ is a* RPF *on $G(\lambda)$ with at least one hyperbolic pole, then q_λ is not an eigenfunction of the induced Hecke operator $\hat{T}_\lambda(n)$ for any $n > 1$.*

Proof. We give a proof by contradiction, which is accomplished in two steps: for any integer $n > 1$ such that $\lambda^2 \nmid n$, and for $n = (\lambda^2)^s n'$, where $s \ge 1$, $n' \ge 1$ and $\lambda^2 \nmid n'$.

Step 1. $n > 1$ is a integer with $\lambda^2 \nmid n$. By way of contradiction, suppose $\hat{T}_\lambda(n)q_\lambda = Cq_\lambda$ for some $C \neq 0$ in $\mathbf{C}$. Then by Corollary 7.2 (a),

$$\hat{\psi}\left(\hat{T}_\lambda(n)q_\lambda\right) = \hat{T}(n)\hat{\psi}(q_\lambda), \tag{29}$$

and by assumption,

$$\hat{\psi}\left(\hat{T}_\lambda(n)q_\lambda\right) = \hat{\psi}(Cq_\lambda) = C\hat{\psi}(q_\lambda) \tag{30}$$

so that

$$\hat{T}(n)\hat{\psi}(q_\lambda) = C\hat{\psi}(q_\lambda). \tag{31}$$

By Proposition 6.1, $\hat{\psi}(q_\lambda)$ has a hyperbolic pole, and hence Theorem 2.4 applies, and therefore equation (31) gives a contradiction. Specifically, by Theorem 2.4, $\hat{\psi}(q_\lambda)$ is not an eigenfunction of $\hat{T}(n)$.

Step 2. $n = n'(\lambda^2)^s$, where s and n' are positive integers, and $\lambda^2 \nmid n'$. By way of contradiction, suppose $\hat{T}_\lambda(n'(\lambda^2)^s)q_\lambda = Cq_\lambda$ for some $C \neq 0$ in $\mathbf{C}$, so that $\hat{\psi}(\hat{T}_\lambda(n'(\lambda^2)^s)q_\lambda) = C\hat{\psi}(q_\lambda)$. Since the induced Hecke operator is multiplicative and by Corollary 7.2 (a), we have

$$\begin{aligned}\hat{\psi}\left(\hat{T}_\lambda\left(n'(\lambda^2)^s\right)q_\lambda\right) &= \hat{\psi}\left(\hat{T}_\lambda(n')\hat{T}_\lambda\left((\lambda^2)^s\right)q_\lambda\right) \\ &= \hat{T}(n')\hat{\psi}\left(\hat{T}_\lambda(\lambda^2)^s q_\lambda\right).\end{aligned}$$

In total, with our original assumption, we have

$$\hat{T}(n')\hat{\psi}\left(\hat{T}_\lambda\left((\lambda^2)^s\right)q_\lambda\right) = C\hat{\psi}(q_\lambda). \tag{32}$$

By Lemma 7.3, $\hat{\psi}(\hat{T}_\lambda((\lambda^2)^s)q_\lambda)$ has a hyperbolic pole, z_s such that $\mathrm{disc}(z_s) > \mathrm{disc}(z_0)$ for any hyperbolic pole z_0 of $\hat{\psi}(q_\lambda)$. Moreover, by Corollary 4.7, $\hat{T}(n')\hat{\psi}(\hat{T}_\lambda((\lambda^2)^s)q_\lambda)$ has a hyperbolic pole Z_n such that $\mathrm{disc}(Z_n) > \mathrm{disc}(z_s)$. In other words, since $\mathrm{disc}(Z_n) > \mathrm{disc}(z_0)$, $\hat{T}(n')\hat{\psi}(\hat{T}_\lambda(\lambda^2)^s)q_\lambda)$ has a pole, Z_n, which cannot be a pole of $\hat{\psi}(q_\lambda)$, and this contradicts equation (32).

Therefore, q_λ is not an eigenfunction of $\hat{T}_\lambda(n)$ for any integer $n > 1$, as desired. □

9. Conclusion

That RPFs on $\Gamma(1)$, $G(\sqrt{2})$ and $G(\sqrt{3})$ with hyperbolic poles are not Hecke eigenfunctions has now been established. Moreover, RPFs defined on $\Gamma(1)$

have been completely classified, independently, in [CZ] and [Pa2], where an explicit construction is provided for all RPFs of a given positive weight. In [As], A. Ash has also looked at the classification of RPFs on $\Gamma(1)$ from a cohomological point of view. However, there is still work to be done in the classification of RPFs on *all* of the Hecke groups.

T. Schmidt is taking the cohomological approach to this problem. The author and A. Parson have a constructive approach to the classification, and these results will appear in subsequent papers.

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