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SPECTRAL PROPERTIES OF WEIGHTED COMPOSITION OPERATORS AND HYPERBOLICITY OF LINEAR SKEW-PRODUCT FLOWS

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1. Introduction

A weighted composition operator is an operator T that acts by the rule $(Tf)(x) = a(x)f(\phi x)$ on a space of vector-valued functions f, defined on a set X. Here ϕ is a given mapping of X, and $a(\cdot)$ is a given operator-valued function. These operators have been studied with different purposes and from different points of view (see [3], [4], [7], [9], [12], [16], [18], [23], [24] and literature, cited therein).

Weighted composition operators are widely used in the description of asymptotic properties of dynamical systems and differential equations. A well-known example is provided by the celebrated Mather Theorem [19]. This theorem states that a diffeomorphism ϕ of a finite dimensional smooth manifold X is Anosov (is hyperbolic, see the definition below) if and only if the associated weighted composition operator T is hyperbolic, that is $\sigma(T) \cap \mathbb{T} = \emptyset$ for the spectrum $\sigma(T)$ and unite circle $\mathbb{T}$. Here T acts in the space of continuous sections f of the tangent bundle over X, a is the differential of ϕ.

This theorem was generalized in several directions (see [1], [2], [5], [14], [18]), and, in particular, for an arbitrary linear skew-product flow. To give the definition of the linear skew-product flow (LSPF) we consider a homeomorphism ϕ of a compact metric space X and a continuous function a: $X \to L(H)$ with values in the algebra $L(H)$ of operators, bounded on a Hilbert space H. Let Φ: $X \times \mathbb{Z}_+ \to L(H)$ be a cocycle over ϕ, defined by the rule $\Phi(x, n) = a(\phi^{n-1}x) \cdot \ldots \cdot a(x)$. The linear skew-product flow, associated with Φ, is the map

$$(1) \qquad \hat{\phi}^n\colon X \times H \to X \times H\colon (x, v) \mapsto (\phi^n x, \Phi(x,n)v), \quad n \in \mathbb{Z}_+.$$

The LSPFs are one of the major objects in studying the asymptotics of variational differential equations $v' = A(\phi^t x)v, x \in X$, where A: $X \to L(H)$, and ϕ^t is a flow on X (see [10], [15], [21], [22] and the literature therein). One can think of $\Phi(x, t)$ as the solving operator for the differential equation: $v'(t) = \Phi(x, t)v(0), t \in \mathbb{R}, x \in X$.

One of the main problems here is the existence of exponential dichotomy (hyperbolicity) for the LSPF (1) with continuous with respect to x dichotomy projection (see [6], [8], [10], [15], [20], [21]). It means the existence of a continuous

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projection-valued function $P\colon X \to L(H)$ which gives for each $x \in X$ a splitting $H = \operatorname{Im} P(x) \dot{+} \operatorname{Ker} P(x)$ of H in the direct sum of stable $\mathbb{S}_x = \operatorname{Im} P(x)$ and unstable $\mathbb{U}_x = \operatorname{Ker} P(x)$ subspaces such that for $v \in \mathbb{S}_x$ (resp. $v \in \mathbb{U}_x$) the norm $\|\Phi(x, n)v\|$ approaches 0 (resp. ∞) with an exponential rate as $n \to \infty$.

We will characterize the hyperbolicity of the LSPF (1) in spectral terms for the weighted composition operator

$$(Tf)(x) = \left(\frac{d\mu \circ \phi^{-1}}{d\mu}\right)^{1/2} a(\phi^{-1}x) f(\phi^{-1}x), \quad x \in X.$$

The operator T acts on the space $L_2 = L_2(X, \mu; H)$ of functions on X with values in H, where μ is a given Borel ϕ-quasi-invariant finite measure on X. We assume, that $\operatorname{supp} \mu = X$ and ϕ is aperiodic, $\mu(\operatorname{Per} \phi) = 0$.

The central new effect in the present paper is the following. The Mather Theorem and its generalizations from [1], [2], [5], [14], [18] show that for *finite* dimensional H the hyperbolicity of the LSPF (1) is equivalent to the hyperbolicity of the operator T. In this sense $\sigma(T)$ plays for the LSPF (1) the same role as the spectrum of monodromy operator does for periodic differential equations. As it was pointed out by R. Rau [20], for *infinite* dimensional H, the Mather Theorem and its generalizations are not always valid. The hyperbolicity of T always implies the hyperbolicity of the LSPF (1). The converse statement is true provided the values of a are invertible or compact operators in H. In general, the condition $\sigma(T) \cap \mathbb{T} = \emptyset$ is implied by the hyperbolicity of the LSPF (1) together with some additional condition on the LSPF (of the invertibility of $a(x)$ on unstable subspaces $\mathbb{U}_x$ for all $x \in X$). In the present paper, however, we were able to characterize the hyperbolicity of the LSPF (1) in other spectral terms for T. This characterization is the following.

The hyperbolicity of the LSPF (1) is equivalent to the existence of T−invariant splitting $L_2(X, \mu; H) = \operatorname{Im} \mathcal{P} \dot{+} \operatorname{Ker} \mathcal{P}$ into direct sum of "stable" subspace $\operatorname{Im} \mathcal{P}$ and "unstable" subspace $\operatorname{Ker} \mathcal{P}$. The spectrum of the "stable" part of the operator T has to be inside the disk $\mathbb{D} = \{z\colon |z| < 1\}$. The "unstable" part $T_u = T \mid \operatorname{Ker} \mathcal{P}$ of the operator T has to be left-invertible. The spectrum of the left-inverse operator for T_u^N must belong to $\mathbb{D}$ for some $N > 1$. And, finally, the set of those functions in $\operatorname{Ker} \mathcal{P}$, that do not have preimages with respect to all powers of T_u, has to withstand the multiplication by continuous scalar functions. Using a C^*-algebra technique from [1, 2, 18], we prove, under this conditions, that any projection $\mathcal{P}$ on $L_2(X, \mu; H)$, that gives the described T-invariant splitting, has a form $(\mathcal{P}f)(x) = P(x)f(x)$, where $P(\cdot)$ defines the hyperbolicity of the LSPF (1).

Similar results can be proved also for so-called evolutionary semigroups (see [13], [17], [20]). The evolutionary semigroup $\{T^t\}$ is the semigroup of operators, acting on $L_2(\mathbb{R}; H)$ by the rule $(T^t f)(x) = U(x, x - t) f(x - t)$. Here $\{U(x, s)\}_{x \geq s}$ is an evolutionary family on H, that can be viewed as the propagator of a differential equation $v' = A(x)v$, $x \in \mathbb{R}$. Clearly, T^t is the weighted composition operator with $X = \mathbb{R}$, $a(x) = U(x, x - t)$, and $\phi x = x - t$.

Also, one can prove similar results for a *strongly* (versus *uniformly*) continuous operator-function $a(\cdot)$. This situation for the evolutionary semigroup corresponds to a *strongly* continuous propagator $U(x, s)$ for the differential equation $v' = A(x)v$ with, generally, unbounded operators $A(x)$, $x \in \mathbb{R}$. Thus, the exponential dichotomy (hyperbolicity) of any well-posed differential equation in Hilbert space can be expressed in terms of the spectral properties of the weighted composition operators, described here.

Section 2 contains some definitions and our main result. We use several lemmas in its proof. These lemmas are proved in Section 3.

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2. Results

Consider the LSPF (1), generated by a continuous cocycle Φ: $X \times \mathbb{Z}_+ \to L(H)$ over a homeomorphism ϕ of a compact metric space X: $\Phi(x, n+k) = \Phi(\phi^n x, k)\Phi(x, n)$, $n, k \in \mathbb{Z}_+$, and $\Phi(x, 0) = I$, $x \in X$.

Definition. The LSPF (1) is called *hyperbolic* if there exist a continuous projection-valued function P: $X \to L(H)$ and constants $M, \lambda > 0$ such that for all $x \in X$ and $n \in \mathbb{Z}_+$ the following is fulfilled:

(i) $\Phi(x, n)P(x) = P(\phi^n x)\Phi(x, n)$;
(ii) $\|\Phi(x, n)v\| \leq Me^{-\lambda n}\|v\|$, $v \in \operatorname{Im} P(x)$,
$\|\Phi(x, n)v\| \geq M^{-1}e^{\lambda n}\|v\|$, $v \in \operatorname{Ker} P(x)$.

The LSPF is called *spectrally hyperbolic*, if, in addition to that,

(iii) $\operatorname{Im}\left(\Phi(x, n)\big|\operatorname{Ker} P(x)\right)$ is dense in $\operatorname{Ker} P(\phi^n x)$.

If Φ takes invertible values, this definition coincides with the definition of exponential dichotomy for the LSPF (1) (cf. [8], [10], [20], [21]). Note, that the second condition in (ii) implies the left-invertibility of the restriction $\Phi(x, n)|\operatorname{Ker} P(x)$ as an operator from $\operatorname{Ker} P(x)$ to $\operatorname{Ker} P(\phi^n x)$, while (iii) implies its both-sided invertibility. Regarding the hyperbolicity and the spectral theory of linear skew-product flows see [5], [14], [15], [22], where the situation $\dim H < \infty$ was considered. Let us stress, that if $\dim H < \infty$, then (ii) automatically implies (iii). See also [11, Definition 7.6.1], and [6], [21] for the case of infinite dimensional H.

The following fact (see [18, Theorem 3.2]) explains the term *spectral* hyperbolicity.

THEOREM 1. *The spectral hyperbolicity of the LSPF* (1) *is equivalent to the hyperbolicity of the operator T on $L_2(X, \mu, H)$.*

Note, that condition (iii) was missing in Definition 3.1 in [18] (see [20]).

The following example (cf. [20]) shows that the hyperbolicity of the LSPF (1) (that is (i) and (ii) without (iii)) does not imply the hyperbolicity of T.

Example 1. Let X be a single-point set, ϕ be the identity map, $H = l_2(\mathbb{Z}_+)$, and $a \equiv a(x)$ be a weighted unilateral shift on $l_2(\mathbb{Z}_+)$, that is, a: $(v_0, v_1, \dots) \mapsto (0, e^{\lambda} v_0, e^{\lambda} v_1, \dots), \lambda > 0$. Note that $\|a^n v\| = e^{\lambda n} \|v\|$ and the LSPF (1) is hyperbolic with $P = 0$. However, $\sigma(T) = \sigma(a) = \{z\colon |z| \leq e^{\lambda}\}$ contains $\mathbb{T}$. $\square$

Similar examples can be constructed to give two LSPFs (1) such that one of them is hyperbolic, another one is nonhyperbolic but the spectra (and even approximate point spectra) of the corresponding weighted composition operators are equal.

Note that under one of the following additional assumptions the spectral hyperbolicity coincides with the hyperbolicity:

1. $a(x)$ is an invertible operator for all $x \in X$. Then $\Phi(x, n)\big|\operatorname{Ker} P(x)$ is also invertible, which implies (iii).
2. $a(x)$ is a compact operator for all $x \in X$. Then the multiplicative ergodic theorem implies (see [18]) that $\dim \operatorname{Ker} P(x) < \infty$. The second inequality in (ii) gives the left invertibility of the matrix $\Phi(x, n)\big|\operatorname{Ker} P(x)$. Hence, this matrix is invertible, which implies (iii).

Let us formulate now the main result of the paper that describes the hyperbolicity of (1), that is conditions (i) and (ii), in the spectral terms for T. To this end for a left-invertible operator A let us denote by $A^{\dagger}$ its left inverse, defined as $A^{\dagger} u = v$ if $u = Av \in \operatorname{Im} A$ and $A^{\dagger} u = 0$ if $u \perp \operatorname{Im} A$. Let $\|A\|_{\bullet} = \inf\{\|Au\|\colon \|u\| = 1\}$, and let $\big|$ denote restriction of an operator.

THEOREM 2. *The LSPF* (1) *is hyperbolic if and only if there exists a projection* $\mathcal{P}$ *on* $L_2(X, \mu, H)$ *such that*:

(a) $T\mathcal{P} = \mathcal{P}T$;
(b) $\sigma(T \,|\, \operatorname{Im} \mathcal{P}) \subset \mathbb{D}$;
(c) *The operator* $T_u = T\big|\operatorname{Ker}\mathcal{P}$ *is left-invertible in* $\operatorname{Ker}\mathcal{P}$ *and for some* $N \in \mathbb{Z}_+$ *one has* $\sigma((T_u^N)^{\dagger}) \subset \mathbb{D}$;
(d) *The subspace* $\operatorname{Ker}\mathcal{P} \ominus \bigcap_{n \geq 0} \operatorname{Im} T_u^n$ *of* $L_2(X, \mu, H)$ *is invariant under the multiplications by scalar continuous functions on* X.

Any projection $\mathcal{P}$ *that satisfies* (*a*), (*b*), (*c*), *and* (*d*) *has a form* $(\mathcal{P} f)(x) = P(x) f(x)$ *for a continuous projection-valued function* $P\colon X \to L(H)$.

Proof of Theorem 2. Let us note that a change of variables gives the equation

$$\|T^n f\|_{L_2} = \|\Phi(\cdot, n) f\|_{L_2}, \quad f \in L_2. \tag{2}$$

Since $\operatorname{supp}\mu = X$, from (4) one has

(3) $\|T^n\| = \max\{\|\Phi(x,n)\|\colon x \in X\}, \quad \|T^n\|_\bullet = \min\{\|\Phi(x,n)\|_\bullet\colon x \in X\}.$

Assume that LSPF (1) is hyperbolic; that is, conditions (i) and (ii) are fulfilled. We will show that (a)–(d) are fulfilled. Define a projection $\mathcal{P}$ by the rule $(\mathcal{P}f)(x) = P(x)f(x)$. Then (i) implies (a). Having applied (5) to $T^n|\operatorname{Im}\mathcal{P}$, one has (b) from the first inequality in (ii). The second inequality in (ii) by the same reason gives $\|T^n\big|\operatorname{Ker}\mathcal{P}\|_\bullet \geq M^{-1}e^{\lambda n}$. Now (c) is a consequence of the following simple fact (see proofs of all lemmas below in Section 3).

LEMMA 1. *For an operator A in a Hilbert space $\mathcal{H}$ the following are equivalent:*

(1) $\|A^n v\| \geq C\gamma^n\|v\|$ *for all $n \in \mathbb{Z}_+$, $v \in \mathcal{H}$ and some $\gamma > 1$ and $C > 0$;*
(2) *A is left-invertible, and $\sigma((A^N)^\dagger) \subset \mathbb{D}$ for some $N \in \mathbb{Z}_+$.*

Let us derive (d) from (i) and (ii). To this end for any continuous $m\colon X \to \mathbb{R}$ we will denote also by m the operator of multiplication by m in L_2, that is $(mf)(x) = m(x)f(x)$. Then $\mathcal{P}m = m\mathcal{P}$ since $\mathcal{P}$ is an operator of multiplication by $P(\cdot)$. Hence $mf \in \operatorname{Ker}\mathcal{P}$ provided $f \in \operatorname{Ker}\mathcal{P}$. Denote $K = \bigcap_{n\geq 0}\operatorname{Im}T_u^n$. Obviously, $T_u^n(m \circ \phi^n) = mT_u^n$. If $f \in K$ then $f = T_u^n g_n$ for some $g_n \in \operatorname{Ker}\mathcal{P}$ and all $n \in \mathbb{Z}_+$. But now $mf = T_u^n(m \circ \phi^n)\,g_n \in K$, and (d) is proved.

Assume now that (a), (b), (c), (d) are fulfilled. We will show the hyperbolicity of (1). To this end, basically, we need to prove that $\mathcal{P}$ is an operator of multiplication by a continuous projection-valued function $P(\cdot)$. Indeed, let us assume that this fact has been already proved. Then (i) follows from (a). The first estimate in (ii) follows from (5), applied to $T^n\,\big|\operatorname{Im}\mathcal{P}$ and $\Phi(x,n)\big|\operatorname{Im}P(x)$. By Lemma 1 also $\|T_u^n f\| \geq C\gamma^n\|f\|$ for some $\gamma > 1$ and $C > 0$ provided (c). Now the second estimate in (ii) follows from (5) applied to T_u^n and $\Phi(x,n)\big|\operatorname{Ker}P(x)$, and the hyperbolicity of (1) is proved.

In order to prove that $(\mathcal{P}f)(x) = P(x)f(x)$ we need to know that the decomposition $L_2 = \operatorname{Im}\mathcal{P} \dotplus \operatorname{Ker}\mathcal{P}$ is invariant under the multiplication by any continuous function m. We formulate this fact as a lemma.

LEMMA 2. *Conditions (a), (b), (c), (d) imply $\mathcal{P}m = m\mathcal{P}$ for any continuous function $m\colon X \to \mathbb{R}$.*

We will show that conditions (a), (b), (c), and Lemma 2 implies $(\mathcal{P}f)(x) = P(x)f(x)$.

We start from the following heuristic remark. Assume $\sigma(T) \cap \mathbb{T} = \emptyset$. Then (a), (b), (c) are fulfilled for the Riesz projection $\mathcal{P}$ of the operator T, that corresponds to the part $\sigma(T) \cap \mathbb{D}$ of its spectrum. In this case the Riesz integral formula (see, e.g., [8]) allows one to calculate $\mathcal{P}$ via the resolvent $(z - T)^{-1}$ for $z \in \mathbb{T}$. From this formula, using the technique from [18], one can derive that $\mathcal{P}$ is an operator of multiplication. Let now (a), (b), (c) are fulfilled. Of course, these conditions do not imply the two-sided invertibility of $z - T$ for $z \in \mathbb{T}$. We will see, however, that the operator $z - T$ is a left-invertible operator for any $z \in \mathbb{T}$. Moreover, we will compute $\mathcal{P}$ by the formula, similar to the integral formula for a Riesz projection. Then we will apply a modification of the usual technique from [18] to prove that $\mathcal{P}$ is an operator of multiplication.

Having in mind these arguments, we formulate the following result.

LEMMA 3. *Condition (a), (b), (c) imply the left-invertibility of $z - T$ for all $z \in \mathbb{T}$, and the formula*

$$\mathcal{P} = \frac{1}{2\pi i} \int_{\mathbb{T}} (z - T)^{\dagger}\, dz. \tag{4}$$

Let us prove that $\mathcal{P}$ from (6) is an operator of multiplication. To this end let us consider the Banach algebra $\mathcal{B}$ of the operators b on $L_2(X, \mu; H)$ of the form

$$b = \sum_{k=-\infty}^{\infty} a_k T_1^k, \quad \text{with} \quad \|b\|_1 = \sum_{k=-\infty}^{\infty} \|a_k\| < \infty, \tag{5}$$

where

$$(T_1 f)(x) = \left(\frac{d\mu \circ \phi^{-1}}{d\mu}\right)^{1/2} f(\phi^{-1} x),$$

and a_k are operators of multiplication by continuous functions $a_k \colon X \to L(H)$, that is $(a_k f)(x) = a_k(x) f(x)$. It is clear, that $z - T = z - T_1 a \in \mathcal{B}$ for all $z \in \mathbb{T}$. Note that $\mathcal{B}$ is inverse-closed (see [18, Proposition 2.3]). This means, that if $b \in \mathcal{B}$ is an invertible operator in $L(L_2)$ then its inverse operator b^{-1} also belongs to $\mathcal{B}$. Moreover, the following fact is valid.

LEMMA 4. *If $b \in \mathcal{B}$ is a left-invertible operator on $L_2(X, \mu; H)$, then $b^{\dagger} \in \mathcal{B}$.*

Now, in accordance with Lemma 3 and (6) we conclude that $\mathcal{P} \in \mathcal{B}$. In accordance with (5), $\mathcal{P}$ can be written as $\mathcal{P} = \sum a_k T_1^k$. Let us use Lemma 2 and rewrite the equation $m\mathcal{P} - \mathcal{P}m = 0$ term-by-term: $m a_k - a_k \left(m \circ \phi^{-k}\right) = 0$, $k \in \mathbb{Z}$. Recall that $\operatorname{Int} \operatorname{Per} \phi = \emptyset$. Hence for any $k \neq 0$ and any $x \notin \operatorname{Per} \phi$ one can choose m such that $m(x) - m \circ \phi^{-k}(x) \neq 0$. Then $a_k = 0$ for $k \neq 0$, and $\mathcal{P} = a_0$ is an operator of multiplication. $\square$

3. Proofs of Lemmas

Proof of Lemma 1. To prove 2) $\Rightarrow$ 1) note that for the spectral radius $r = r((A^N)^{\dagger})$ one has $r < 1$ provided 2). For small $\epsilon > 0$ then $\|(A^N)^{\dagger k}\| \leq c(r + \epsilon)^k$ for some $c > 0$ and all $k \in \mathbb{Z}_+$. For any $n \in \mathbb{Z}_+$ take $n = Nk + k_0$, $0 \leq k_0 < N$. Then, since $(A^N)^\dagger \cdot A^N = I$, one has

$$\|v\| = \|(A^{k_0})^\dagger (A^N)^{\dagger k} A^n v\| \leq \|(A^{k_0})^\dagger\| \cdot c(r+\epsilon)^k \|A^n v\|,$$

and (1) is proved.

To prove (1) $\Rightarrow$ (2) let us denote by $v_{(\cdot)}$ the orthoprojection of $v \in \mathcal{H}$ onto subspace $(\cdot)$. Choose N such that $C\gamma^N > \gamma_0$ for some $\gamma_0 > 1$. Then

$$\begin{aligned} \|v\| &\geq \|v_{\operatorname{Im} A^N}\| = \|A^N (A^N)^\dagger v_{\operatorname{Im} A^N}\| \\ &\geq C\gamma^N \|(A^N)^\dagger v_{\operatorname{Im} A^N}\| > \gamma_0 \|(A^N)^\dagger v\|, \end{aligned}$$

and $\sigma((A^N)^\dagger) \subset \mathbb{D}$. Here we have used the fact that $A^\dagger v_{\operatorname{Im} A} = A^\dagger v$ and $AA^\dagger u = u$ for $u \in \operatorname{Im} A$. $\square$

Proof of Lemma 2. Denote $Q = I - \mathcal{P}$, $T_s = T \mid \operatorname{Im} \mathcal{P}$, $T_u = T \mid \operatorname{Im} Q$. Let us point out, that

$$\operatorname{Im} \mathcal{P} = \{f \in L_2(X, \mu; H) \colon \|T^n f\| \to 0, n \to \infty\}. \tag{6}$$

Indeed, for $f \in \operatorname{Im} \mathcal{P}$ and small $\epsilon > 0$ one has $\|T^n f\| = \|T_s^n f\| \leq c(r(T_s) + \epsilon)^n \|f\| \to 0$, since the spectral radius $r(T_s) < 1$ provided (b). By Lemma 1 for $A = T_u$ condition (c) implies that $\|T_u^n Qf\| \geq C\gamma^n \|Qf\|$, $\gamma > 1$. If $\|T^n f\| \to 0$ and $f = \mathcal{P}f + Qf$ then

$$\begin{aligned} \|Qf\| &\leq C^{-1}\gamma^{-n} \|T_u^n Qf\| = C^{-1}\gamma^{-n}(\|T^n f - T_s^n \mathcal{P} f\|) \\ &\leq C^{-1}\gamma^{-n}(\|T^n f\| + \|T_s^n \mathcal{P} f\|) \to 0, \end{aligned}$$

as $n \to \infty$, and (8) is proved.

Let us fix a continuous $m \colon X \to \mathbb{R}$. Note that $T^n mf = \left(m \circ \phi^{-n}\right) T^n f$. But $\|m \circ \phi^{-n}\| = \|m\| = \max\{|m(x)| \colon x \in X\}$ for all $n \in \mathbb{Z}_+$. Then for $f \in \operatorname{Im} \mathcal{P}$ one has $\|T^n mf\| \to 0$ as $n \to \infty$ and $mf \in \operatorname{Im} \mathcal{P}$ by (8).

In order to prove that $mf \in \operatorname{Ker} \mathcal{P}$ provided $f \in \operatorname{Ker} \mathcal{P}$ let us denote $K = \bigcap_{n \geq 0} \operatorname{Im} T_u^n$. Note that $\operatorname{Ker} \mathcal{P} \ominus K$ is invariant under multiplications by m in accordance with (d), and we need to show only that $mK \subset K$.

Note that $mf \in \operatorname{Im} T^n$ provided $f \in \operatorname{Im} T^n$, since for $f = T^n g$ one has

$$T^n \left(m \circ \phi^n\right) g = mT^n g = mf \in \operatorname{Im} T^n.$$

Fix $f \in K$ and show that $mf \in K$. Let us denote, for brevity, $B = (T_u^\dagger)^N$. Consider $f_n = B^n f$, $n = 0, 1, \ldots$. Since $f \in K$ one has $f = T_u^{nN} f_n$. Also, $\sigma(B) \subset \mathbb{D}$ provided (c). Then $\|f_n\| \leq c(r(B) + \epsilon)^n \|f\| \to 0$ as $n \to \infty$ for small $\epsilon > 0$. Let us assume $mf \notin \operatorname{Ker} \mathcal{P}$. Consider the functions $g_n = m \circ \phi^n \cdot f_n$. Obviously $\|g_n\| \leq \|m\| \|f_n\| \to 0$ as $n \to \infty$, and $T^{nN} g_n = mf$. Decompose $g_n = \mathcal{P} g_n + Q g_n$.

Since $\operatorname{Ker}\mathcal{P}$ and $\operatorname{Im}\mathcal{P}$ are T-invariant by (a), $T_s^{nN}\mathcal{P}g_n = \mathcal{P}mf$, $T_u^{nN}Qg_n = Qmf$. By Lemma 1 one has $\|Qmf\| = \|T_u^{nN}Qg_n\| \geq C\gamma^{nN}\|Qg_n\|$ and $\|Qg_n\| \to 0$ as $n \to \infty$.

Since, as we have seen, $\|g_n\| \to 0$, we conclude that $\|\mathcal{P}g_n\| \to 0$. However by (b), $\|\mathcal{P}mf\| = \|T_s^{nN}\mathcal{P}g_n\| \leq c(r(T_s)+\epsilon)^{nN}\|\mathcal{P}g_n\| \to 0$ for small $\epsilon > 0$, which contradicts the assumption $\mathcal{P}mf \neq 0$. □

Proof of Lemma 3. Let $T_s = T\big|\operatorname{Im}\mathcal{P}$, $T_u = T\big|\operatorname{Im}Q$, and $Q = I - \mathcal{P}$. Decompose

$$zI - T = (z\mathcal{P} - T_s)\dot{+}(zQ - T_u),\ z \in \mathbb{T}.$$

The operator $z\mathcal{P} - T_s$ is invertible in $\operatorname{Im}\mathcal{P}$ provided (b). Also one has

$$(z\mathcal{P} - T_s)^{-1} = z^{-1}(\mathcal{P} - z^{-1}T_s)^{-1} = \sum_{k=0}^{\infty} z^{-(k+1)}T_s^k. \tag{7}$$

Having denoted $B = (T_u^N)^\dagger$ for N from (c), one has

$$z^N Q - T_u^N = z^N B T_u^N - T_u^N = (z^N B - Q)T_u^N.$$

But $z^N B - Q$ is an invertible operator in $\operatorname{Ker}\mathcal{P}$ provided (c). Since B is the left inverse for T_u^N, one has

$$(z^N Q - T_u^N)^\dagger = B(z^N B - Q)^{-1} = B\sum_{k=0}^{\infty}(z^N B)^k.$$

From the identity

$$z^N Q - T_u^N = (z^{N-1}Q + z^{N-2}T_u + \cdots + T_u^{N-1})(zQ - T_u) \tag{8}$$

one has

$$(zQ - T_u)^\dagger = B\sum_{k=0}^{\infty} z^{Nk}B^k(z^{N-1}Q + z^{N-2}T_u + \cdots + T_u^{N-1}), \tag{9}$$

and finally $(z - T)^\dagger = (z\mathcal{P} - T_s)^{-1}\dot{+}(zQ - T_u)^\dagger$. Both series (9) and (11) converge absolutely for $|z| = 1$. Since $\int_{\mathbb{T}} z^k\,dz = 0$ for $k \neq -1$, integration of $(z - T)^\dagger$ gives the only one nonzero term which corresponds to the value $k = 0$ in (9), that is to the operator $T_s^0 = \mathcal{P}$, and (6) is proved. □

Proof of Lemma 4 (told to the author by R. Exel). Note that $T_1^* = T_1^{-1}$. Then in accordance with (7), $b \in \mathcal{B}$ implies $b^* \in \mathcal{B}$, and $b^*b \in \mathcal{B}$. Since b is a left-invertible operator in $L(L_2)$, the Hermitian operator b^*b is invertible in $L(L_2)$. Indeed, if $b^*bv = 0$ for some v, then $\|bv\|^2 = (b^*bv, v) = 0$ and $v = 0$. But (see [18, Proposition 2.3]) $\mathcal{B}$ contains inverse operators for each of its element, that is invertible in $L(L_2)$. Hence, $(b^*b)^{-1} \in \mathcal{B}$ and also $(b^*b)^{-1}b^* \in \mathcal{B}$. To finish the proof, let us note, that $b^\dagger = (b^*b)^{-1}b^*$. Indeed, $b^\dagger bv = (b^*b)^{-1}b^*bv = v$. If $u \perp \operatorname{Im} b$ then $u \in \operatorname{Ker} b^*$ and $b^\dagger u = (b^*b)^{-1}b^*u = 0$. □

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