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A TIGHT CLOSURE APPROACH TO ARITHMETIC MACAULAYFICATION

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1. Introduction

In this paper we consider the existence of arithmetic Macaulayfications. Let R be a commutative Noetherian ring with identity. We say that R has an *arithmetic Macaulayfication* if there exists an ideal I such that the Rees algebra $R[It]$ is Cohen-Macaulay. In this case, it is clear that $X = \operatorname{Proj} R[It]$ will be a Cohen-Macaulay scheme and the natural projection from X to $Y = \operatorname{Spec} R$ will be what is called a *Macaulayfication* of Y. However, the statement that $R[It]$ is Cohen-Macaulay is significantly stronger than saying X is Cohen-Macaulay.

The most general class of known Macaulayfications comes from desingularization. A *desingularization* of a reduced scheme Y is a regular scheme X together with a proper birational map from X to Y. Relaxing the requirement that X be regular and requiring instead that it be Cohen-Macaulay, we arrive at the definition of a Macaulayfication. Of course, any desingularization of Y is necessarily a Macaulayfication of Y. Hironaka [Hi] proved the existence of a desingularization for any reduced scheme essentially of finite type over a field of characteristic zero. In particular, when $Y = \operatorname{Spec} R$ is affine, Hironaka's work shows that a desingularization X can be obtained by blowing up a suitable ideal I, so that X will be the projective R-scheme associated to the Rees algebra $R[It]$. By comparison, the existence of an arithmetic Macaulayfication, which is really a property of the embedding of X into projective space over R, is not known in this generality. Moreover, even the existence of a desingularization is not known at the present time for arbitrary reduced R.

Work has also been done on Macaulayfications independent of desingularizations; both Faltings [Fa] and Brodmann [Br1] proved the existence of a Macaulayfication for special types of rings, without any assumptions about a ground field. Brodmann [Br2,6.2] and Goto and Yamagishi [GY,7.11] prove the existence of an arithmetic Macaulayfication for any local ring whose completion is equidimensional and Cohen-Macaulay on the punctured spectrum. Of course, any Cohen-Macaulay ring trivially has an arithmetic Macaulayfication: take I to be the zero ideal. For a slightly less trivial example, let $I \subset R$ be any ideal generated by a non-zero-divisor of R; the Rees ring $R[It]$ is then isomorphic to a polynomial ring in one indeterminate over R and is therefore Cohen-Macaulay if and only if R is.

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Nothing else is presently known concerning the existence of arithmetic Macaulayfications. We conjecture the existence of an arithmetic Macaulayfication, at least for local excellent domains.

We stress that there is a significant difference between the existence of a Macaulayfication and the existence of an arithmetic Macaulayfication. For example, let R be a domain essentially of finite type over the complex numbers. One might naively hope that if we take an ideal I such that Proj $R[It]$ is a desingularization of Spec R, then I or some power of I gives an arithmetic Macaulayfication. However, recent work of Lipman [L] shows that $R[It]$ cannot be Cohen-Macaulay unless R is a rational singularity. In particular, if $R[It]$ is an arithmetic Macaulayfication of R, then Proj $R[It]$ cannot be a desingularization unless R is already Cohen-Macaulay! Thus the problem of finding an arithmetic Macaulayfication is quite different from the problem of resolution of singularities.

Lipman's result requires use of the vanishing theorem of Grauert and Riemenschneider [GrR]. In its dual form, this vanishing theorem states that for $i <$ dimension R, $H^i_Y(\mathcal{O}_X) = 0$, where X is a desingularization Proj $R[It]$ of Spec R and Y is the closed fiber. It is worth noting (see [HS]) that, quite generally, this vanishing theorem holds on an arbitrary Cohen-Macaulay blow-up $X =$ Proj $R[It]$ if and only if for some power of I, which we relabel as I, the ideal $IR[It]$ is Cohen-Macaulay as an $R[It]$ module. It seems quite natural to conjecture the existence of such an ideal I for excellent local domains. When such an I exists, we will say that R has a *coMacaulayfication*. In this language, Paul Roberts [Ro1] proved that if R has a coMacaulayfication, then the Improved New Intersection Conjecture is true for R. Despite the progress of Roberts [Ro2] on the Intersection Conjectures, this notorious problem is open in mixed characteristic. Although the conjectures of the existence of an arithmetic Macaulayfication and a coMacaulayfication seem very similar, there is a significant sense in which they are dichotomous. In fact, both can hold for the same ideal only if R is already Cohen-Macaulay. More profoundly, the existence of an arithmetic Macaulayfication has to do with the local cohomology of the associated graded ring being zero except in degrees -1 (see (4.2.1)), while a coMacaulayfication has to do with the pieces of degree -1 being zero (see [HS]).

In this paper our main result is the existence of an arithmetic Macaulayfication for rings of characteristic p having an isolated singularity (this last condition is weakened in our last section to include rings which have an isolated non-F-rational point).[1] Our assumption that R has an isolated singularity implies that R has finite local cohomology, so that our result is already implied by the results of [Br2] and [GY]. What is new in this paper is the construction of the ideal I giving the arithmetic Macaulayfication, and the ensuing proof. Both the construction and the proof use the theory of tight closure (see [HH1–4]). Our proof is very simple in concept, although some details still require work. What makes tight closure so useful in the context of arithmetic

[1] Since this paper was written, related work has been done by Aberbach [Ab3], Brodmann [Br3], and Kurano [K].

Macaulayfications is a property which is called "colon-capturing." Roughly speaking, colon capturing enables us to manipulate ideals generated by monomials in a system of parameters for R as if the system of parameters were actually a regular sequence. We make heavy use of the following form of colon capturing: if $m_1, \ldots, m_k, n$ are monomials in a system of parameters for an equidimensional excellent local ring R, then the colon ideal $(m_1, \ldots, m_k){:}_R\, n$ is contained in the tight closure of the answer one would get formally for this colon ideal if the system of parameters were indeterminates in a polynomial ring.

We hope that our results will not only be of independent interest to those studying tight closure, but also furnish ample proof that this theory gives insight into the existence of arithmetic Macaulayfications. In particular, we hope that the theory of tight closure can be used to prove these results for more general classes of rings, and also shed light upon the existence of a coMacaulayfication.

We close the introduction by giving a short summary of the structure of this paper. Section 2 contains the background material on tight closure, including a discussion and precise statement of colon capturing. Section 3 gives a proof of the Main Lemma 3.2, which is really the crucial point in the entire argument. It deals with the tight closure of certain ideals generated by monomials in parameters which are test elements. Section 4 is devoted to the proof of the main theorem on the existence of arithmetic Macaulayfications, and we close in Section 5 by extending the main theorem to excellent local rings with an isolated non F-rational point. This section is of independent interest in the theory of tight closure.

2. Background on tight closure

In its main setting, tight closure is an operation performed on ideals in a commutative ring of prime characteristic. There are also various ways to define tight closure for rings containing fields of arbitrary characteristic [HH4]. Most of the results of this paper would be valid if several basic theorems on tight closure were proved for rings containing fields of characteristic zero. In particular the existence of sufficiently many test elements is crucial for the applications of our main theorem. This theory is being worked out in [HH4]. For simplicity we prove our results only in positive characteristic. In this section, we recall the definition of tight closure in characteristic $p > 0$ and a few elementary facts.

Let R denote a commutative, Noetherian ring of prime characteristic p, and let R^0 denote the subset of elements not in any minimal prime of R.

DEFINITION 2.1. *The tight closure of an ideal $I \subset R$, denoted I^*, consists of all elements $z \in R$ for which there exists $c \in R^0$ such that*

$$cz^q \in I^{[q]} \quad \textit{for all } q = p^e \gg 0$$

where $I^{[q]}$ denotes the ideal generated by the q^{th} powers of all the elements (equivalently, the generators) of I.

The tight closure I^* of I is easily seen to be an ideal of R containing I. We say that I is *tightly closed* if $I^* = I$.

The reader is referred to [HH1] for more about tight closure.

The element c in the preceding definition used to "test" whether z is contained in I^* is allowed to depend on I and even on the element z we are testing for inclusion in I^*. However, much of the usefulness of tight closure arises from the fact that in fact, there exist certain elements c which can be used in *any* tight closure test. We say that $c \in R^0$ is a *test element for I* if

$$z \in I^* \quad \text{if and only if} \quad cz^q \in I^{[q]} \quad \text{for all } q = p^0, p^1, p^2, \ldots.$$

An element $c \in R^0$ is a *test element* if it is a test element for all ideals $I \subset R$. In Section 5 we will discuss other types of test elements.

Fortunately, test elements are known to exist. In particular, in Section 6 of [HH2], it is shown that:

THEOREM 2.2. *If R is a reduced excellent local ring, and $c \in R^0$ is any element such that R_c is regular, then some power of c is a test element for R.*

The assumption that R_c is regular can be weakened. In Section 5 we will prove an analog of this theorem for a special class of test elements. Our analogue involves the concept of F-rationality. A local Noetherian ring of characteristic p is *F-rational* if every parameter ideal is tightly closed. This concept is closely related to the concept of rational singularities (or more generally to pseudorational rings). We refer the reader to [LT], [S2], [Fe], and [W] for information concerning the relationship between F-rationality and rational singularities.

Colon Capturing A key feature of tight closure is, roughly speaking, that it "captures the failure of a ring to be Cohen-Macaulay." As an example of the most basic form of "colon capturing," let $x_1, \ldots, x_d$ be any system of parameters for a complete local domain R. Then for all $i = 0, 1, \ldots, d-1$, we have

$$(x_1, x_2, \ldots, x_i) \colon (x_{i+1}) \subset (x_1, x_2, \ldots, x_i)^* \colon (x_{i+1}) \subset (x_1, x_2, \ldots, x_i)^*.$$

A more general statement of colon-capturing follows [HH1, (7.6) and (7.14)].

THEOREM 2.3. *Let (R, m) be an equidimensional excellent local ring. Let $x_1, \ldots, x_d$ be any system of parameters for R and let I and J be any two ideals of the (polynomial) subring $A = \frac{\mathbb{Z}}{p\mathbb{Z}}[x_1, \ldots, x_d]$ of R generated by monomials in the variables. Then*

$$(IR)^* :_R JR \subset ((I :_A J)R)^*.$$

$$(IR)^* \cap (JR)^* \subset ((I \cap J)R)^*.$$

We have slightly changed the assumptions of Theorem 7.6 in [HH1]. In [HH1, (7.6)] there is an assumption that the ring be a homomorphic image of a Cohen-Macaulay ring. This condition ensures that tight closure "captures the colon" for parameter ideals. However, since it is now known that excellent local reduced rings have test elements, this difficulty can be surmounted by passing to the completion of R. For details, see [HH2, (6.28)].

The point of Theorem 2.3 is that since I and J are generated by monomials in $x_1, \dots, x_d$, the intersection and colon are easily computed as

$$(I \cap J)R = \{\, \text{l.c.m. } (m_i, m_j') \,\}R$$

$$\begin{aligned}(I :_A \ (m_1', \dots m_k'))R &= (\cap_{j=1}^{k}(I :_A \ m_j'))R \\ &= \left(\cap_{j=1}^{k}\left\{\frac{m_i}{\text{g.c.d. } (m_i, m_j')},\ i = 1, 2, \dots, l\right\}\right)R\end{aligned}$$

where m_i runs through a monomial generating set for I, and m_j' runs through a monomial generating set for J (g.c.d. means greatest common divisor and l.c.m. means least common multiple). We refer to $(I :_A J)R$ as the "formal colon ideal" and write $(I :_f \ J)$ to indicate this ideal of R whenever I and J are ideals generated by monomials in parameters for R. Similar conventions will apply to intersections. The reader is referred to [HH1] and [HH3] for more about colon capturing.

REMARK 2.4. Throughout the ensuing text will be many manipulations with "colon capturing." Since many of the more routine computations will be left to the reader, further words on the philosophy of colon capturing may be warranted. Suppose that J is an ideal generated by monomials $m_1, m_2, \dots, m_k$ in the parameters $x_1, \dots, x_d$ for R. Let m be some other monomial. Then the colon ideals $(J : m)$ or $(J^* : m)$ will always be contained in the tight closure of the formal colon ideal $(J :_f \ m)$, which is the ideal generated by the monomials $\frac{m_i}{g.c.d.\,(m_i, m)}$ for $i = 1, 2, \dots k$. Because this formal colon ideal is generated by monomials in $x_1, \dots, x_d$, at the very least we know that

$$(2.4.1.) \qquad (m_1, m_2, \dots, m_k)^*\colon m \subset (x_1, \dots, x_d)^*$$

unless it is the unit ideal, which happens if and only if some m_i formally divides m. This idea will be used again and again.

As an example, we apply this idea to reprove a result from [HH3]: if $I = (x_1, \dots, x_d)$ is a parameter ideal then

$$(2.4.2.) \qquad I^{N-1} \cap (I^N)^* = I^{N-1}I^*.$$

We need only check that if $z = \sum u_\alpha m_\alpha \in (I^N)^*$, where m_α is a monomial of degree $N-1$ in $x_1, \dots, x_d$, then $u_\alpha \in I^*$. Focusing on a particular u_α, we see that

$$u_\alpha m_\alpha \in (I^N)^* + (m_\beta \mid m_\beta \text{ monomial of degree } N-1 \text{ other than } m_\alpha).$$

Letting J denote the monomial ideal above on the right, we see

$$u_\alpha \in J^*:_f m_\alpha \subset I^*$$

since $m \notin J$. This completes the proof of (2.4.2), a typical example of the sort of routine colon capturing arguments left to the reader.

3. Tight closure of ideals of monomials in parameters

In this section we demonstrate the usefulness of "colon capturing" for re-writing tight closures of ideals generated by monomials in parameters. The following lemmas, especially Lemma 3.2, are the key ingredients in the proof of arithmetic Macaulayfication appearing in Section 4. The first lemma is an easy application of colon capturing.

LEMMA 3.1. *Let $x_1, \ldots, x_d$ be parameters in an equidimensional reduced excellent local ring (R, m), where $d \geq 1$, and let I denote the ideal they generate. Assume that each x_i is a test element. Then*:

(*i*) $(I^N)^* = I^{N-1}I^*$.
(*ii*) $(x_1, \ldots, x_k)^* \cap I \subset (x_1, \ldots, x_k)$.

Proof. (*i*) Induce on the length d of the parameter system. Assume that $d = 1$, and choose $z \in ((x_1)^N)^*$. Since x_1 is a test element it follows that $x_1 z \in (x_1^N)$. Then there is an element v such that $x_1(z - x_1^{N-1}v) = 0$. As x_1 is a parameter and R is reduced, it follows that $z = x_1^{N-1}v \in ((x_1)^N)^*$, and by colon-capturing it then follows that $v \in (x_1)^*$. Thus $z \in (x_1^{N-1})(x_1)^*$. Assume that $d \geq 2$. Let $z \in (I^N)^*$. Since each x_i is a test element, we obtain that $x_d z \in I^N$. Hence there exists an element $u \in I^{N-1}$ such that $x_d(z-u) \in J^N$, where $J = (x_1, \ldots, x_{d-1})$. Using colon capturing we see that $z - u \in (J^N)^*$. By induction $(J^N)^* = J^{N-1}J^*$. In particular it follows that $z \in I^{N-1} \cap (I^N)^* = I^{N-1}I^*$, the last equality following from (2.4.2).

(*ii*) Induce on $d - k$. The statement is obvious when $d - k = 0$. Assume that u is in $(x_1, \ldots, x_k)^*$ but not $(x_1, \ldots, x_k)$. This assumption is unchanged by subtracting off elements of $(x_1, \ldots, x_k)$; we will do this throughout the proof and repeatedly relabel the offending element u.

Write $u = \sum_{j=k+1}^d x_j u_j \in (x_1, \ldots, x_k)^*$. Then

$$u' = u - x_{k+1}u_{k+1} \in (x_1, \ldots, x_k, x_{k+1})^* \cap I.$$

By the inductive assumption, we know that $u' \in (x_1, \ldots, x_k, x_{k+1})$. This says that, after subtracting off an element in the desired ideal $(x_1, \ldots, x_k)$, we may assume that $u = x_{k+1}v$ for some $v \in R$. Colon capturing now forces $v \in (x_1, \ldots, x_k)^*$, and since x_{k+1} is a test element, it follows that $u = x_{k+1}v \in (x_1, \ldots, x_k)$, as needed. This completes the proof. □

MAIN LEMMA 3.2. *Let $x_1, \dots, x_d$ be parameters in an equidimensional excellent local ring R and let I denote the ideal they generate. Assume that each x_i is a test element. Then for any integers $N, t_i \geq 1$ and any $k \leq d$:*

$$(I^N + (x_1^{t_1}, x_2^{t_2}, \dots, x_k^{t_k}))^* = (I^N)^* + \sum (x_{i_1}^{t_{i_1}-1} x_{i_2}^{t_{i_2}-1} \cdots x_{i_j}^{t_{i_j}-1})(x_{i_1}, x_{i_2}, \dots, x_{i_j})^*,$$

where the sum here ranges over all subsets $\{i_1, i_2, \dots, i_j\}$ of $\{1, 2, \dots, k\}$.

REMARK. Actually, it is not necessary to assume that the $x_1, \dots, x_d$ are test elements. The proof only uses that they are test elements for ideals generated by monomials in parameters. In Section 5, we will show (under weak hypotheses) that any $c \in R^0$ such that R_c is F-rational has a power which will be a test element for ideals generated by monomials in parameters.

Proof. Throughout this proof, I_i denotes the ideal generated by all the $x_1, \dots, x_d$ except x_i. The proof proceeds by induction on $\sum t_i$.

Base case. If each $t_i = 1$, we need to show that

$$((x_1, \dots, x_d)^N + (x_1, \dots, x_k))^* = ((x_1, \dots, x_d)^N)^* + (x_1, \dots, x_k)^*.$$

Induct on $d - k$. If $k = d$, there is nothing to prove. Assume that $k < d$ and take any $z \in ((x_1, \dots, x_d)^N + (x_1, \dots, x_k))^*$. Since x_d is a test element, we see that $x_d z \in ((x_1, \dots, x_d)^N + (x_1, \dots, x_k))$. It follows that

$$x_d(z - u) \in (I_d)^N + (x_1, \dots, x_k),$$

for some $u \in I^{N-1}$. By using colon capturing (*cf.* Remark 2.4) and the inductive hypothesis (on $d - k$), we see that

$$z - u \in ((I_d)^N + (x_1, \dots, x_k))^* = (I_d^N)^* + (x_1, \dots, x_k)^*.$$

This shows that $z - u$ is in the desired ideal, whence we may assume that $z = u \in (x_{k+1}, x_{k+2}, \dots, x_d)^{N-1}$. Write $u = \sum u_\alpha m_\alpha$ where m_α is a monomial of degree $N - 1$ in $x_{k+1}, x_{k+2}, \dots, x_d$. A crucial observation is that, again using colon capturing as in Remark 2, one easily checks that $u_\alpha \in (x_1, \dots, x_d)^*$. We conclude that $z \in (I^N)^* + (x_1, \dots, x_k)^*$ as needed to complete the initial step in the induction.

Now assume that some $t_i \geq 2$ and the lemma holds whenever $\sum t_i$ is smaller. Change notation so as to assume without loss of generality that $t_1 \geq 2$. Say that $z \in ((x_1, \dots, x_d)^N + (x_1^{t_1}, x_2^{t_2}, \dots, x_k^{t_k}))^*$. Since x_1 is a test element, this implies that

$$x_1 z \in (x_1, \dots, x_d)^N + (x_1^{t_1}, x_2^{t_2}, \dots, x_k^{t_k}).$$

This inclusion can be re-written as

$$x_1(z - u - x_1^{t_1-1}v) \in I_1^N + (x_2^{t_2}, \dots, x_k^{t_k}),$$

where $u \in I^{N-1}$ and v is just some element of R. Using colon capturing, we see that

$$(z - u - x_1^{t_1-1}v) \in (I_1^N + (x_2^{t_2}, \dots, x_k^{t_k}))^*,$$

which, by the inductive hypothesis, is in the desired ideal. We may therefore assume that $z = u + x_1^{t_1-1}v$ for $u = \sum u_\alpha m_\alpha$, where m_α is a monomial of degree $N-1$ in $x_1, \dots, x_d$; without loss of generality, we can assume that the exponent of x_i is strictly less than t_i, and the exponent of x_1 is strictly less than $t_1 - 1$. Again, it is an easy consequence of colon capturing that $u_\alpha \in (x_1, \dots, x_d)^*$, so that $u \in (I^N)^*$, and hence is in the desired ideal. We can therefore assume that

$$z = x^{t_1-1}v \in ((x_1, \dots, x_d)^N + (x_1^{t_1}, x_2^{t_2}, \dots, x_k^{t_k}))^*.$$

By colon capturing, we have

$$v \in ((x_1, \dots, x_d)^{N-t_1+1} + (x_1, x_2^{t_2}, \dots, x_k^{t_k}))^*,$$

which we may expand using the inductive hypothesis in order to conclude that $x_1^{t_1-1}v$ is in the desired ideal. This completes the proof. □

COROLLARY 3.3. *Let $x_1, \dots, x_d$ be parameters in an equidimensional excellent local ring R and let I denote the ideal they generate. Assume that each x_i is a test element. For arbitrary integers N and t, with $N \geq t \geq 1$,*

$$(I^N)^* \cap (x_1^t, x_2^t, \dots, x_k^t) = (I^{N-t})^*(x_1^t, x_2^t, \dots, x_k^t).$$

for any $k \leq d$.

Proof. We use induction on k. The statement follows immediately from colon capturing when $k = 1$. Write $w = \sum_{1 \leq i \leq k} r_i x_i^t \in (I^N)^*$. Then

$$r_k \in ((I^N)^* + (x_1^t, x_2^t, \dots x_{k-1}^t))\colon x_k^t.$$

By colon capturing this latter ideal is contained in $(I^{N-t} + (x_1^t, x_2^t, \dots x_{k-1}^t))^*$. An application of Lemma 3.2 therefore guarantees that

$$r_k \in (I^{N-t})^* + \sum (x_{i_1}^{t-1} x_{i_2}^{t-1} \cdots x_{i_j}^{t-1})(x_{i_1}, x_{i_2}, \dots, x_{i_j})^*,$$

where the sum here ranges over all subsets $\{i_1, i_2, \dots, i_j\}$ of $\{1, 2, \dots, k-1\}$. Because x_k is a test element, we see that $r_k x_k^t = x_k^t b_k + s$, where $b_k \in (I^{N-t})^*$ and $s \in (x_1^t, \dots, x_{k-1}^t)$. Thus $w - r_k x_k^t \in (x_1^t, \dots, x_{k-1}^t) \cap (I^N)^*$ which is equal to $(I^{N-t})^*(x_1^t, \dots, x_{k-1}^t)$ by inductive assumption. The corollary follows. □

We will also need a stronger version of the Main Lemma 3.2 with more restrictive hypotheses.

LEMMA 3.4. *Let $x_1, \ldots, x_d$ be parameters in an equidimensional excellent local ring R and let I denote the ideal they generate. Assume that each x_i is a test element. Then for any integers $N, t_i \geq 1, k \leq d$ and for any integer a with $\sum_{1 \leq i \leq k}(t_i - 1) + 2 \leq a \leq N$:*

$$(I^N + (x_1^{t_1}, x_2^{t_2}, \ldots, x_k^{t_k}))^* \cap (I^a)^* = (I^N)^* + \sum_{1 \leq i \leq k} x_i^{t_i}(I^{a-t_i})^*.$$

Proof. The right hand side is clearly in the left hand side. The reverse containment is proved by induction on the sum of the t_i. If all $t_i = 1$, then the Main Lemma proves that

$$((x_1, \ldots, x_d)^N + (x_1, \ldots, x_k))^* = ((x_1, \ldots, x_d)^N)^* + (x_1, \ldots, x_k)^*.$$

When we intersect with $(I^a)^*$, with $a \geq 2$, we can then apply Lemma 3.1 and Corollary 3.3 to get the desired conclusion. Without loss of generality, assume that $t_1 \geq 2$. The induction gives us

$$(I^N + (x_1^{t_1-1}, x_2^{t_2}, \ldots, x_k^{t_k}))^* \cap (I^a)^* = (I^N)^* + \sum_{2 \leq i \leq k} x_i^{t_i}(I^{a-t_i})^* + x_1^{t_1-1}(I^{a-t_1+1})^*. \tag{3.4.1}$$

Let $u \in (I^N + (x_1^{t_1}, x_2^{t_2}, \ldots, x_k^{t_k}))^* \cap (I^a)^*$. By using (3.4.1) we can write $u = v + x_1^{t_1-1}w$, where $v \in (I^N)^* + \sum_{2 \leq i \leq k} x_i^{t_i}(I^{a-t_i})^*$ and $w \in (I^{a-t_1+1})^*$. By Lemma 2.3,

$$\begin{aligned} w &\in (((I^N + (x_1^{t_1}, x_2^{t_2}, \ldots, x_k^{t_k}))^*) :_R x_1^{t_1-1}) \cap (I^{a-t_1+1})^* \\ &\subseteq (I^{N-t_1+1} + (x_1, x_2^{t_2}, \ldots, x_k^{t_k}))^* \cap (I^{a-t_1+1})^*. \end{aligned}$$

It then follows by induction that $w \in (I^{N-t_1+1})^* + \sum_{2 \leq i \leq k} x_i^{t_i}(I^{a-t_1+1-t_i})^* + x_1(I^{a-t_1})^*$. Multiplying by $x_1^{t_1-1}$ then gives the desired conclusion. □

4. Arithmetic Macaulayfication

Given a local ring (R, m), when does there exist a filtration F: $R = I_0 \supset I_1 \supset I_2 \supset \cdots$ such that the corresponding Rees ring

$$R_F = R \oplus I_1 t \oplus I_2 t^2 \oplus \cdots \tag{4.1}$$

is Cohen-Macaulay? Such a ring is called an *arithmetic Macaulayfication* of R. In practice, one often requires that the filtration comes from the powers of a particular ideal I, in which case we write $R[It]$ instead of R_F. Included in the definition of 'Cohen-Macaulay' is the assumption that the ring be Noetherian. In this case, there will be an integer k such that $I_{kn} = (I_k)^n$ for all $n \geq 1$. The subring $R[I_k t^k]$ of R_F is a direct summand of R_F, and R_F is finite over this subring. It follows that (cf. [BH, (6.4.5)]) if R_F is Cohen-Macaulay, then so is $R[I_k t]$. Hence if there is an arithmetic Macaulayfication involving a filtration, there is also one where the filtration is given by powers of an ideal. So there is no loss of generality in considering only this case, although it may be more convenient to use the more general case.

Let $x_1, \ldots, x_d$ be a system of parameters for R and consider the filtration F: $R \supset (I^N)^* \supset (I^{2N})^* \supset (I^{3N})^* \supset \cdots$ where I is the ideal generated by $x_1, \ldots, x_d$ and N is a fixed integer. Because of the colon capturing properties of tight closure, one might expect the associated Rees ring R_F to have nice properties, maybe even be an arithmetic Macaulayfication for R. In fact, the next theorem shows that this is indeed the case with some additional assumptions on R.

THEOREM 4.1. *Let (R, m) be an excellent normal local ring of dimension d. Let $x_1, \ldots, x_d$ be any system of parameters such that each x_i is a test element. Then the Rees algebra*

$$R[Jt] = R \oplus J \oplus J^2 \oplus J^3 \oplus \cdots$$

is Cohen-Macaulay, where J is the ideal $((x_1, \ldots, x_d)^{d-2})^*$.

COROLLARY 4.2. *Let (R, m) be an excellent normal local domain of dimension d. Assume that R has an isolated non-F-rational point (e.g., R has an isolated singularity). Then there exists some integer T such that given any system of parameters $x_1, \ldots, x_d$ for R, the Rees ring $R[Jt]$ is Cohen-Macaulay where $J = ((x_1^t, x_2^t, \ldots, x_d^t)^{d-2})^*$ for any $t \geq T$.*

Proof of 4.2. For the case where R has an isolated singularity, the result follows from the theorem by taking T sufficiently large so that $(x_1^T, x_2^T, \ldots, x_d^T)$ is contained in the test ideal for R, which is known to be m-primary by Theorem 2.2. To complete the proof of the corollary in its full generality, we need to know that the ideal of test elements for all ideals generated by monomials in parameters is contained, up to radical, in the ideal defining the non-F-rational locus of R. We will prove this in Section 5. □

Before embarking on the proof of Theorem 4.1, we recall a criterion (see [L], [IT], and [Vi]) for the Cohen-Macaulayness of Rees algebras R_F. Let (R, m) be a Noetherian local ring of dimension d with filtration F determining the Rees ring R_F as in 4.1 above. Let G_F denote the associated graded algebra $G_F = R/I_1 \oplus I_1/I_2 \oplus I_2/I_3 \oplus \cdots$. Let $\tilde{m}$ denote the distinguished maximal ideal $m/I_1 \oplus I_1/I_2 \oplus \cdots$ of G. Then the Cohen-Macaulayness of the Rees ring R_F is equivalent to the following two conditions on the local cohomology modules for G:

(1) $H^d_{\tilde{m}}(G) = 0$ in all non-negative degrees.

(4.2.1.) (2) $H^i_{\tilde{m}}(G) = 0$ except in degree -1, for all $i < d$.

We will distinguish the (non-zero) homogeneous elements of degree a in G by writing $\tilde{r}$ for the element $r + I_a/I_{a+1}$ whenever $r \in I_a - I_{a+1}$. Note that if $\tilde{r}$ has degree a and $\tilde{s}$ has degree b, then their product $\tilde{r}\tilde{s}$ is either the degree $a+b$ element $\widetilde{rs}$ or it is zero, which happens if and only if $rs \in I_{a+b+1}$.

Let $\tilde{y}_1, \ldots, \tilde{y}_d$ be a homogeneous system of parameters for G. Local cohomology with support in the ideal generated by $\tilde{y}_1, \ldots, \tilde{y}_d$ may be computed as a limit of Koszul cohomology, and thus as the cohomology of the complex

$$(4.2.2.) \qquad 0 \longrightarrow G \xrightarrow{\delta_0} G_{\tilde{y}_1} \oplus \cdots \oplus G_{\tilde{y}_d} \xrightarrow{\delta_1} \cdots \xrightarrow{\delta_{d-1}} G_{\tilde{y}_1\tilde{y}_2\cdots\tilde{y}_d} \xrightarrow{\delta_d} 0$$

where the maps are all essentially just localization (up to sign) and the signs are assigned as in the Koszul (cohomology) complex: the first map sends $\tilde{r}$ to $(\frac{\tilde{r}}{1}, \frac{\tilde{r}}{1}, \ldots \frac{\tilde{r}}{1}) \in \oplus_i G_{\tilde{y}_i}$, and the subsequent maps are the same as in the standard Čech complex for the open cover of the punctured spectrum of G consisting of the affines where $\tilde{y}_i$ does not vanish. Because G is graded and the $\tilde{y}_i$'s are homogeneous, this complex is graded, inducing a natural grading on the local cohomology modules. More generally, of course, it is not necessary to assume that the $\tilde{y}_i$'s are a system of parameter or that d is the dimension of G: the cohomology of complex (4.2.2.) always yields the local cohomology of G with support in the ideal generated by the $\tilde{y}_i$'s. We will be applying this when the $\tilde{y}_i$'s are a partial system of parameters for G.

It will be convenient to re-write conditions (4.2.1.) in a different form. With notation as above, the Cohen-Macaulayness of the Rees ring R_F is equivalent to the following two conditions on the local cohomology modules for G:

$$(4.2.3.) \qquad \begin{array}{l} (1)\ H^d_{\tilde{m}}(G) = 0 \ \text{ in all non-negative degrees.} \\ (2)\ H^{k-1}_{(\tilde{y}_{i_1},\ldots,\tilde{y}_{i_k})}(G) = 0 \ \text{ except in degree } -1, \quad \text{for all } i < d, \\ \qquad \text{for all subsets } \{i_1, \ldots, i_k\} \subset \{1, \ldots, d\}. \end{array}$$

This equivalent form of (4.2.1.) follows immediately from the next fact.

LEMMA 4.3. *Let G be a $\mathbb{N}$-graded ring, and let $z_1, \ldots, z_d$ be any homogeneous elements of G of positive degree. Let S be any subset of $\mathbb{Z}$ that is bounded above. Suppose that for each subset $\{i_1, \ldots, i_k\} \subset \{1, 2, \ldots d\}$ of cardinality k, the module $H^{k-1}_{(z_{i_1},\ldots,z_{i_k})}(G)$ is zero in all degrees $\delta \notin S$. Then for all i, $H^i_n(G)$ is zero in degree $\delta \notin S$, where n is any ideal of G generated by more than i of the elements z_j.*

Proof of 4.3. There is a standard long exact sequence for local cohomology

$$(4.3.1.) \qquad \xrightarrow{\partial} H^{i-1}_{(z_{i_1},\ldots,z_{i_{k-1}})}(G_{z_{i_k}}) \longrightarrow H^i_{(z_{i_1},\ldots,z_{i_k})}(G) \longrightarrow H^i_{(z_{i_1},\ldots,z_{i_{k-1}})}(G) \xrightarrow{\partial}$$

where the unlabeled arrows are degree preserving maps. This complex arises by thinking of the functor Γ_E = "global sections supported on E" where E is the closed subscheme of Spec G given by the ideal $(z_{i_1}, \ldots, z_{i_{k-1}})$: the map induced by restricting this functor to the open set Spec $G_{z_{i_k}}$ (i.e., the open set where z_{i_k} does not vanish) has $H^0_{(z_{i_1},\dots,z_{i_k})}(G)$ as its kernel and extends to a long exact cokernel complex as above.

In particular, this long exact sequence is easily derived via the snake lemma (the maps ∂ are the connecting homomorphisms) from the short exact sequence induced by the inclusion of the complex of the form (4.2.2.) for $G_{z_{i_k}}$ on the elements $z_{i_1},\dots,z_{i_{k-1}}$ (with index shifted by -1) into the similar complex on the elements $z_{i_1},\dots,z_{i_k}$. The cokernel complex is easily checked to be a complex of the form (4.2.2.) for G on the elements $z_{i_1},\dots,z_{i_{k-1}}$.

Assuming the lemma fails to hold, choose k smallest possible such that n is generated by k of the z_i's, relabeled $z_1, z_2, \dots, z_k$, and such that $H^i_n(G)$ is non-zero in some degree $\delta \notin \mathcal{S}$ for some $i < k$. Because $H^{k-1}_n(G)$ is zero in degree δ, we may assume that $i < k-1$. Because of the minimality assumption on k, $H^i_{(z_1,z_2,\dots,z_{k-1})}(G)$ is zero in degree δ. The inductive assumption also guarantees that $H^{i-1}_{(z_1,z_2,\dots,z_{k-1})}(G)$ is zero in all degrees $\delta \notin \mathcal{S}$; in particular $H^{i-1}_{(z_1,z_2,\dots,z_{k-1})}(G)$ is zero in all sufficiently large degrees and thus $H^{i-1}_{(z_1,z_2,\dots,z_{k-1})}(G_{z_k}) = H^{i-1}_{(z_1,z_2,\dots,z_{k-1})}(G)\otimes G_{z_k}$ is the zero module. It follows from the long exact sequence (4.3.1.) that $H^i_n(G) = 0$ in all degrees $\delta \notin \mathcal{S}$. $\square$

We are now ready to prove the main theorem.

Proof of Theorem 4.1. Let $J = ((x_1,\dots,x_d)^{d-2})^*$ and fix the notation as in the preceding paragraphs, with respect to the filtration F: $R \supset J \supset J^2 \supset \dots$. Set $y_i = x_i^{d-2}$. The degree one elements $\tilde{y}_i$ for $i = 1, 2, \dots, d$ form a homogeneous system of parameters for G. We will show that $R[Jt]$ is Cohen-Macaulay by checking the criterion (4.2.3.) on the local cohomology modules $H^{k-1}_{\tilde{n}}(G)$, computed via the system of parameters $\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_d$.

In general, even if (R, m) is assumed only to be reduced, excellent and equidimensional and $x_1,\dots,x_d$ is an arbitrary system of parameters, the elements $\widetilde{x_i^N}$ are always non-zero-divisors on the associated graded ring with respect to the tight closure filtration for $I^N = (x_1,\dots,x_d)^N$: $R/(I^N)^* \oplus (I^N)^*/(I^{2N})^* \oplus (I^{2N})^*/(I^{3N})^* \dots$. For if $\widetilde{x_i^N}\tilde{z} = 0$ in G, where $\tilde{z}$ is assumed to be homogeneous of degree a, then since $\widetilde{x_i^N}$ is homogeneous of degree 1, we must have $x_i^N z \in ((I^N)^{a+2})^*$. But colon capturing then implies that $z \in ((x_1,\dots,x_d)^{N(a+1)})^*$, contrary to the fact that $\tilde{z}$ had degree a.

Applied to our situation, this says that each $\tilde{y}_i$ is a non-zero-divisor on G. Using Lemma 3.1, we see that for all integers N,

$$((x_1,\dots,x_d)^N)^* = (x_1,\dots,x_d)^{N-1}(x_1,\dots,x_d)^* = ((x_1,\dots,x_d)^*)^N;$$

in particular, we see that

$$J^N = (((x_1,\dots,x_d)^{(d-2)})^*)^N = (J^N)^* \tag{4.3.2.}$$

whence that each $\tilde{y}_i$ is a non-zero-divisor on G follows. Note that the local cohomology module $H^0_{\tilde{n}}(G)$ is therefore the zero module, when $\tilde{n}$ is an ideal generated by any subset of the $\tilde{y}_i$'s. This takes care of condition (4.2.1.) when $i = 0$, or alternatively, condition (4.2.3.) when $k = 1$.

We next show that $H^1_{\tilde{n}}(G)$ is zero using only the hypotheses of normality and excellence, where $\tilde{n}$ is generated by any subset of $\{\tilde{y}_1, \dots, \tilde{y}_d\}$, provided it has cardinality k at least two. Let $\eta = (\frac{\tilde{r}_1}{\tilde{y}_1^t}, \frac{\tilde{r}_2}{\tilde{y}_2^t}, \dots, \frac{\tilde{r}_k}{\tilde{y}_k^t}) \in \oplus_{i=1}^k G_{\tilde{y}_i}$ represent an element of the cohomology module $H^1_{\tilde{n}}(G)$. Without loss of generality, we may assume that each $\tilde{r}_i$ is degree a, so that η is homogeneous of degree $a - t$. That η is in the kernel of δ_1 is equivalent to

$$\frac{\tilde{r}_i}{\tilde{y}_i^t} - \frac{\tilde{r}_j}{\tilde{y}_j^t} = 0 \quad \text{in } G$$

for all $i, j = 1, 2, \dots, k$. Because each y_i is a non-zero-divisor, this is equivalent to

$$\tilde{r}_i \tilde{y}_j^t - \tilde{r}_j \tilde{y}_i^t = 0 \quad \text{in } G, \tag{4.3.3.}$$

which is in turn equivalent to

$$r_i y_j^t - r_j y_i^t \in J^{a+t+1} \quad \text{in } R.$$

Using Lemma 3.2, we see that then

$$r_i \in (J^{a+t+1} + (y_i^t)) \colon y_j^t \subset (J^{a+1})^* + y_i^{t-1}(y_i)^*.$$

The normality of R ensures that principal ideals are tightly closed, whence $r_i \in J^{a+1} + (y_i^t)$. Thus each $\tilde{r}_i \in (\tilde{y}_i^t)G$, so we may assume that η has the form $(\frac{\tilde{s}_1}{1}, \frac{\tilde{s}_2}{1}, \dots, \frac{\tilde{s}_k}{1})$. But then because η is in the kernel of δ_1, we conclude, as in (4.3.3.) above, that $\tilde{s}_i = \tilde{s}_j = \tilde{s}$ for all i, j. This implies that $\eta = \delta_0(\tilde{s})$, whence η represents the zero element in $H^1_{\tilde{n}}(G)$. Since η was an arbitrary homogeneous element, we conclude that $H^1_{\tilde{n}}(G) = 0$.

We now show that $H^d_{\tilde{m}}(G)$ is zero in all non-negative degrees. Since $H^d_{\tilde{m}}(G)$ is the cokernel of the last non-trivial map in the complex (4.2.2.), each homogeneous element is represented by

$$\eta = \frac{\tilde{z}}{(\tilde{y}_1 \tilde{y}_2 \cdots \tilde{y}_d)^t}$$

where the degree of η is equal to $a - dt$ for $a =$ degree of $\tilde{z}$. We need to show that if $a \geq dt$, then $\eta = 0$.

It is easy to see that in general, if an ideal K is generated by d elements $z_1, \dots, z_d$, and $b \geq ds - d + 1$, then $K^b = (z_1^s, \dots, z_d^s)K^{b-s}$. Hence provided $a(d-2) - 1 \geq d(d-2)t - d + 1$ (which holds if $a \geq dt$), we have

$$I^{a(d-2)-1}(I)^* = (x_1^{(d-2)t}, \dots, x_d^{(d-2)t}) I^{a(d-2)-(d-2)t-1}(I)^*,$$

where I is the ideal $(x_1, \dots, x_d)R$. Now let $z \in (J^a)^*$, where $a \geq dt$. This says that $z \in (I^{a(d-2)})^* = I^{a(d-2)-1}(I)^*$, whence $z \in (y_1^t, \dots, y_d^t)J^{a-t}$. Thus we can write any degree a element $\tilde{z} \in G$ as $\tilde{z} = \tilde{r}_1\tilde{y}_1^t + \tilde{r}_2\tilde{y}_2^t + \cdots + \tilde{r}_d\tilde{y}_d^t$. This shows that η represents zero in the cokernel of δ_d of (4.2.2.), and hence in $H^d_{\tilde{m}}(G)$. This completes the proof that $H^d_{\tilde{m}}(G)$ is zero in non-negative degrees.

The proof is now complete when R has dimension two or less, so we assume that $d \geq 3$. Take any $k \geq 3$ of the elements from the set $\{\tilde{y}_1, \ldots, \tilde{y}_d\}$, which for convenience we re-label as $\tilde{y}_1, \ldots, \tilde{y}_k$. The ideal they generate will be denoted $\tilde{n}$. We need to show that, as in condition (4.2.3.), the module $H^{k-1}_{\tilde{n}}(G)$ is zero in all degrees except possibly degree -1.

Suppose that

$$\eta = \left[\frac{\tilde{y}_1^t\tilde{r}_1}{(\tilde{y}_1\tilde{y}_2\cdots\tilde{y}_k)^t}, \frac{\tilde{y}_2^t\tilde{r}_2}{(\tilde{y}_1\tilde{y}_2\cdots\tilde{y}_k)^t}, \ldots, \frac{\tilde{y}_k^t\tilde{r}_k}{(\tilde{y}_1\tilde{y}_2\cdots\tilde{y}_k)^t}\right] \in \oplus_{i=1}^k G_{\tilde{y}_1\cdots\hat{\tilde{y}}_i\cdots\tilde{y}_k}$$

represents a non-zero element of $H^{k-1}_{\tilde{n}}(G)$ of degree $\neq -1$, with notation as in complex (4.2.2.). We make the following inductive assumption: the representative cocycle η is chosen so that it has as many zero components $\frac{\tilde{y}_i^t\tilde{r}_i}{(\tilde{y}_1\tilde{y}_2\cdots\tilde{y}_k)^t}$ as possible. Relabel so that $\tilde{r}_{h+1} = \tilde{r}_{h+2} = \cdots = \tilde{r}_k = 0$ with h minimal (if $h = k$, this condition is vacuous).

Without loss of generality, we may assume that each $\tilde{r}_i$ is homogeneous of degree a, so that η has degree $a - (k-1)t$. Because $\eta \in$ kernel δ_{k-1}, we know that

$$\frac{\tilde{y}_1^t\tilde{r}_1 + \tilde{y}_2^t\tilde{r}_2 + \cdots + \tilde{y}_h^t\tilde{r}_h}{(\tilde{y}_1\tilde{y}_2\cdots\tilde{y}_k)^t} = 0 \text{ in } G_{\tilde{y}_1\tilde{y}_2\cdots\tilde{y}_k}.$$

Since the $\tilde{y}_i$'s are all non-zero-divisors, we know that $\tilde{y}_1^t\tilde{r}_1 + \tilde{y}_2^t\tilde{r}_2 + \cdots + \tilde{y}_h^t\tilde{r}_h = 0$ in G, which we may interpret, equivalently, as

$$\text{(4.3.4.)} \qquad y_1^tr_1 + y_2^tr_2 + \cdots + y_h^tr_h \in J^{a+t+1} \text{ in } R.$$

Rearranging and applying colon capturing, we see that

$$r_h \in ((J^{a+t+1} + (y_1^t, \ldots, y_{h-1}^t)) \colon (y_h^t)) \cap J^a \subset (J^{a+1} + (y_1^t, \ldots, y_{h-1}^t))^* \cap J^a;$$

in terms of the x_i and $I = (x_1, \ldots, x_d)$, we have

$$\text{(4.3.5.)} \qquad r_h \in (I^{(d-2)(a+1)} + (x_1^{(d-2)t}, \ldots, x_{h-1}^{(d-2)t}))^* \cap (I^{a(d-2)})^*.$$

Our goal is to show that $\eta = 0$, provided its degree is not -1. We first treat the case where the degree of η is non-negative. In this case, $a - (k-1)t \geq 0$, so that $a(d-2) \geq (k-1)t(d-2) \geq (d-2)t(k-1) - (k-1) + 2$ (because $k \geq 3$). Applying Lemma 3.4, we conclude that $r_h \in (I^{(d-2)(a+1)})^* + \sum_{j<h} x_j^{(d-2)t}(I^{(d-2)(a-t)})^*$, which implies that $\tilde{r}_h \in (\tilde{y}_1^t, \ldots, \tilde{y}_{h-1}^t)G$.

Write $\tilde{r}_h = \tilde{a}_1\tilde{y}_1 + \cdots + \tilde{a}_{h-1}\tilde{y}_{h-1}$, for some elements $\tilde{a}_i \in G$. One easily checks that there exist boundaries (elements in the image of δ_{k-2}) of the form

$$\mu = \left[\frac{\tilde{y}_1^t\tilde{r}_1'}{(\tilde{y}_1\tilde{y}_2\cdots\tilde{y}_k)^t}, \frac{\tilde{y}_2^t\tilde{r}_2'}{(\tilde{y}_1\tilde{y}_2\cdots\tilde{y}_k)^t}, \ldots, \underbrace{\frac{\tilde{y}_h^t(\tilde{a}_1\tilde{y}_1^t + \cdots + \tilde{a}_{h-1}\tilde{y}_{h-1}^t)}{(\tilde{y}_1\cdots\tilde{y}_k)^t}}_{(h)^{th}\text{slot}}, \right.$$
$$\left. 0,\ 0, \ldots, 0 \right]$$

for some choice of $\tilde{r}_i'$. Subtracting μ from η, we get a different cocycle representing the same cohomology class, contradicting the minimality of h. This shows that $H_{\tilde{n}}^{k-1}(G)$ is zero in non-negative degree.

We now treat the case where the degree of η is at most -2. Applying the Main Lemma to (4.3.5.) above, we see that

$$\text{(4.3.6.)} \qquad r_h \in (J^{a+1})^* + \sum (x_{i_1}x_{i_2}\cdots x_{i_j})^{t(d-2)-1}(x_{i_1}, x_{i_2}, \ldots, x_{i_j})^*,$$

where the sum ranges over all subsets $\{i_1, i_2, \ldots, i_j\}$ of $\{1, \ldots, h-1\}$. In fact, (4.3.6.) can be refined as follows: for $h < k$, we may assume that $r_h \in J^{a+1} + (y_1^t, \ldots, y_{h-1}^t)$; whereas for $h = k$, we may assume that

$$r_k - (x_1x_2\cdots x_{k-1})^{t(d-2)-1}u \in J^{a+1} + (y_1^t, \ldots, y_{k-1}^t),$$

for some $u \in (x_1, x_2, \ldots x_{k-1})^*$. The reason is that (multiplying numerators and denominators of the components of η by $\tilde{y}_1\cdots\tilde{y}_k$) we may rewrite η as

$$\left[\frac{\tilde{y}_1^{t+1}\tilde{s}_1}{(\tilde{y}_1\tilde{y}_2\cdots\tilde{y}_k)^{t+1}}, \frac{\tilde{y}_2^{t+1}\tilde{s}_2}{(\tilde{y}_1\tilde{y}_2\cdots\tilde{y}_k)^{t+1}}, \ldots, \frac{\tilde{y}_k^{t+1}\tilde{s}_k}{(\tilde{y}_1\tilde{y}_2\cdots\tilde{y}_k)^{t+1}} \right] \in \oplus_{i=1}^k G_{\tilde{y}_1\cdots\hat{\tilde{y}}_i\cdots\tilde{y}_k}$$

where each $s_i = y_1\cdots\hat{y}_i\cdots y_k r_i = (x_1\cdots\hat{x}_i\cdots x_k)^{d-2}r_i$. Since each x_j is a test element, we see that when $j < k$ and N and M are any integers,

$$(x_1\cdots\hat{x}_i\cdots x_k)^N(x_{i_1}x_{i_2}\cdots x_{i_j})^{M-1}(x_{i_1}, x_{i_2}, \ldots, x_{i_j})^*$$
$$\underbrace{\subset}_{\text{re-group}} \underbrace{(x_{l_1}x_{l_2}\cdots x_{l_h})^N}_{c}(x_{i_1}x_{i_2}\cdots x_{i_j})^{N+M-1}(x_{i_1}, x_{i_2}, \ldots, x_{i_j})^*$$
$$\underbrace{\subset}_{c\text{ is a test element}} (x_{i_1}^{N+M}, x_{i_2}^{N+M}, \ldots, x_{i_j}^{N+M})$$

where $\{i_1, \ldots, i_j\}$ is a *proper* subset of $\{1, \ldots, \hat{i}, \ldots, k\}$ and $\{l_1, \ldots, l_h\}$ is its (non-empty) complement. In particular, multiplying (4.3.6.) by $y_1 \cdots \hat{y}_h \cdots y_k$, we see that the element s_h is in the ideal

$$(J^{a+k})^* + (x_1^{(d-2)(t+1)}, \ldots, x_{h-1}^{(d-2)(t+1)})$$

in the event that $h < k$; similarly, when $h = k$, we see that s_h is in

$$(J^{a+k})^* + (x_1^{(d-2)(t+1)}, \ldots, x_{k-1}^{(d-2)(t+1)}) + (x_1 \cdots x_{k-1})^{(d-2)(t+1)-1}(x_1, \ldots, x_k)^*.$$

Noting that $(J^N)^* = J^N$ for all integers N, and changing notation so as to replace s_h by r_h and $t+1$ by t (and implicitly changing a also), we conclude the verification of this refinement. In other words, in the notation for the cocycle η, we may assume that $r_h \in J^{a+1} + (y_1^t, \ldots, y_{h-1}^t) + (x_1 x_2 \cdots x_{k-1})^{t(d-2)-1}(x_1, x_2, \ldots, x_{k-1})^*$, whether or not $h = k$.

Furthermore, we can assume that $r_h \in J^{a+1} + (y_1^t, \ldots, y_{h-1}^t)$, even when $h = k$. To see this, consider an element $s = xu$ where $x = (x_1 x_2 \cdots x_{k-1})^{t(d-2)-1}$ and $u \in (x_1, x_2, \ldots x_{k-1})^*$. Clearly,

$$\begin{aligned} s &\in ((x_1, x_2, \ldots, x_{k-1})^{(k-1)(d-2)t-(k-1)+1})^* \\ &\subset ((x_1, x_2, \ldots, x_{k-1})^{(d-2)})^{(k-1)t-1})^* \\ &\subset J^{a+1}, \end{aligned}$$

since $(k-1)(d-2)t - (k-2) \geq (d-2)((k-1)t-1) \geq (d-2)(a+1)$, the latter inequality following since the degree of η, which is $a-(k-1)t$, is at most -2. We conclude that r_h may be assumed in $J^{a+1} + (y_1^t, \ldots, y_{h-1}^t)$. (The same argument shows an analogous statement for each r_i, but we won't need this fact.)

Finally, because $r_h \in (J^{a+1} + (y_1^t, \ldots, y_{h-1}^t)) \cap J^a$, we can use Corollary 3.4 to conclude that $r_h \in J^{a+1} + (y_1^t, \ldots, y_{h-1}^t)J^{a-t}$. This means that $\tilde{r}_h$ may be assumed to be in $(\tilde{y}_1^t, \ldots, \tilde{y}_{h-1}^t)G$. The argument that $H_{\bar{n}}^k(G)$ vanishes in degrees less than -1 can now be concluded exactly as above for the non-negative degree case, by subtracting off the boundary μ to contradict the minimality of h. This completes the verification of condition (2) of (4.2.3.) and the proof of Theorem 4.1. □

5. Parameter test elements

In this section we contribute to the understanding of test elements, proving that, under some mild assumptions on R, any element $c \in R^0$ such that R_c is F-rational has a power which is a test element for any ideal generated by monomials in parameters $x_1, \ldots, x_d$ for R. This fact enables us to complete the proof of Corollary 4.2. In fact, what we actually show is something stronger: any such c will be a test element for any m primary ideal I of R such that R/I has finite *phantom* projective dimension. Velez has shown that some power of c will be a test element for any ideal generated by parameters. Our improvement is a technical improvement of his result, building on the ideas of Aberbach [Ab2], which in turn stem from Hochster and Huneke's results on *phantom acyclicity.* We refer the reader to [Ab1] for the basic definitions and properties of phantom projective dimension and to Section 9 of [HH1] for more on phantom acyclicity. For more information on parameter test elements, the reader is referred to Section 8 of [HH3], [S1], and [V].

THEOREM 5.1. *Let (R, m) be a reduced and equidimensional excellent local ring. If $c \in R^0$ is such that R_c is F-rational, then some power of c is a test element for all m primary ideals I such that R/I has finite phantom projective dimension.*

COROLLARY 5.2. *Let (R, m) be a reduced, equidimensional excellent local ring and suppose that R_c is F-rational. Then some power of c is a test element for all ideals generated by monomials in any system of parameters for R.*

Proof. In general, if $I = \bigcap_\lambda I_\lambda$ and $d \in R$ multiplies each I_λ^* into I_λ, then d multiplies I^* into I. Moreover, if $I^{[q]} = \bigcap_\lambda I_\lambda^{[q]}$, then any test element for $\{I_\alpha\colon \lambda \in \Lambda\}$ must be a test element for I.

Fix the power c^N of c guaranteed by Theorem 5.1 to be a test element for all m primary ideals of finite phantom projective dimension, a class of ideals that includes all m-primary ideals generated by monomials in any system of parameters. Take any system of parameters $x_1, \dots, x_d$ and consider an arbitrary monomial ideal I in these parameters. Any such I is an intersection of m primary ideals generated by monomials in $x_1, \dots, x_d$; in particular, $I = \bigcap_{t\in\mathbb{N}}(I + (x_1^t, \dots, x_d^t))$. For each q, $I^{[q]} = \bigcap_{t\in\mathbb{N}}(I^{[q]} + (x_1^{tq}, \dots, x_d^{tq}))$, and these ideals are also generated by monomials in $x_1, \dots, x_d$. We conclude that c^N is a test element for I. The corollary is proved. □

Corollary 5.2 was exactly the technical improvement we needed to finish off the proof of Corollary 4.2. It remains to prove the main theorem of this section. The proof follows that of Aberbach in [Ab2].

Proof of 5.1. We first show we may assume without loss of generality that R is complete. Suppose that $c \in R^0$ such that R_c is F-rational. Since $R_c \longrightarrow R_c \otimes \hat{R} = (\hat{R})_c$ is a smooth map, it follows from [V] that $(\hat{R})_c$ is F-rational. Assume Theorem 5.1 holds for complete rings and let c^N be the power of c that is a test element for all ideals $\mathcal{I}$ of $\hat{R}$ with $\hat{R}/\mathcal{I}$ of finite phantom projective dimension over $\hat{R}$. Suppose that $z \in I^*$ in R where $I \subset R$ is such that R/I has a finite phantom resolution. After tensoring with $\hat{R}$, we get a finite phantom resolution for $\hat{R}/I\hat{R}$. Since the image of z in $\hat{R}$ is in $(I\hat{R})^*$, we see that $c^N z^q \in I^{[q]}\hat{R}$ in $\hat{R}$. It follows from the faithful flatness of $R \subset \hat{R}$ that $c^N z^q \in I^{[q]}$ in R as well. We conclude that c^N is a test ideal for all ideals $I \subset R$ such that R/I has finite phantom projective dimension.

Assume R is a complete local equidimensional ring of dimension d. By Theorem 11.8 of [HH1], there is an ideal $\mathcal{A} \subset R$ such that $\mathcal{A}H^i(\mathbb{F}.) = 0$ whenever the homology of the complex $\mathbb{F}.$ is phantom. The radical of this ideal $\mathcal{A}$ defines the non-Cohen-Macaulay locus of R. Let $\mathcal{B}$ be the parameter test ideal of R. By definition, the elements of $\mathcal{B}$ are the test elements for any ideal generated by any parameters (equivalently, any full system of parameters) for R. We claim that $\mathcal{B}\mathcal{A}^{2d}$ is contained in the test ideal for all m-primary ideals I such that R/I has a finite phantom resolution. That is, $\mathcal{B}\mathcal{A}^{2d}(I^*)^{[q]} \subset I^{[q]}$, for all such I. Since the class of ideals I such that R/I has finite phantom projective dimension is closed under Frobenius powers, this is equivalent to $\mathcal{B}\mathcal{A}^{2d}(I^*) \subset I$, for all such I.

To prove this claim, let $\mathbb{F}.$ be a finite phantom resolution of R/I, so that $F_0 = R$ and the subsequent F_i are all free R modules. Using Aberbach's "Phantom Auslander-Buchsbaum" theorem [Ab1], we know that the length of $\mathbb{F}.$ is exactly d = dimension R. Choose any system of parameters $x_1, \dots, x_d$ contained in I. Let $\mathbb{K}.$ denote the Koszul complex on the parameters $x_1, \dots, x_d$. Take any $u \in I^* - I$. By Lemma 9.16 of [HH1], we know that for any $a \in \mathcal{A}^d$, the map $K_0 = R \longrightarrow F_0 = R$ given by sending the generator $1 \in K_0$ to the element au extends to a map of the complexes $\theta\colon \mathbb{K}. \longrightarrow \mathbb{F}.$ Applying the functor $\otimes_R {}^eR$ to this map of complexes preserves the phantom exactness. We denote by $\theta^{[q]}\colon {}^e\mathbb{K}. \longrightarrow {}^e\mathbb{F}.$ the new map of complexes. The abutment map is the map $R/(x_1^q, \dots, x_d^q)R \longrightarrow R/I^{[q]}$ given by sending the equivalence class of 1 to the equivalence class of $(au)^q$.

Let c' be any test element for R. Since $u \in I^*$, we know that $c'(au)^q \in I^{[q]}$ for all q. It follows that both the zero map and the map $c'\theta^{[q]}$ from ${}^e\mathbb{K}.$ to ${}^e\mathbb{F}.$ extend the zero map on the abutments $R/(x_1^q, \dots, x_d^q)R \longrightarrow R/I^{[q]}$. Lemma 9.16 of [HH1] then guarantees the existence of a phantom homotopy: for each $a' \in \mathcal{A}^d$ and each $q = p^e$, there exists a homotopy $h_i\colon {}^eK_i \longrightarrow {}^eF_{i+1}$ such that $a'c'\theta_i{}^{[q]} = \phi_{i+1}{}^{[q]}h_i + h_{i-1}\alpha_i{}^{[q]}$ where $\alpha.$ is the boundary map for $\mathbb{K}.$ and $\phi.$ is the boundary map for $\mathbb{F}.$.

Consider what this says about the last map $\theta_d\colon K_d \cong R \longrightarrow F_d$ in the original map $\theta\colon \mathbb{K}. \longrightarrow \mathbb{F}.$ of complexes. For each q, we have a commutative diagram coming from the homotopy h:

$$\begin{array}{ccc} {}^eF_d & \xleftarrow[h_{d-1}]{} & {}^eK_{d-1} \\ \Big\uparrow{\scriptstyle a'c'\theta_d^{[q]}} & & \Big\| \\ {}^eK_d \cong R & \xrightarrow[\alpha_d{}^{[q]}]{} & {}^eK_{d-1} \end{array} \tag{5.2.1.}$$

Let $\check{}$ indicate the dual functor $\mathrm{Hom}_R(-, R)$. Applying this to (5.2.1.), we get

$$\begin{array}{ccc} {}^e\check{F}_d & \xrightarrow{\check{h}_{d-1}} & {}^e\check{K}_{d-1} \\ {\scriptstyle a'c'\theta_d^{[q]\check{}}}\Big\downarrow & & \Big\| \\ {}^e\check{K}_d \cong R & \xleftarrow[\alpha_d{}^{[q]\check{}}]{} & {}^e\check{K}_{d-1} \end{array} \tag{5.2.2.}$$

Using the commutativity of diagram (5.2.2.), we see that the image of the map $a'c'\check{\theta}_d^{[q]} \subset {}^e\check{K}_d \cong R$ is contained in the image of $\check{\alpha_d^{[q]}} = (x_1^q, x_2^q, \dots, x_d^q)$. We conclude that the image of $\check{\theta}_d$ is contained in the tight closure of $(x_1, \dots, x_d)$ in R.

Consider again the original map of complexes θ: $\mathbb{K}. \longrightarrow \mathbb{F}.$ which extends the map from $K_0 \cong R$ to $F_0 \cong R$ given by multiplication by au. Let c be any parameter test element. Multiply the map θ by c and apply the dual functor $\check{}$. We get a map of complexes $c\check{\theta}$: $\check{\mathbb{F}.} \longrightarrow \check{\mathbb{K}.}$. The map on the abutments $\check{F}_d/(\mathrm{im}\,\check{\phi}_d) \longrightarrow \check{K}_d/(\mathrm{im}\,\check{\theta}_d) \cong R/(x_1, \dots, x_d)$ induced by $c\check{\theta}_d$ is the zero map, since $(\mathrm{im}\ \check{\phi}_d) \subset (x_1, \dots, x_d)^*$ and c is a parameter test element.

Again using Lemma 9.16 of [HH1], we see that after multiplication by $a'' \in \mathcal{A}^d$, the map $(c\theta)\check{}$ is homotopic to the zero map. Again we consider what happens at the tail end of the complexes. The last map in the homotopy $h.'$: $\check{\mathbb{F}.} \longrightarrow \check{\mathbb{K}}._{-1}$ cuts diagonally across the following diagram from $\check{F}_1$ to $\check{K}_0 \cong R$:

(5.2.3.)
$$\begin{tikzcd}
\check{F}_1 \arrow[d, "a''c\check{\theta}_1"] & \check{F}_0 \cong R \arrow[l, "\check{\phi}_1"'] \arrow[d, "a''ca\check{u}"] \\
\check{K}_1 \cong R & \check{K}_0 \cong R \arrow[l, "\check{\alpha_1}"']
\end{tikzcd}$$

The upper right hand triangle here is a commutative diagram (although the lower left triangle may not be). Dualizing again, and now writing in the map induced by the homotopy h', we get a commutative diagram coming from the upper right triangle of diagram (5.2.3.):

$$\begin{tikzcd}
F_1 \arrow[r, "\phi_1"'] & R \\
R \arrow[u, "\check{h'}"] \arrow[r, equal] & R \arrow[u, "a''cau"']
\end{tikzcd}$$

This final commutative diagram shows that $aa''cu \in im\,\phi_1 = I \subset R$. Since u was an arbitrary element of I^*, a and a'' are arbitrary elements of $\mathcal{A}^d$ and c an arbitrary parameter test element, we conclude that $\mathcal{A}^{2d}\mathcal{B}$ multiplies I^* into I. Thus $\mathcal{A}^{2d}\mathcal{B}$ is contained in the test ideal for all m-primary ideals of I such that R/I has finite phantom projective dimension.

It remains only to show that if R_c is F-rational, then some power of c is in $\mathcal{A}^{2d}\mathcal{B}$. Juan Velez has shown that some power of c is a parameter test element. On the other hand, since F-rational rings are Cohen-Macaulay, we know that c is in the defining ideal for the non-Cohen-Macaulay locus for R. In particular, some power of c is in $\mathcal{A}$. It follows that some power of c is in $\mathcal{A}^{2d}\mathcal{B}$, and the proof is complete. □

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