ILLINOIS JOURNAL OF MATHEMATICS
Volume 40, Number 2, Summer 1996

ENERGY MINIMIZING SECTIONS OF A FIBER BUNDLE

SHIAH-SEN WANG

0. Introduction

Interior partial regularity for minimizers of functionals having nonquadratic growth between Riemannian manifolds has been extensively studied. See [2], [6], [8], [9] and references therein for details. Here we study sections of a fiber bundle X that locally minimize the L^p norm of the gradient among all $L^{1,p}_{\text{loc}}$ sections when $p \in (1, \infty)$.We show that such a local minimizing section is Hölder continuous everywhere except a closed subset Z of the base manifold M, and that the set Z has Hausdorff dimension at most $m - [p] - 1$, where m is the dimension of M.

It is a well-known topological fact that there is no continuous unit tangent vector field on an even-dimensional sphere; thus continuity of a local minimizing section on *all* of M may be impossible by topological obstructions. In the trivial bundle case, i.e., $X = M \times N$ with N as the fiber, and $p = 2$, the problem studied here can be easily reduced to study minimizing harmonic maps from M to N; therefore, continuity of local minimizing sections may be impeded by energy considerations (see [7]), even without the topological obstructions.

In contrast with harmonic sections (see [1], 2.39), we *do* include the "horizontal" energy in the energy functional. This causes a major problem in proving the partial regularity for minimizing sections of the simplest form of functionals having non-quadratic growth discussed here because we have to deal with the map constraint—the projection map π of the fiber bundle.

The methods used to prove the results are described as follows:

In Section 1, first we locally associate an $L^{1,p}$ section $\widetilde{v}$ with each map $v \in L^{1,p}(\Omega, N)$ for some bounded open subset Ω of M by the local trivialization property of the bundle. Then we construct a new functional $\mathcal{G}$ defined on $L^{1,p}(\Omega, N)$ from the original one—the L^p norm of the gradient. Via this reformulation, we can study $\mathcal{G}$-minimizers with submanifold N constraint instead of p-energy minimizers with the mapping constraint π.

In Section 2, we prove that small normalized p-Dirichlet energy of a $\mathcal{G}$-minimizer u implies Hölder continuity using the De Giorgi blowing up argument outlined in Luckhaus' paper [9]—where he studies general functionals with nice blow-ups. The key ingredients of the proof are Lemma 2 and Lemma 3. We show the blow-up functional $\mathcal{F}$ of $\mathcal{G}$ is nice (in fact, our blow-up functional $\mathcal{F}$ is nicer than the one

Received September 8, 1994.
1991 Mathematics Subject Classification. Primary 58E20; Secondary 49K20

studied in Luckhaus' paper) in Lemma 2 by applying Tolksdorff's results on systems of degenerate elliptic p.d.e.'s (see [13]). Then we show energy decay inequality in Lemma 3 by the De Giorgi blowing up argument. In order to use this argument, we also use Luckhaus' comparison map lemma (see Lemma 1) and rescale both the domain and the target manifolds as in Proposition 1 of [9]. Once the energy decay inequality is established, we can iterate this inequality and get Morrey's growth estimate, and so the Hölder continuity follows. Also, by a standard covering argument and partial Hölder continuity result, we see that the singular set Z defined by

$$Z = \left\{ x \in M \mid \Theta(x) = \limsup_{r \to 0} r^{p-m} \int_{B_r(x)} |\nabla u|^p > 0 \right\},$$

is a closed subset of Ω and has Hausdorff dimension at most $m - p$.

In Section 3, we show that the Hausdorff dimension estimate on Z can be improved. By rescaling the domain near a point in Z, we show that the blow-up map $\hat{v}$ of u minimizies a functional $\mathcal{D}$, and a monotonicity formula hold for u_ρ, obtained from u by rescaling the domain; and thus $\hat{v}$ is radially homogeneous of order 0, i.e., $\partial_r \hat{v} = 0$. Hence Federer's dimension reduction argument can be applied here, and so the assertion on the Hausdorff dimension of Z follows.

Concerning the higher regularity of a $\mathcal{G}$-minimizer u where u is Hölder continuous, we can quote the results in Giaquinta and Modica's paper in case $p \geq 2$ (see [6] in which they study maps between coordinate neighborhoods) by the fact that we already establish Hölder continuity. Unfortunately we are not able to extend this result to the case when $p \in (1, 2)$ at this moment because some technical inequalities are not true when $p \in (1, 2)$ (see Section 2 of [6]).

1. Preliminary setup and notations

Suppose $\mathcal{B}$ is a fiber bundle consisting of:

(1) a base space M—an m-dimensional C^2 Riemannian manifold;
(2) a fiber space N—a closed n-dimensional C^2 submanifold of some Euclidean space $\mathbb{R}^k$;
(3) a total space X—an $(m+n)$-dimensional C^2 Riemannian manifold;
(4) a projection map $\pi: X \to M$—a C^2 submersion from X onto M so that $N_x = \pi^{-1}\{x\}$ is C^2 diffeomorphic to N for all $x \in M$.

Let $N_\tau = \{y \in \mathbb{R}^k \mid \operatorname{dist}(x, N) < \tau\}$ for some $\tau > 0$, be a neighborhood of N so that the unique nearest point projection $\xi: N_\tau \to N$ is well defined and let Γ_1 be a positive constant depending on N only so that

$$\|\nabla(\xi(y) - \Lambda_{\xi(y)})\| \leq \Gamma_1 |y - \xi(y)|,$$

where $\Lambda_{\xi(y)}$ is the orthogonal projection of $\mathbb{R}^k$ onto $\operatorname{Tan}(N, \xi(y))$.

For a point a in M, let $\Omega \subset\subset M$ be a neighborhood of a equipped with the standard Euclidean metric of $\mathbb{R}^m$, and let $h: \Omega \times \mathbb{R}^k \to X$ be a C^2 map so that $g = h\,|_{\Omega\times N}$ is a C^2 diffeomorphism from $\Omega \times N$ onto $V = g(\Omega \times N)$, a neighborhood of N_a in X. Let $\pi_2: \Omega \times \mathbb{R}^k \to \mathbb{R}^k$ be the standard coordinate projection.

For $p \in (1, \infty)$, define

$$SL^{1,p}(\Omega, X) = \{\widetilde{w} \in L^{1,p}(\Omega, X) \mid \pi \circ \widetilde{w} = x \ a.e.\ x \in \Omega\},$$

and

$$L^{1,p}(\Omega, N) = \{w \in L^{1,p}(\Omega, \mathbb{R}^k) \mid w(x) \in N \ a.e.\ x \in \Omega\}.$$

Define $\Phi: L^{1,p}(\Omega, N) \to SL^{1,p}(\Omega, X)$ by

$$\Phi(w)(x) = g(x, w(x)), \qquad \forall x \in \Omega \text{ and } w \in L^{1,p}(\Omega, N).$$

Clearly, Φ is bijective with inverse map defined by

$$\Phi^{-1}(\widetilde{w})(x) = \pi_2 \circ g^{-1} \circ \widetilde{w}(x), \qquad \forall x \in \Omega \text{ and } \widetilde{w} \in SL^{1,p}(\Omega, X).$$

Define $\mathcal{E}: SL^{1,p}(\Omega, X) \to \mathbb{R}$ by

$$\mathcal{E}(\widetilde{w})(x) = \int_\Omega |\nabla \widetilde{w}|^p, \qquad \forall \widetilde{w} \in SL^{1,p}(\Omega, X).$$

Let $M(m, k)$ be the space of linear maps from $\mathbb{R}^m$ to $\mathbb{R}^k$ and define $\mathcal{G}: L^{1,p}(\Omega, N) \to \mathbb{R}$ by

$$\mathcal{G}(w) = \int_\Omega G(x, w, \nabla w)dx, \qquad \forall w \in L^{1,p}(\Omega, N),$$

with

$$G(x, y, \eta) = |A(x, y) + B(x, y)\eta|^p, \qquad \forall (x, y, \eta) \in \Omega \times \mathbb{R}^k \times M(m, k),$$

where

$$\begin{aligned} A(x, y) &= D_x h(x, y)\colon \mathrm{Tan}(\Omega, x) \to \mathrm{Tan}(X, h(x, y)), \\ B(x, y) &= D_y h(x, y)\colon \mathbb{R}^k \to \mathrm{Tan}(X, h(x, y)), \end{aligned}$$

$\forall (x, y) \in \Omega \times \mathbb{R}^k$. Notice that if $(x, y) \in \Omega \times N$, then

$$\mathrm{rank} A(x, y)\big|_{\mathrm{Tan}(\Omega,x)} = m, \text{ and } \mathrm{rank} B(x, y)\big|_{\mathrm{Tan}(N,y)} = n$$

This fact will be used in proving Lemma 2.2. Observe that u is a $\mathcal{G}$-minimizer in $L^{1,p}(\Omega, N)$ iff $\Phi(u)$ is a $\mathcal{E}$-minimizer in $SL^{1,p}(\Omega, X)$ because $\mathcal{G}(u) = \mathcal{E}(\Phi(u))$.

Let Γ_2 and Γ_3 be two positive numbers so that

$$\Gamma_2^{-1} \leq \|Dh(x, y)\|, \ \|Dg^{-1}(z)\| \leq \Gamma_2, \quad \forall (x, y) \in \Omega \times N, \forall z \in g^{-1}(\Omega \times N), \tag{1}$$

and

$$\Gamma_3^{-1} \leq \|A|_{\mathrm{Tan}(\Omega,x)}\|, \ \|B|_{\mathrm{Tan}(N,y)}\| \leq \Gamma_3, \qquad \forall x \in \Omega, \forall y \in N \tag{2}$$

Next, we make some observations to simplify our exposition of the following two sections.

(1) The compactness assumption on N can be replaced by the hypothesis that the image of a small ball for an $\mathcal{G}$-minimizer is contained in $\widehat{N} \subset\subset N$ (compare [12], Theorem I).
(2) It can be easily checked that the integrand G of the functional $\mathcal{G}$ defined above satisfies all the growth conditions studied by Luckhaus in [9] and [10], also studied by Giaquinta and Modica in [6] for the case when $p \geq 2$ and when the maps are between coordinate neighborhood. However, we will carry out most of the computation in the proof, since some of them are simpler here without referring to the general hypotheses of G these papers imposed; i.e., here we provide an example that these hypotheses on G are "natural".
(3) For notational simplicity, we will confine our study to the case that $B_2(0) \subset \Omega \subset \mathbb{R}^m$ with the standard Euclidean metric, because the general case can be easily modified by first shrinking Ω if necessary so that the general metric is C^1 close to the Euclidean one and then by rescaling the Euclidean metric as in the minimizing p-harmonic case studied by Hardt and Lin in [8], Section 7.

We complete this section with a discussion on *scaling*:
For $w \in L^{1,p}(B_r(a), \mathbb{R}^k)$ with $a \in B_1(0)$ and $r \in (0, 1)$, the expression

$$w_{r,a}(x) = w(rx + a) \quad (= w_r(x), \text{ when } a = 0), \qquad \forall x \in B_1(0) = B$$

defines a map in $L^{1,p}(B, N)$.

If u is a $\mathcal{G}$-minimizer in $L^{1,p}(B, N)$, then $u_{r,a}$ is minimizes the functional $\mathcal{G}_{r,a}$ among maps in $L^{1,p}(B, N)$, defined by

$$\mathcal{G}_{r,a}(w) = \int_B G_{r,a}(x, w, \nabla w)dx, \qquad w \in L^{1,p}(B, N),$$

where

$$G_{r,a}(x, y, \eta) = G(rx + a, y, \eta/r), \quad \forall (x, y, \eta) \in \Omega \times \mathbb{R}^k \times M(k, m).$$

For $w \in L^{1,p}(B_r(a), \mathbb{R}^k)$, write

$$E_{r,a}(w) = r^{p-m} \int_{B_r(a)} |\nabla w|^p dx \quad (= E_r(w), \text{ when } a = 0).$$

Note that

$$E_\rho(w_{r,a}) = E_{r\rho,a}(w), \qquad \rho \in (0, 1].$$

2. Small energy implies Hölder continuity

The following lemma is due to Luckhaus (see [9], Lemma 1), which extends Lemma 4.3 of [12].

LEMMA 2.1. *For $\beta \in ([p-1]/p, 1)$, there is a positive constant $c_1 = c(m, k, p, \beta)$ so that if $0 < t \le 1/2$, $0 < \rho, \epsilon < 1$, $a \in B$, and $v_1, v_2 \in L^{1,p}(\partial B_\rho(a), N)$, then there is a map $w \in L^{1,p}(B_\rho(a) - B_{(1-t)\rho}(a), \mathbb{R}^k)$ satisfying*

$$w(x) = \begin{cases} v_1(x) & x \in \partial B_\rho(a); \\ v_2((x-a)/(1-t)+a) & x \in \partial B_{(1-t)\rho}(a). \end{cases}$$

$$\operatorname{dist}(w(x), N) \le r, \qquad x \in B_\rho(a) - B_{(1-t)\rho}(a),$$

and

$$\int_{B_\rho(a)-B_{(1-t)\rho}(a)} |\nabla w|^p \le c_1 K^p (1 + (\epsilon/\rho)^p) t, \tag{3}$$

where $r = c_1 K \epsilon^{1-\beta} \rho^{[p-1]-(n-1)/p}$, $K^p = \int_{\partial B_\rho(a)} (|\nabla_{\tan} v_1|^p + |\nabla_{\tan} v_2|^p + |v_1 - v_2|^p/\epsilon^p)$.

The next lemma shows that the blow-up function is nice (actually Hölder continuity is sufficient to apply Luckhaus' results; see Hypothesis (A2) in [9]).

LEMMA 2.2. *There is a positive constant $c_2 = c(m, k, p, \Gamma_3)$ such that if $(x_0, y_0) \in \Omega \times N$, $v \in L^{1,p}(B, \mathbb{R}^k)$ with $v(B) \subset \operatorname{Tan}(N, y_0) (\simeq \mathbb{R}^n)$, and v minimizes $\mathcal{F}$ among maps in $L^{1,p}(B, \operatorname{Tan}(N, y_0))$, then $v \in C^{1,\gamma}$ for some $\gamma \in (0, 1)$ and*

$$\|\nabla v\|_{L^\infty(B_{2R}(a))} \le c_2 \fint_{B_{3R}(a)} (1 + |\nabla v|)^p \tag{4}$$

whenever $B_{3R}(a) \in B$, where

$$\mathcal{F}(w) = \int_B F(w), \quad \forall w \in L^{1,p}(B, \operatorname{Tan}(N, y_0)),$$

with

$$F(\eta) = |B(x_0, y_0)\eta|^p, \qquad \eta \in M(k, m+n).$$

Proof. Since v minimizes the functional $\mathcal{F}$ among maps in $L^{1,p}(B, \operatorname{Tan}(N, y_0))$, it satisfies the following degenerate elliptic system of equations:

$$\int_B |B(x_0, y_0)\nabla v|^{p-2}\langle B(x_0, y_0)\nabla v, B(x_0, y_0)\nabla\varphi\rangle = 0, \quad \forall\varphi \in L_0^{1,p}(B, \operatorname{Tan}(N, y_0)).$$

Hence, the conclusions follow immediately from Theorem 5.1 and Theorem 6.1 of [13]. □

LEMMA 2.3. *For $\alpha \in (0, 1)$, there are positive constants $c_3 > 1$, ϵ_0 depending only on m, p, k, $\Gamma_{i=1,2,3}$, and α, such that if u is a $\mathcal{G}$-minimizer and $r^{\alpha p} < E_{r,a}(u) < \epsilon_0^p$ with $0 < r < 1, a \in B$, then we have*

$$E_{r/c_3,a}(u) \leq c_3^{-\alpha p} E_{r,a}(u). \tag{5}$$

Proof. Were the conclusion false, there would exist a sequence of balls $\{B_{r_i}(a_i)\}_i$, for some constant c_3 to be chosen later, so that

$$\begin{cases} E_{r_i,a_i}(u) = \epsilon_i^p \\ E_{s_i,a_i}(u) \geq c_3^{-\alpha p}\epsilon_i^p, \quad \text{where } s_i = r_i/c_3, \\ r_i^\alpha < \epsilon_i \end{cases} \tag{6}$$

and that as $i \to \infty$, $\epsilon_i \to \infty$.

Let $\overline{y}_i = \fint_{B_{r_i}(a_i)} u$. Then we have

$$\operatorname{dist}(\overline{y}_i, N) \leq c_4\epsilon_i^p,$$

for some $c_4 = c(m, p, k)$ by the Poincaré inequality. Hence $y_i = \xi(\overline{y}_i)$ is well defined, when i is sufficiently large.

For these i's, let

$$v_i = (u(r_i x + a_i) - y_i)/\epsilon_i \text{ and } u_i = \epsilon_i v_i + y_i.$$

By the Poincaré inequality again, we have

$$\|v_i\|_{L^{1,p}(B,\mathbb{R}^k)} \leq c_5 \qquad \text{for some } c_5 = c(m, p, k, \Gamma_1).$$

Note that N is compact. Passing to subsequences without changing notations, we may assume that

$$\begin{cases} a_i \to x_0 \in B, & \\ y_i \to y_0 \in N, & \\ v_i \to v & \text{strongly in } L^p(B, \mathbb{R}^k) \text{ norm and pointwise a.e. on } B, \\ \nabla v_i \to \nabla v & \text{weakly in } L^p(B, M(m, k)), \\ \alpha_i^{-p}G(a_i, y_i, \alpha_i\eta) \to F(\eta) & \text{for all } \eta \in \mathbb{R}^k, \text{ where } \alpha_i = \epsilon_i/r_i. \end{cases}$$

Claim. v satisfies the hypothesis of Lemma 2.2 (and so its conclusion), and for each $\rho \in (0, 1)$,

$$\epsilon_i^{-p} r_i^{p-m} \int_{B_{\rho r_i}(a_i)} G(x, u, \nabla u)\, dx \to \int_{B_\rho(0)} F(v). \tag{7}$$

Assume the claim is true for the moment. Then by Lemma 2.2 (4) with $a = 0$, $R = 1/3$, and requiring $c_3 \geq 3/2$, we have

$$E_{1/c_3}(v) < c_6, \qquad \text{with } c_6 = (m, k, p, \Gamma_3).$$

It follows that

$$c_3^{m-p} \int_{B_{1/c_3}} F(v) < c_7 c_3^{-p} \qquad \text{where } c_7 = c(m, k, p, \Gamma_3).$$

Thus by (7), we see that

$$\lim_{i\to\infty} \frac{s_i^{p-m} \int_{B_{s_i}(a_i)} G(x, u, \nabla u)\, dx}{r_i^{p-m} \int_{B_{r_i}(a_i)} G(x, u, \nabla u)\, dx} < c_7 c_3^{-p}$$

By taking i sufficiently large, one has

$$s_i^{p-m} \int_{B_{s_i}(a_i)} G(x, u, \nabla u)\, dx \leq 2 c_7 c_3^{-p} r_i^{p-m} \int_{B_{r_i}(a_i)} G(x, u, \nabla u)\, dx,$$

so

$$s_i^p + E_{s_i, a_i}(u) \leq c_8 c_3^{-p} (r_i^p + E_{r_i, a_i}(u)) \qquad \text{for some } c_8 = c(m, p, \Gamma_2, \Gamma_3)$$

Hence by (6) and the definition of s_i, we have

$$c_9 c_3^{-p} \epsilon_i^p \geq c_3^{-\alpha p} \epsilon_i^p \qquad \text{for some } c_9 = (m, p, \Gamma_{i=1,2,3}).$$

This leads to a contradiction to (6) by further requiring that $c_9 c_3^{(\alpha-1)p} \leq 1$.

Proof of the claim. The proof here is due to Luckhaus (see p. 358 of [9]). Let $\hat{v}$ be any comparison function coinciding with v in $B - B_{1-\lambda}$ for some $\lambda \in (0, 1)$. By Fatou's lemma and Fubini's theorem, we may assume that there is a $\hat{\rho} \in (1 - \lambda, 1)$ such that

$$\int_{\partial B_{\hat\rho}} |v_i - \hat{v}|^p \to 0, \text{ as } i \to \infty, \qquad \int_{\partial B_{\hat\rho}} \left(|\nabla v_i|^p + |\nabla \hat{v}|^p \right) \leq c_{10} < \infty$$

Furthermore, choose a sequence of positive numbers $R_i \to \infty$ as $i \to \infty$ such that

$$\|(\mathrm{Id} - \xi)\Big|_{B_{\epsilon_i R_i}(y_i) \cap \left(\mathrm{Tan}(N, y_i) + y_i\right)}\|_{L^\infty} = o(\epsilon_i)$$

$$\|\nabla(\mathrm{Id} - \xi)\Big|_{B_{\epsilon_i R_i}(y_i) \cap \left(\mathrm{Tan}(N, y_i) + y_i\right)}\|_{L^\infty} = o(\epsilon_i)/\epsilon_i$$

Define

$$\hat{v}_i = R_i\hat{v}/\max(|v_i|, R_i), \quad \hat{u}_i = \xi(y_i + \epsilon_i\hat{v}_i), \quad u_i = y_i + \epsilon_i v_i,$$

and apply Lemma 2.1 to $\hat{u}_i, u_i, \lambda_i$ and $\hat{\rho}$ in place of v_1, v_2, t and ρ. We find a map $\widetilde{w}_i$ such that

$$\widetilde{w}_i(x) = \begin{cases} \hat{u}_i(x/(1-\lambda_i)) & \text{for } |x| < \hat{\rho}(1-\lambda_i), \\ u_i(x) & \text{for } |x| > \hat{\rho}, \end{cases}$$

$$\int_{B_{\hat{\rho}} - B_{\hat{\rho}(1-\lambda_i)}} |\nabla\widetilde{w}_i|^p \le c_{11}\lambda_i\epsilon_i^p,$$

$\lambda_i \to 0$ and dist $(y, N) \to 0$ uniformly as $i \to \infty$.

Thus

$$\begin{aligned} \int_{B_\rho} F(\nabla\hat{v}) &= \lim_{i\to\infty} \alpha_i^{-p} \int_{B_\rho} G(a_i, y_i, \alpha_i\nabla\hat{v}_i) \\ &= \lim_{i\to\infty} r_i^p/\epsilon_i^p \int_{B_\rho} G(a_i + r_i x, \xi(\widetilde{w}_i), \nabla(\xi(\widetilde{w}_i))/r_i) \\ &\ge \lim_{i\to\infty} r_i^p/\epsilon_i^p \int_{B_\rho} G(a_i + r_i x, u_i, \nabla u_i/r_i) \\ &\ge \int_{B_\rho} F(\nabla v), \end{aligned}$$

where the last inequality follows from the lower semicontinuity of $\mathcal{F}$. Hence the claim holds by taking $\hat{v} = v$ in the above inequalities. □

THEOREM 2.4. *If u is a $\mathcal{G}$-minimizer, then for $\alpha \in (0, 1)$ and $B_r(a) \in B$, we have the following*:

(1) *If $E_{a,r}(u) \le r^{\alpha p}$, then u is C^α in $B_r(a)$.*
(2) *If $r^{\alpha p} < E_{a,r}(u) < \epsilon_0^p$ with ϵ_0 given as in Lemma 2.3, then there is a positive constant c_{12}, depending on m, k, p, α and Γ_i, for $i = 1, 2, 3$, such that*

$$E_{B_{r'}(b)}(u) \le c_{12}\big(r'/r\big)^{\alpha p}, \qquad \forall b \in B_r(a) \text{ and } r' \in (0, r)$$

and u is C^α on $B_{r/2}(a)$.

Proof. Case (1) follows immediately from Morrey's growth estimate Lemma (see [11], 3.5.2). The first assertion of Case (2) is from iterating (5), and the second one is from the first one and Morrey's growth estimate Lemma. □

COROLLARY 2.5. *Any $\mathcal{G}$-minimizer is Hölder continuous on $M - Z$, where Z is defined as in Section 0. Moreover, Z is relatively closed in M and has $(m-p)$-dimensional Hausdorff measure zero in case $1 < p \leq m$.*

Proof. The closedness of Z and Hölder continuity of u follow immediately from Theorem 2.4, and the Hausdorff dimension estimate on Z is from the standard covering argument. □

3. Improvement on the Hausdorff dimension of the singular set Z

In this section we will assume that $p \in (1, m)$. By Theorem 2.4, we know that u is Hölder continuous at a point in M if the normalized p-energy E tends to 0 when the radius tends to 0. Hence we only need to study the case when $a \in Z$, where Z is defined as in Section 0 and

$$E_{r,a}(u) \geq \epsilon_0, \qquad \forall r \in (0,1). \tag{8}$$

For simplicity, we assume that $a = 0 \in Z$. Define the functional $\mathcal{D}$: $L^{1,p}(B, N) \to R$ by

$$\mathcal{D}(w) = \int_B |B(0, w)\nabla w|^p \qquad w \in L^{1,p}(B, N),$$

(compare Hypothesis (A3) in [9]). We write $\mathcal{D}_t(w) = \int_{B_t} |B(0, w)\nabla w|^p, \forall t \in (0, 1)$.

THEOREM 3.1. *There exists a sequence $r_i \to 0$ so that the rescaled maps $u_{r_i} \in L^{1,p}(B, N)$ satisfy the following properties:*

$$\begin{aligned} &u_{r_i} \to u_0 \in L^{1,p}(B, N) \text{ strongly in } L^p \text{ norm},\\ &\nabla u_{r_i} \to \nabla u_0 \text{ weakly in } L^p(B, M(m,k)),\\ &r_i{}^{p-m} \int_{B_{r_i}} G(x, u, \nabla u) \to \mathcal{D}(u_0), \end{aligned} \tag{9}$$

as $i \to \infty$. Furthermore, u_0 minimizes $\mathcal{D}$ with respect to a fixed trace on ∂B, and $\partial_r u_0 \equiv 0$.

Proof. To show that u_{r_i} converges to some u_0 weakly in $L^{1,p}(B, N)$, we notice that, by (2),

$$\Gamma_3^{-1} \int_{B_t} |\nabla u_r|^p \leq \mathcal{D}_t(u_r) \leq \Gamma_3 \int_{B_t} |\nabla u_r|^p, \quad \forall t, r \in (0, 1].$$

Hence, it is sufficient to show that $\mathcal{D}(u_r)$ is bounded by Rellich compactness theorem.

Choose a constant $c_{13} = c(m, p, k, N, \Gamma_{i=1,2,3}) > 1$ such that

$$\left|r^p G_{r,0}(x, y, \eta) - |B(0, y)\eta|^p\right| \leq c_{13} r^p (1+|\eta|^p), \qquad \forall (x, y, \eta) \in B \times N \times M(m, k).$$

Let $t \in (0, 1)$ and define

$$u_{r,t} = \begin{cases} u_r(x) & \text{for } |x| > t, \\ u_r(tx/|x|) & \text{for } |x| \leq t. \end{cases}$$

Thus, by the homogeneity on the gradient variable of the integrand for $\mathcal{D}$ and Fubini's theorem, we have

$$\int_{B_t} |B(0, u_{r,t}) \nabla u_{r,t}|^p = t/(m-p) \int_{\partial B_t} |B(0, u_r) \nabla u_r|^p. \tag{10}$$

Note that by (1), (2), and (10), we have

$$(1 - c_{14} r^p) r^p G_{r,0}(x, u_{r,t}, \nabla u_{r,t}) \leq |B(0, u_{r,t}) \nabla u_{r,t}|^p + c_{14} r^p, \tag{11}$$

$$(1 - c_{14} r^p) |B(0, u_r) \nabla u_r|^p \leq r^p G(x, u_r, \nabla u_r) + c_{14} r^p, \tag{12}$$

where $c_{14} = \max(c_{13}, c_{13}\Gamma_2^p, c_{13}\Gamma_3^p)$.

By (11), (12), and $\mathcal{G}_{r,0}$-minimality of u_r, we have

$$(1 - c_{15} r^p) \int_{B_t} |B(0, u_r) \nabla u_r|^p \leq t/(m-p) \int_{\partial B_t} |B(0, u_r) \nabla u_r|^p + c_{15} r^p t^m,$$

where $c_{15} = c_{14}^2 \mathcal{L}^m(B)$. By (8) and taking larger c_{15}, the last term in the above inequality can be absorbed into the left-hand side to get

$$(1 - c_{15} r^p) \mathcal{D}_t(u_r) \leq t/m - p \int_{\partial B_t} |B(0, u_r) \nabla u_r|^p.$$

Hence we obtain the following *monotonicity inequality*:

$$\partial_t \left(\log[t^{p-m} \mathcal{D}_t(u_r)] + c_{15} r^p \log t \right) \geq 0. \tag{13}$$

Apply (13) to the sequences $r_i = e^{-i}$, $t_i = r_{i+1}/r_i = e^{-1}$ in place of r, t respectively. Write $u_i = u_{r_i}$. Note that $e^{m-p}\mathcal{D}_{e^{-1}}(u_i) = \mathcal{D}(u_{i+1})$ and that

$$-\sum_i r_i^p \log (r_{i+1}/r_i) = \sum_i r_i^p < \infty.$$

Therefore $\lim\limits_{i\to\infty} \mathcal{D}(u_i)$ exists, $\mathcal{D}(u_i)$ is bounded and

$$\lim_{i\to\infty} r_i^{p-m} \mathcal{G}_{r_i,0}(u) = \lim_{i\to\infty} \mathcal{D}(u_i).$$

Hence, by passing to a subsequence without changing notation, we have

$$u_i \to u_0 \text{ strongly in } L^p(B, N) \text{ and } u_i \rightharpoonup u_0 \text{ weakly in } L^{1,p}(B, N).$$

This completes our proof of the first assertion.

To show u_0 minimizes $\mathcal{D}$, let $v \in L^{1,p}(B, N)$ be such that $u_0 - v = 0$ on $B - B_{1-t}$ for some $t \in (0, 1)$. Choosing another subsequence of u_i and $\rho \in (1-t, 1)$, if necessary by Fatou's lemma and Fubini's theorem, we may assume that

$$\int_{\partial B_\rho} (|\nabla u_i|^p + |\nabla v|^p) \le c_{16} < \infty \text{ and } \int_{\partial B_\rho} |u_i - v|^p \to 0 \text{ as } i \to \infty.$$

By Lemma 2.1, we have a sequence $w_i \in L^{1,p}(B, N)$, so that

$$w_i(x) = \begin{cases} v(x/(1-\lambda_i)) & \text{for } x \in B_{(1-\lambda_i)\rho}, \\ u_i(x) & \text{for } x \in B - B_\rho, \end{cases}$$

$$\int_{B_\rho - B_{(1-\lambda_i)\rho}} |w_i|^p \le c_{17}\lambda_i,$$

where $\lambda_i \to 0$ as $i \to \infty$. By (10), the lower semicontinuity of $\mathcal{D}$, and the $\mathcal{G}_{r_i,0}$-minimality of u_i, we obtain

$$\mathcal{D}(u_0) \le \lim_{i\to\infty} \mathcal{D}(u_i) \le \lim_{i\to\infty} \mathcal{D}(w_i) = \mathcal{D}(v).$$

Hence u_0 minimizes $\mathcal{D}$.

Notice that (13) also implies that $\partial_r r^{p-m}\mathcal{D}_r(u_0) = 0$, and thus

$$r^{p-m}\mathcal{D}_r(u_0) = \mathcal{D}(u_0). \tag{14}$$

Since u_0 minimizes $\mathcal{D}$, compare with

$$u_{0,r}(x) = \begin{cases} u_0(x) & \text{for } x \in B - B_r, \\ u_0(rx/|x|) & \text{for } x \in B_r; \end{cases}$$

we have

$$\mathcal{D}_r(u_0) \le r/m - p \int_{\partial B_r} |B(0, u_0)\nabla_{\tan} u_0|^p.$$

This inequality and (14) imply that $\partial_r u_0 \equiv 0$. □

COROLLARY 3.2. *Suppose that $u \in L^{1,p}_{\mathrm{loc}}(M, N)$ locally minimizes the functional $\mathcal{G}$; then it is locally Hölder continuous on $M - Z$, where Z is defined as in Section* 0. *Furthermore, the singular set Z has Hausdorff dimension at most $m - [p] - 1$. In particular, Z is discrete if $m = [p] + 1$.*

Proof. The proof is as in 4.5 of [8] or [3]. □

Remarks. (1) Concerning the higher regularity of a $\mathcal{G}$-minimizer u where it is continuous, we can quote the results in Giaquinta and Modica's paper in case $p \geq 2$ (see [6] in which they study maps between coordinate neighborhoods) by the fact that we already establish Hölder continuity in Corollary 2.3. Unfortunately we are not able to extend this result to the case when $p \in (1, 2)$ at this moment because some technical inequalities are not true when $p \in (1, 2)$ (see Section 2 of [6]).

(2) In case the base manifold M has nonempty smooth boundary, the boundary regularity of the corresponding Dirichlet problem can be obtained by arguing as in Section 5 of [8] with some necessary modifactions.

REFERENCES

1. J. Eells and L. Lemaire, *Another report on harmonic maps*, Bull. London Math. Soc. **20** (1987), 385–524.
2. M. Fuchs, *p-Harmonic obstacle problems* Part I: *Partial regularity theory*, Ann. Math. Pura Appl. **156** (1990), 127–158.
3. H. Federer, *The singular set of area minimizing rectifiable currents with codimension one and of area minimizing flat chains modulo two with arbitrary codimension*, Bull. Amer. Math. Soc. **79** (1970), 761–771.
4. M. Giaquinta, *Multiple integrals in the calculus of variations and nonlinear elliptic systems*, Princeton University Press, Princeton, 1983.
5. M. Giaquinta and E. Giusti, *Differentiability of minima of nondifferentiable functional*, Invent. Math. **72** (1983), 285–298.
6. M. Giaquinta and G. Modica, *Remarks on the regularity of minimizers of certain degenerate functionals*, Manuscripta Math. **57** (1986), 55–99.
7. R. Hardt and F. H. Lin, *A remark on H^1 mappings*, Manuscripta Math. **56** (1986), 1–10.
8. ———, *Mappings minimizing the L^p norm of the gradient*, Comm. Pure Appl. Math. **11** (1987), 555–588.
9. S. Luckhaus, *Partial Hölder continuity for minima of certain energies among maps into a Riemannian manifold*, Indiana Univ. Math. J. **37** (1988), 349–367.
10. ———, *Convergence of minimizers for the p-Dirichlet integral*, preprint, 1991.
11. C. B. Morrey, Jr., *Multiple integrals in the calculus of variations*, Springer-Verlag, New York, 1966.
12. R. Schoen and K. Uhlenbeck, *A regularity theory for harmonic maps*, J. Diff. Geom. **18** (1982), 307–335.
13. P. Tolksdorff, *Everywhere regularity for some quasi-linear systems with a lack of ellipticity*, Ann. Math. Pura Appl. **134** (1983), 241–261.

NATIONAL CHIAO TUNG UNIVERSITY
HSINCHU, TAIWAN