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RIEMANNIAN SUBMERSIONS WHICH PRESERVE THE EIGENFORMS OF THE LAPLACIAN

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Let $\pi\colon Z \to Y$ be a Riemannian submersion where Y and Z are closed Riemannian manifolds. Let $E(\lambda, \Delta_p^Y) \subset C^\infty \Lambda^p Y$ and $E(\lambda, \Delta_p^Z) \subset C^\infty \Lambda^p Z$ be the eigenspaces of the p form valued Laplacians on Y and on Z. We say the pullback

$$\pi^*\colon C^\infty \Lambda^p Y \to C^\infty \Lambda^p Z \tag{1}$$

preserves the p eigenforms of the Laplacian if for any $\lambda \in \mathbb{R}$, there exists $\mu(\lambda) \in \mathbb{R}$ so that

$$\pi^* E(\lambda, \Delta_p^Y) \subseteq E(\mu(\lambda), \Delta_p^Z); \tag{2}$$

in other words $\pi^*\Phi$ is an eigenform of Δ_p^Z, although with a possibly different eigenvalue, for every eigenform Φ of Δ_p^Y.

THEOREM 1. *The following conditions are equivalent*:

(a) *The fibers of π are minimal submanifolds.*
(b) $\Delta_0^Z \pi^* = \pi^* \Delta_0^Y$.
(c) *π^* preserves the eigenfunctions of the Laplacian Δ_0^Y.*

THEOREM 2. *The following conditions are equivalent*:

(a) *The fibers of π are minimal submanifolds and the horizontal distribution of π is integrable.*
(b) *For all $0 \le p \le \dim(Y)$, $\Delta_p^Z \pi^* = \pi^* \Delta_p^Y$.*
(c) *There exists p with $1 \le p \le \dim(Y)$ such that π^* preserves the p eigenforms of the Laplacian Δ_p^Y.*

These results deal with the totality of the eigenspaces; the following result deals with a single eigenform.

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THEOREM 3. (a) *If* $\pi: Z \to Y$ *is a Riemannian submersion and if there exists* $0 \neq F \in E(\lambda, \Delta_0^Y)$ *so that* $\pi^* F \in E(\mu, \Delta_0^Z)$, *then* $\mu = \lambda$.

(b) *If* $2 \leq p$, *then there exists a Riemannian submersion* $\pi: Z \to Y$ *and an eigenform* $\Phi \in E(\lambda, \Delta_p^Y)$ *so that* $\pi^*\Phi \in E(\mu, \Delta_p^Z)$ *for* $\lambda < \mu$.

Watson [9] proved that (a) and (b) of Theorem 1 are equivalent; see also [10] for related work. Goldberg and Ishihara [5] generalized Watson's result and showed (a) and (b) of Theorem 2 are equivalent; see also Y. Muto [8] for related results. Furutani [2] proved that maps which intertwine elliptic differential and even pseudo-differential operators are necessarily Riemannian submersions. Ishihara [6] characterized maps which preserve local harmonic functions. We refer to Bergery and Bourguignon [1] for a careful discussion of the relationship between the complete spectrum of Δ_0^Y and Δ_0^Z if the fibers of π are totally geodesic. The major result of this paper is the equivalence of (a,b) with (c) in Theorem 2. This shows that one can not in fact intertwine all the eigenspaces unless all the eigenvalues are equal; if π^* preserves the eigenforms of the Laplacian, then π^* intertwines the Laplacians and preserves the eigenvalues as well. We will use the Hopf fibration to show it is possible to change eigenvalues and prove Theorem 3; we refer to Y. Muto [7] for other examples. We acknowledge with gratitude helpful conversations with J. Leahy about this paper.

Establish the following notational conventions. Decompose the tangent bundle $TZ = V \oplus H$ into vertical and horizonal distributions where

$$V_z := \text{kernel}(\pi_*: T_zZ \to T_{\pi z}Y) \text{ and } H_z := V_z^\perp; \tag{3}$$

by assumption $\pi_*: H_z \to T_{\pi(z)}Y$ is an isometry. Dually, decompose the cotangent bundle $T^*Z = V^* \oplus H^*$ where

$$H_z^* := \text{image}(\pi^*: T_{\pi z}^*Y \to T_z^*Z) \text{ and } V_z^* := (H_z^*)^\perp. \tag{4}$$

We use the metric to identify the tangent and cotangent spaces henceforth. Let ρ^H be orthogonal projection on H.

If $\Phi \in C^\infty \Lambda Y$ is a p form on Y, let $\phi = \pi^*\Phi$ be the pull back to Z. Let indices $\{a, b\}$ range from 1 to $\dim(Y)$ and index local orthonormal frames $\{F_a\}$ for $T^*Y = TY$. Let $f_a = \pi^* F_a$ be the corresponding local orthonormal frames for $H^* = H$. Similarly, let indices $\{i, j\}$ range from $\dim(Y) + 1$ to $\dim(Z)$ and index local orthonormal frames $\{e_i\}$ for $V^* = V$. We adopt the Einstein convention and sum over repeated indices. Let $\Gamma^Y_{\dots}$ and $\Gamma^Z_{\dots}$ be the Christoffel symbols of the Levi-Civita connections on Y and on Z. The mean curvature vector is defined by

$$\theta := \rho^H(\nabla_{e_i} e_i) = \Gamma^Z_{iia} f_a; \tag{5}$$

we omit the usual normalizing factor of $\dim(V)^{-1}$ as it plays no role. Then θ vanishes if and only if the fibers of π are minimal; as noted in [1], this means the structure group of the fiber bundle reduces to the group of volume preserving diffeomorphisms of the fiber. Let

$$\omega = \omega_{abi} := \tfrac{1}{2}(\Gamma^Z_{abi} - \Gamma^Z_{bai}). \tag{6}$$

The tensor ω vanishes if and only if the horizontal distribution H is integrable.

Let d^Y and d^Z be exterior differentiation on Y and Z respectively;

$$\pi^* d^Y = d^Z \pi^* \tag{7}$$

by naturality. Let δ^Y and δ^Z be the adjoint operators on the spaces of smooth forms $C^\infty \Lambda Y$ and $C^\infty \Lambda Z$. Let ext$^\cdot$ denote left exterior multiplication; if α is a cotangent vector and if β is a differential form, then $\mathrm{ext}^\cdot(\alpha)\beta = \alpha \wedge \beta$. Let int$^\cdot$ denote the adjoint, left interior multiplication. Since π is a Riemannian submersion,

$$\begin{aligned}&\pi^* \circ \mathrm{ext}^Y(F) = \mathrm{ext}^Z(\pi^* F) \circ \pi^*, \text{ and}\\ &\pi^* \circ \mathrm{int}^Y(F) = \mathrm{int}^Z(\pi^* F) \circ \pi^*\end{aligned} \tag{8}$$

for all $F \in T^*Y$. If $\xi, \eta \in T^*Z$, then

$$\begin{aligned}|\xi|^2 &= \mathrm{ext}^Z(\xi)\mathrm{int}^Z(\xi) + \mathrm{int}^Z(\xi)\mathrm{ext}^Z(\xi)\\ 0 &= \mathrm{ext}^Z(\xi)\mathrm{ext}^Z(\eta) + \mathrm{ext}^Z(\eta)\,\mathrm{ext}^Z(\xi)\\ 0 &= \mathrm{int}^Z(\xi)\mathrm{int}^Z(\eta) + \mathrm{int}^Z(\eta)\mathrm{int}^Z(\xi).\end{aligned} \tag{9}$$

We shall need the following technical lemma to prove Theorems 1 and 2; although it follows from the work of [5, 9], we shall give a proof to keep this article self-contained and also to establish some necessary notation.

LEMMA 4. *Let $\Omega = \omega_{abi} ext^Z(e_i) int^Z(f_a) int^Z(f_b)$. Then*

$$\delta^Z \pi^* = \pi^* \delta^Y + int^Z(\theta)\pi^* + \Omega\pi^*.$$

Proof. We expand

$$\delta^Z = -\mathrm{int}^Z(e_i)\nabla^Z_{e_i} - \mathrm{int}^Z(f_a)\nabla^Z_{f_a}; \tag{10}$$

see for example Gilkey [3] for details. Let $\{F_A\}$ be the corresponding local orthonormal frame for ΛY where $A = \{a_1 < \cdots < a_p\}$ is a multi-index. We expand $\Phi \in C^\infty \Lambda Y$ in the form $\Phi = \Phi_A F_A$. Then

$$\begin{aligned}\delta^Z \pi^* \Phi = &-\mathrm{int}^Z(e_i)\{e_i(\pi^*\Phi_A)\pi^* F_A + \Gamma^Z_{iab}\,\mathrm{ext}^Z(f_b)\mathrm{int}^Z(f_a)\pi^*\Phi\} &(11.1)\\ &-\mathrm{int}^Z(e_i)\Gamma^Z_{iaj}\mathrm{ext}^Z(e_j)\mathrm{int}^Z(f_a)\pi^*\Phi &(11.2)\\ &-\mathrm{int}^Z(f_a)\{f_a(\pi^*\Phi_A)\pi^* F_A + \Gamma^Z_{abc}\mathrm{ext}^Z(f_c)\mathrm{int}^Z(f_b)\pi^*\Phi\} &(11.3)\\ &-\mathrm{int}^Z(f_a)\Gamma^Z_{abi}\mathrm{ext}^Z(e_i)\,\mathrm{int}^Z(f_b)\pi^*\Phi. &(11.4)\end{aligned}$$

Since horizontal covector fields are annihilated by $\mathrm{int}^Z(e_i)$, (11.1) vanishes. Furthermore in (11.2), $i = j$ so

$$(11.2) = \Gamma^Z_{iia}\mathrm{int}^Z(f_a)\pi^*\Phi = \mathrm{int}^Z(\theta)\pi^*\Phi. \tag{12}$$

Since $\Gamma^Y_{abc} = \Gamma^Z_{abc}$. (11.3) yields $\pi^*\delta^Y$. Finally, we anti-symmetrize to see

$$(11.4) = \omega_{abi}\ \mathrm{ext}^Z(e_i)\mathrm{int}^Z(f_a)\mathrm{int}^Z(f_b)\pi^*\Phi. \tag{13}$$

□

Proof of Theorem 1. Since Ω vanishes on $\Lambda^1 Z$ and $\Delta_0 = \delta d$, by Lemma 4,

$$\Delta_0^Z\pi^* - \pi^*\Delta_0^Y = \mathrm{int}^Z(\theta)d^Z\pi^* \tag{14}$$

where we regard θ as a horizontal 1 form; let Θ be the dual horizontal vector field. We use (14) to see (a) implies (b) in Theorem 1. It is immediate that (b) implies (c). Suppose (c) holds in Theorem 1. Let $\epsilon(\lambda) := \mu(\lambda) - \lambda$. Then

$$\mathrm{int}^Z(\theta)d^Z\pi^*\Phi = \epsilon(\lambda)\pi^*\Phi\ \forall\lambda, \forall\Phi \in E(\lambda, \Delta_0^Y). \tag{15}$$

Let $d\nu^Z$ be the Riemannian measure on Z and let $\phi = \pi^*\Phi$. Since $\mathrm{int}^Z(\theta)\phi$ is the pointwise inner product of θ with ϕ,

$$\begin{aligned}\epsilon(\lambda)\int_Z \phi^2 d\nu^Z &= \int_Z (\theta, d^Z\phi)\phi d\nu^Z = \tfrac{1}{2}\int_Z (\theta, d^Z(\phi^2))d\nu^Z\\ &= \tfrac{1}{2}\int_Z \phi^2\delta^Z\theta d\nu^Z.\end{aligned} \tag{16}$$

Let $||\delta^Z\theta||_\infty$ be the maximum value of $\delta^Z\theta$. By (16),

$$|\epsilon(\lambda)|\int_Z \phi^2 d\nu^Z \le \tfrac{1}{2}||\delta^Z\theta||_\infty \int_Z \phi^2 d\nu^Z. \tag{17}$$

Thus $|\epsilon(\lambda)| \le \frac{1}{2}||\delta^Z\theta||_\infty$ is uniformly bounded. Expand $\Psi \in C^\infty(Y)$ in a Fourier series $\Psi = \Sigma c_\lambda\Phi_\lambda$ for $\Phi_\lambda \in E(\lambda, \Delta_0^Y)$. Then

$$||\Psi||^2_{L^2(Y)} = \Sigma c_\lambda^2. \tag{18}$$

Let $\mathcal{E}(\Psi) = \Sigma\epsilon(\lambda)c_\lambda\Phi_\lambda$. Then

$$||\mathcal{E}(\Psi)||^2_{L^2(Y)} = \Sigma\epsilon(\lambda)^2 c_\lambda^2 \le \tfrac{1}{4}||\delta^Z\theta||^2_\infty||\Psi||^2_{L^2(Y)} \tag{19}$$

so $\mathcal{E}$ is a bounded operator on $L^2(Y)$. Furthermore

$$\Theta(\pi^*\Psi) = (\theta, d^Z\pi^*\Psi) = \pi^*\mathcal{E}(\Psi). \tag{20}$$

Since π^* is a continuous operator from $L^2(Y)$ to $L^2(Z)$,

$$\begin{aligned} ||\Theta(\pi^*\Psi)||_{L^2(Z)} &= ||\pi^*\mathcal{E}\Psi||_{L^2(Z)} \le C||\mathcal{E}\Psi||_{L^2(Y)} \\ &\le C||\delta^Z\theta||_\infty ||\Psi||_{L^2(Y)} \;\forall\Psi \in C^\infty(Y). \end{aligned} \tag{21}$$

Such an estimate is not possible for a non-trivial horizontal vector field Θ. We complexify and let $\Psi_n = e^{in\Phi}$ for $\Phi \in C^\infty(Y)$. Then

$$\begin{aligned} n||\Theta(\pi^*\Phi)||_{L^2(Z)} &= ||\Theta(\pi^*\Psi_n)||_{L^2(Z)} \le C||\delta^Z\theta||_\infty ||\Psi_n||_{L^2(Y)} \\ &= C||\delta^Z\theta||_\infty \mathrm{vol}(Y) \;\forall n \in \mathbb{N} \end{aligned} \tag{22}$$

This shows $\Theta(\pi^*\Phi) = 0$ for all $\Phi \in C^\infty(Y)$ so $\Theta \equiv 0$ and dually $\theta \equiv 0$. □

Proof of Theorem 2. Suppose (a) holds. Then $\delta^Z\pi^* = \pi^*\delta^Y$ so $\Delta_p^Z\pi^* = \pi^*\Delta_p^Y$ for all p and (b) holds. Clearly (b) implies (c). Suppose (c) holds for some p with $1 \le p \le \dim(Y)$. Let $\epsilon(\lambda) := \mu(\lambda) - \lambda$. Then

$$(\Delta_p^Z\pi^* - \pi^*\Delta_p^Y)\Phi = \epsilon(\lambda)\pi^*\Phi \;\forall\lambda, \forall\Phi \in E(\lambda, \Delta_p^Y). \tag{23}$$

We use Lemma 4 to get the expansion

$$\Delta_p^Z\pi^* - \pi^*\Delta_p^Y = d^Z(\mathrm{int}^Z(\theta) + \Omega)\pi^* + (\mathrm{int}^Z(\theta) + \Omega)d^Z\pi^*. \tag{24}$$

Let $\rho^\perp$ be orthogonal projection on the orthogonal complement of $\Lambda H^* = \pi^*\Lambda Y$;

$$\rho^\perp(\Delta_p^Z\pi^* - \pi^*\Delta_p^Y)\Phi = \rho^\perp\epsilon(\lambda)\pi^*\Phi = 0 \;\forall\Phi \in E(\lambda, \Delta_p^Y). \tag{25}$$

Since the span of the eigenforms of Δ_p^Y is dense in $C^\infty\Lambda^pY$, (25) holds for all $\Phi \in C^\infty\Lambda^pY$. Fix $z_0 \in Z$ and choose $\Phi \in C^\infty\Lambda^pY$ so $\Phi(\pi z_0) = 0$. We apply (25) and suppress all terms which do not involve derivatives of Φ since $\pi^*\Phi(z_0) = 0$. Then

$$\begin{aligned} 0 &= \rho^\perp(\Delta_p^Z\pi^* - \pi^*\Delta_p^Y)\Phi \\ &= \{\rho^\perp(\mathrm{int}^Z(\theta)\mathrm{ext}^Z(e_i) + \mathrm{ext}^Z(e_i)\,\mathrm{int}^Z(\theta))\nabla_{e_i}\pi^*\Phi && (26.1)\\ &\quad +\rho^\perp(\mathrm{int}^Z(\theta)\,\mathrm{ext}^Z(f_a) + \mathrm{ext}^Z(f_a)\,\mathrm{int}^Z(\theta))\nabla_{f_a}\pi^*\Phi && (26.2)\\ &\quad +\rho^\perp(\mathrm{ext}^Z(e_i)\Omega + \Omega\,\mathrm{ext}^Z(e_i))\nabla_{e_i}\pi^*\Phi && (26.3)\\ &\quad +\rho^\perp(\mathrm{ext}^Z(f_a)\Omega + \Omega\,\mathrm{ext}^Z(f_a))\nabla_{f_a}\pi^*\Phi\}(z_0). && (26.4) \end{aligned}$$

At z_0, the vertical covariant derivatives of Φ vanish so (26.1) and (26.3) vanish. At z_0, (26.2) is horizontal so it is annihilated by $\rho^\perp$ and vanishes. We use the commutation relations of (9) to see that

$$\mathrm{ext}^Z(f_a)\Omega + \Omega\mathrm{ext}^Z(f_a) = -2\omega_{abi}\mathrm{ext}^Z(e_i)\mathrm{int}^Z(f_b). \tag{27}$$

This shows that

$$\{\omega_{abi}\mathrm{ext}^Z(e_i)\mathrm{int}^Z(f_b)\nabla_{f_a}\pi^*\Phi\}(z_0) = 0. \tag{28}$$

Since $p \geq 1$, $\omega_{abi} \equiv 0$ and the horizontal distribution H is integrable.

Let

$$\mathcal{F} := \mathrm{int}^Z(\theta)d^Z + d^Z\mathrm{int}^Z(\theta) - \nabla^Z_\Theta. \tag{29}$$

Then $\mathcal{F}$ is a 0^{th} order operator which is bounded on $L^2(Z)$ and

$$\begin{aligned}\Delta^Z_p\pi^*\Phi - \pi^*\Delta^Y_p\Phi &= \nabla^Z_\Theta\pi^*\Phi + \mathcal{F}\pi^*\Phi \\ &= \epsilon(\lambda)\pi^*\Phi \;\forall\lambda \in \mathbb{R}, \forall\Phi \in E(\lambda, \Delta^Y_p).\end{aligned} \tag{30}$$

Let $\phi = \pi^*\Phi$. We show $|\epsilon(\lambda)|$ is uniformly bounded by computing

$$\begin{aligned}|\epsilon(\lambda)| \cdot ||\phi||^2_{L^2(Z)} &\leq ||\mathcal{F}|| \cdot ||\phi||^2_{L^2(Z)} + |(\nabla^Z_\Theta\phi, \phi)_{L^2(Z)}| \\ |(\nabla^Z_\Theta\phi, \phi)_{L^2(Z)}| &= \tfrac{1}{2}|\textstyle\int_Z\Theta(\phi, \phi)d\nu^Z| = \tfrac{1}{2}|\textstyle\int_Z(\theta, d^Z(\phi, \phi))d\nu^Z| \\ &= \tfrac{1}{2}|\textstyle\int_Z\delta^Z\theta \cdot (\phi, \phi)d\nu^Z| \leq \tfrac{1}{2}||\delta^Z\theta||_\infty\textstyle\int_Z(\phi, \phi)d\nu^Z.\end{aligned} \tag{31}$$

Therefore $\nabla_\Theta\pi^*$ is a bounded operator from $L^2(\Lambda^pY)$ to $L^2(\Lambda^pZ)$; the same argument as that given to prove Theorem 1 then shows $\theta = 0$. □

Proof of Theorem 3. Assume there exists $F \in C^\infty(Y)$ so that

$$\Delta^Y_0 F = \lambda F \text{ and } \Delta^Z_0\pi^*F = \mu\pi^*F \tag{32}$$

for $\lambda \neq \mu$. We use Lemma 4 to see that

$$(\mu - \lambda)\pi^*F = \Delta^Z_0\pi^*F - \pi^*\Delta^Y_0 F = \mathrm{int}^Z(\theta)d^Z\pi^*F. \tag{33}$$

Choose $y_0 \in Y$ so that $F(y_0)$ is the minimal value of F and let $\pi z_0 = y_0$. Then $(d^Y F)(y_0) = 0$. This implies $(d^Z\pi^*F)(z_0) = 0$ and hence $(\mathrm{int}^Z(\theta)d^Z\pi^*F)(z_0) = 0$. Since $(\lambda - \mu) \neq 0$, we conclude $0 = \pi^*F(z_0) = F(y_0)$. This shows $F \geq 0$. A similar argument shows $F \leq 0$ and hence $F \equiv 0$; this completes the proof of Theorem 3 (a).

To prove Theorem 3 (b), we construct an example. Suppose first $p = 2$. The Hopf fibration π_h: $S^3 \to CP^1$ is a Riemannian submersion. The fibers of π_h are great circles; these are totally geodesic submanifolds of S^3 and hence minimal so $\theta = 0$. Let χ be the volume element of CP^1; χ is a non-trivial harmonic 2-form on CP^1 generating the second cohomology group. Let e be a global orthonormal section to the vertical distribution V. The transitive action of the unitary group $U(2)$ on S^3 preserves all the structures. This implies there exists a constant c_0 so

$$\Omega = c_0 \cdot \mathrm{ext}^{S^3}(e) \cdot \mathrm{int}^{S^3}(\pi_h^*\chi). \tag{34}$$

Since $d^{S^3}\pi_h^*\chi = \pi_h^* d^{CP^1}\chi = 0$ and $\Delta_2^{CP^1}\chi = 0$, we use Lemma 4 to see

$$\Delta_2^{S^3}\pi_h^*\chi = d^{S^3}(\Omega\pi_h^*\chi) = c_0 d^{S^3}(e). \tag{35}$$

Furthermore,

$$d^{S^3}e = \Gamma_{aib}\,\mathrm{ext}^{S^3}(f_a)\mathrm{ext}^{S^3}(f_b) = c_0\pi_h^*\chi. \tag{36}$$

This shows that $\Delta_2^{S^3}\pi_h^*\chi = c_0^2\pi_h^*\chi$. Since $H^2(S^3) = 0$, there are no harmonic 2-forms on S^3 so $c_0^2 \neq 0$. This completes the proof of (b) if $p = 2$.

If $N = N_1 \times N_2$ is a Riemannian product manifold, we may decompose as follows:

$$\begin{aligned} \Lambda^p N &= \oplus_{r+s=p}\Lambda^r N_1 \otimes \Lambda^s N_2, \text{ and} \\ \Delta_p^N &= \oplus_{r+s=p}(\Delta_r^{N_1} \otimes 1 + 1 \otimes \Delta_s^{N_2}). \end{aligned} \tag{37}$$

Let $p = 2+q$ for $q > 0$ and let M be any compact Riemannian manifold of dimension at least q. Let $Z = S^3 \times M$, $Y = CP^1 \times M$, and $\pi = \pi_h \times \mathrm{id}$. Let $\Phi_q \in E(\lambda, \Delta_q^M)$ be any q eigenform on M. Then by (37),

$$\begin{aligned} \Delta_p^Y(\chi \wedge \Phi_q) &= \Delta_2^{CP^1}(\chi) \wedge \Phi_q + \chi \wedge \Delta_q^M \Phi_q = \lambda\chi \wedge \Phi_q, \\ \Delta_p^Z(\pi^*(\chi \wedge \Phi_q)) &= \Delta_2^{S^3}(\pi_h^*\chi) \wedge \Phi_q + \pi_h^*\chi \wedge \Delta_q^M \Phi_q \\ &= (c_0^2 + \lambda)\pi^*(\chi \wedge \Phi_q). \end{aligned} \tag{38}$$

□

REMARK. In the proof of Theorem 3 (b), Z is the circle bundle associated to a vector bundle of rank 2 over Y. In [4], we show the eigenvalues do not change if Z is the sphere bundle associated to a vector bundle of rank at least 3 over Y.

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