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COANALYTIC FAMILIES OF NORMS ON A SEPARABLE BANACH SPACE

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Introduction

We define a standard Borel structure on the set of all equivalent norms of a separable Banach space through the Effros-Borel structure on the closed subsets of this space. In this frame, R. Kaufman has shown, using tools from harmonic analysis, that the set of rotund norms on $c_0(\mathbb{N})$ is a coanalytic non-Borel set ([K1]). Here, we show by straight geometric methods that an analytic set which contains the norms which are uniformly rotund in every direction (URED) on an infinite-dimensional Banach space Y with a basis contains a norm which is not rotund, and as a corollary we obtain that the set of URED norms is coanalytic non-Borel. It follows that if Y is an infinite dimensional separable Banach space, then the set of rotund norms is coanalytic non-Borel. Thus we obtain that the set of the Gateaux-differentiable norms on a reflexive separable infinite-dimensional Banach space is coanalytic non-Borel.

In the first section, we define a norm $|||\cdot|||$ on $c_0(\mathbb{N})$ which is uniformly rotund in every direction but one. In the second section, following similar lines as in the construction of the James tree space ([J], or see [LS]), to every tree θ on $\mathbb{N}$, we associate a Banach space $E(\theta)$, isomorphic to $c_0(\mathbb{N})$ and such that every branch supports a copy of a segment in the unit sphere of $(c_0(\mathbb{N}), |||\cdot|||)$. If θ is well founded, the norm of $E(\theta)$ is shown to be URED, and if not, it is not rotund. In the third section, we deduce our main results.

We refer to [K2] and [D-G-S] for related results.

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Notation. Let X be a Banach space. We will denote by B_X the closed unit ball of X. If $A \subseteq X$, then $\operatorname{conv}(A)$ denotes its convex hull, $sp_{\mathbf{Q}}(A)$ the $\mathbf{Q}$-vector space spanned by A, $sp(A)$ the vector space spanned by A, $\overline{\operatorname{conv}}(A)$ and $\overline{sp}(A)$ their closures. We will denote by A^{ω} and $A^{<\omega}$ the set of all sequences and the set of all finite sequences in A. By "norm" on X, we always mean equivalent norm. We refer to [K-L] for the definitions of trees, height. We denote by $\mathbb{N}$ the set $\{0, 1, 2, \ldots\}$ and by $\mathbb{N}^*$ the set $\mathbb{N} \setminus \{0\}$. The tree $\omega^{<\omega}$ of finite sequences in $\mathbb{N}$ will be denoted T. The set of trees on $\mathbb{N}$, that is, the set of subtrees of T, is denoted $\mathcal{T}$. A branch of a tree θ means $\sigma \in \omega^{\omega}$ such that $s \in \theta$ if $s \prec \sigma$. The set of well founded trees is denoted

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WF. An interval $[s, t]$ in T, with $s \in T$ and $t \in T$, is the set of the sequences w in T with $s \preceq w \preceq t$. We define a total order on T as follows: for $s \in T$, of length $|s|$, let $\sum(s)$ be the sum of its elements. Then

$$s < s' \text{ if } |s| + \sum(s) < |s'| + \sum(s') \text{ or if } |s| + \sum(s) = |s'| + \sum(s'),$$

and if s is strictly less than s' in the lexicographical order.

This order determines a strictly increasing bijection $s \mapsto \overline{s}$ from T onto $\mathbb{N}$. The inverse image of $n \in \mathbb{N}$ is denoted $\underline{s}_n$. We shall use this order for indexing bases.

We use the notation Σ_1^1 (resp. Π_1^1, Δ_1^1) for analytic (resp. coanalytic, Borel) set. A Π_1^1 which is not Δ_1^1 will be true Π_1^1 (see [K-L] for instance).

We recall that a norm $\|\cdot\|$ on a Banach space X is uniformly rotund in the direction $z \in X \setminus \{0\}$ if one of these two equivalent properties is true (see [D-G-Z], II, 6.1 and 6.2):

(i) If $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are two sequences in X, such that $\lim \|x_n + y_n\| = 2$, and for any $n \in \mathbb{N}$, $\|x_n\| = \|y_n\| = 1$ and $x_n - y_n \in sp(z)$, then $\lim \|x_n - y_n\| = 0$.

(ii) If $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are two sequences in X, and $(\lambda_n)_{n\in\mathbb{N}}$ a sequence in $\mathbb{R}$ such that $(x_n)_{n\in\mathbb{N}}$ is bounded,

$$\lim(2\|x_n\|^2 + 2\|y_n\|^2 - \|x_n + y_n\|^2) = 0,$$

and for any $n \in \mathbb{N}$, $x_n - y_n = \lambda_n z$, then $\lim \lambda_n = 0$.

If the norm is uniformly rotund in every direction $z \in X \setminus \{0\}$, the norm is said to be uniformly rotund in every direction (URED).

Of course, if a norm is URED, it is rotund.

1. Construction and properties of a norm on $c_0(\mathbb{N})$

We denote by E the Banach space $c_0(\mathbb{N})$ equipped with the equivalent rotund norm

$$\|(x(i))_{i\in\mathbb{N}}\| = \sup_{i\in\mathbb{N}} |x(i)| + \left(\sum_{i\in\mathbb{N}} \frac{1}{2^i} x(i)^2\right)^{\frac{1}{2}}.$$

We denote by $(e_i)_{i\in\mathbb{N}}$ the normalized basis of E obtained from the canonical basis of $c_0(\mathbb{N})$. This basis is 1-unconditional.

In this basis a vector x is written $x = \sum_{i\in\mathbb{N}} x(i)e_i$. We define the set

$$E^+ = \{x \in E; \forall\, i \in \mathbb{N}, x(i) > 0\}.$$

If $p \in \mathbb{N}$, let $E_p = sp\{e_i; i \leq p\}$, and π_p the natural projection on E_p.

We shall define two vectors $x_1, x_2 \in E^+$, and show the existence of a countable family $\mathcal{F} = \{f_n; n \in \mathbb{N}^*\}$ in E^*, which separates two vectors of E as soon as their difference is not in $sp(x_2 - x_1)$, and such that, for any $f \in \mathcal{F}$, $f(x_1) = f(x_2) = 1$, $\|f\| < 1$, and $f(e_i) > 0$ if $i \in \mathbb{N}$. Then we shall define on E an equivalent norm which is uniformly rotund in every direction except in the direction $x_2 - x_1$.

We start with a few lemmas.

Let $x_0 \in E^+$ be such that $(x_0(i))_{i\in\mathbb{N}} \in \ell_1(\mathbb{N}), x_0(0) > 1$, hence $\|x_0\| > 1$, and $y_0 = \frac{x_0}{\|x_0\|} \in E^+$. Let $g_0 \in E^*$ be such that $\|g_0\| = g_0(y_0) = 1$, and $f_0 = \frac{1}{\|x_0\|} g_0$.

LEMMA 1. (i) *If* $i \in \mathbb{N}$, $f_0(e_i) > 0$.

(ii) *There exists* $u \in E$ *such that* $x_1 = x_0 - u$ *and* $x_2 = x_0 + u$ *are in* E^+, *and such that for* $p \in \mathbb{N}$, $u \notin E_p$, *and if* x *is in the segment* $[x_1, x_2]$, *then* $x \in E^+, x(0) = x_0(0) > 1$, $f_0(x) = 1$ *and* $(x(i))_{i\in\mathbb{N}} \in \ell_1(\mathbb{N})$.

Proof. (i) Let $i \in \mathbb{N}$, and assume $g_0(e_i) \leq 0$. Since $y_0 \in E^+$,

$$g_0(y_0 - y_0(i)e_i) \geq g_0(y_0) = 1,$$

then

$$\|y_0 - y_0(i)e_i\| \geq 1.$$

Since $(e_i)_{i\in\mathbb{N}}$ is 1-unconditional, we have equality, and

$$g_0(y_0 - y_0(i)e_i) = g_0(y_0) = 1;$$

thus

$$g_0\left(\frac{1}{2}[(y_0 - y_0(i)e_i) + y_0]\right) = 1.$$

Hence

$$\left\|\frac{1}{2}[(y_0 - y_0(i)e_i) + y_0]\right\| = 1$$

which is a contradiction, because the norm $\|\cdot\|$ is rotund. Thus $g_0(e_i) > 0$, and

$$f_0(e_i) = \frac{1}{\|x_0\|} g_0(e_i) > 0.$$

(ii) Let $u(0) = 0$, and for $j \in \mathbb{N}^*$,

$$\begin{aligned} u(2j-1) &= \frac{1}{2^j \|f_0\|} \inf(x_0(2j-1), x_0(2j), 1) f_0(e_{2j}) \\ u(2j) &= \frac{-1}{2^j \|f_0\|} \inf(x_0(2j-1), x_0(2j), 1) f_0(e_{2j-1}). \end{aligned}$$

Since

$$\sum_{i\in\mathbb{N}} |u(i)| \le \sum_{j\in\mathbb{N}^*} \frac{1}{2^j\,\|f_0\|}\left(|f_0(e_{2j-1})| + |f_0(e_{2j})|\right)$$
$$\le \sum_{j\in\mathbb{N}^*} \frac{2}{2^j} = 2,$$

we have $(u(i))_{i\in\mathbb{N}} \in \ell_1(\mathbb{N})$, and $\sum_{i\in\mathbb{N}} u(i)e_i = u$ defines a vector of E.

Since $x_0 \in E^+$, and $f_0(e_i) > 0$ for all i, we have $u(i) \neq 0$ for all $i > 0$, and $u \notin E_p$ for all p. It is easily seen that $f_0(u) = 0$.

Let $x_1 = x_0 - u$, and $x_2 = x_0 + u$. Then

$$f_0(x_0) = f_0(x_1) = f_0(x_2) = 1,$$

and $x_1 \in E^+, x_2 \in E^+$. Indeed,

$$x_1(0) = x_2(0) = x_0(0) > 0,$$

and for $j > 0$,

$$|u(2j-1)| \le \frac{1}{2^j} x_0(2j-1)\frac{f_0(e_{2j})}{\|f_0\|} < x_0(2j-1),$$

$$|u(2j)| \le \frac{1}{2^j} x_0(2j)\frac{f_0(e_{2j-1})}{\|f_0\|} < x_0(2j),$$

then $|u(i)| < x_0(i)$ for all $i > 0$, and $x_1 \in E^+, x_2 \in E^+$. Since $(u(i))_{i\in\mathbb{N}}$ and $(x_0(i))_{i\in\mathbb{N}}$ are in $\ell_1(\mathbb{N})$, $(x_1(i))_{i\in\mathbb{N}}$ and $(x_2(i))_{i\in\mathbb{N}}$ are in $\ell_1(\mathbb{N})$ as well. Thus all our conditions are satisfied for $x \in [x_1, x_2]$. □

We denote by S the segment $[x_1, x_2]$ in E, and $A = \overline{\text{conv}}[B_E \cup \{\pm\pi_p(x_1), \pm\pi_p(x_2); p \in \mathbb{N}\}]$. Then we have $S \subseteq A$, and the Minkowski functional j_A of A is clearly an equivalent norm on E.

LEMMA 2. *There exists a countable family $\mathcal{F} = \{f_n; n \in \mathbb{N}^*\}$ in E^* such that*

(i) *If $y, z \in E$ are such that $z - y \notin sp(u)$, then for some $f \in \mathcal{F}$, $f(y) \neq f(z)$.*
(ii) *For any $f \in \mathcal{F}$, if $x \in S$, the $f(x) = 1$, and if $x \in A\setminus S$, then $-1 \le f(x) < 1$. Thus $j_A(x) = 1$ if $x \in S$.*
(iii) *For any $f \in \mathcal{F}$, $f(e_i) > 0$ for all $i \ge 0$.*

Let $\mathcal{G}$ be the family $\{g \in E^*; g(x_1) = g(x_2) = 1, \|g\| < 1, g(e_i) > 0$ for all $i \ge 0\}$.

We first show

FACT. *If $y, z \in E$ are such that $z - y \notin sp(u)$, then for some $g \in \mathcal{G}$ we have $g(y) \neq g(z)$.*

Proof. If $z - y \notin \ker f_0$, since $f_0 \in \mathcal{G}$, we can take $g = f_0$.

Assume $v = z - y \in \ker f_0$. We look for $h \in E^*$ such that $h(x_0) = 1, h(u) = 0, h(v) \neq 0$, and $h(e_i) > 0$ if $i \in \mathbb{N}$. The three vectors x_0, u and v are linearly independent, because $u, v \in \ker f_0$, $v \notin sp(u)$, and $x_0 \notin \ker f_0$. Then there exists $i_1, i_2, i_3 \in \mathbb{N}$ such that

$$\begin{vmatrix} x_0(i_1) & x_0(i_2) & x_0(i_3) \\ u(i_1) & u(i_2) & u(i_3) \\ v(i_1) & v(i_2) & v(i_3) \end{vmatrix} \neq 0.$$

The system

$$\begin{cases} \sum_{j=1}^{3} x_0(i_j)\xi_j = 0 \\ \sum_{j=1}^{3} u(i_j)\xi_j = 0 \\ \sum_{j=1}^{3} v(i_j)\xi_j = 1 \end{cases}$$

has a unique solution (ξ_1, ξ_2, ξ_3). Let $\alpha \neq 0$ be such that $f_0(e_{i_j}) - \alpha\xi_j > 0$ for $j \in \{1, 2, 3\}$.

We define $h \in E^*$ as follows:

For $j \in \{1, 2, 3\}, h(e_{i_j}) = f_0(e_{i_j}) - \alpha\xi_j$.

If $i \notin \{i_1, i_2, i_3\}, h(e_i) = f_0(e_i)$.

Thus $h(e_i) > 0$ if $i \in \mathbb{N}$. It is easily seen that:

$$\begin{aligned} h(x_0) - f_0(x_0) &= 0; \text{ thus } h(x_0) = 1. \\ h(u) - f_0(u) &= 0; \text{ thus } h(u) = 0. \\ h(v) - f_0(v) &= -\alpha; \text{ thus } h(v) \neq 0. \end{aligned}$$

If $\beta \in [0, 1)$, let $g_\beta = \beta f_0 + (1 - \beta)h$. If $\beta \to 1$, then $g_\beta \to f_0$, and $\|g_\beta\| \to \|f_0\| < 1$. Thus for some $\beta_0 \in [0, 1)$, $\|g_{\beta_0}\| < 1$, and $g = g_{\beta_0}$ clearly satisfies the required conditions. $\square$

We now come back to the proof of Lemma 2.

The set $\mathcal{G}$ is w^*-separable. Let $\mathcal{F} = \{f_n; n \in \mathbb{N}^*\}$ be a w^*-dense sequence in $\mathcal{G}$. If y and z are two vectors of E such that $z - y \notin sp(u)$, from the fact, the set $\{g \in \mathcal{G}; g(z - y) \neq 0\}$ is w^*-open, non empty, and thus (i) is satisfied, and (iii) follows from $\mathcal{F} \subseteq \mathcal{G}$.

It remains to show (ii). Let $f \in \mathcal{F}$. As $f(x_1) = f(x_2) = 1$, if $x \in S$, then $f(x) = 1$. For $j = 1, 2$, we have $x_j \in E^+$, and for $i \in \mathbb{N}$, $f(e_i) > 0$. Therefore, for $p \in \mathbb{N}$, we have

$$0 \leq f(\pi_p(x_j)) \leq f(x_j) = 1$$

and, as $\|f\| < 1$, if $x \in A$, then $-1 \leq f(x) \leq 1$.

Now let $x \in A$ be such that $f(x) = 1$. We are going to show that, for any $\varepsilon > 0$, there exists $z \in S$ such that $\|x - z\| \leq \varepsilon$. Since S is closed, that will show that $x \in S$, and (ii) holds.

Let $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that, if $p \geq N$, then $\left\|x_j - \pi_p(x_j)\right\| \leq \frac{\varepsilon}{2}$ for $j \in \{1, 2\}$.

Let ε_1 be such that $\varepsilon_1 < 1 - \|f\|$ and

$$0 < \varepsilon_1 < \inf\{x_j(i) f(e_i); i \leq N, j \in \{1, 2\}\}.$$

If $p \in \mathbb{N}$ is such that $f(\pi_p(x_j)) \geq 1 - \varepsilon_1$ for $j \in \{1, 2\}$, then $p \geq N$, and $\left\|x_j - \pi_p(x_j)\right\| \leq \frac{\varepsilon}{2}$.

The set

$$\{x_1, x_2\} \cup \{\pi_p(x_j); p \in \mathbb{N}, j \in \{1, 2\}\}$$

is compact; consequently so is the set

$$M_{\varepsilon_1} = \{x_1, x_2\} \cup \{\pi_p(x_j); p \in \mathbb{N}, j \in \{1, 2\}, f(\pi_p(x_j)) \geq 1 - \varepsilon_1\}.$$

We let

$$M'_{\varepsilon_1} = B_E \cup \{\pi_p(x_j); j \in \{1, 2\}, p \in \mathbb{N}, f(\pi_p(x_j)) \leq 1 - \varepsilon_1\}$$

$$\cup \{-\pi_p(x_j); j \in \{1, 2\}, p \in \mathbb{N}\}.$$

We have

$$A = \overline{\operatorname{conv}}(\overline{\operatorname{conv}}(M_{\varepsilon_1}) \cup \overline{\operatorname{conv}}(M'_{\varepsilon_1})),$$

and, as $\overline{\operatorname{conv}}(M_{\varepsilon_1})$ is compact,

$$A = \operatorname{conv}(\overline{\operatorname{conv}}(M_{\varepsilon_1}) \cup \overline{\operatorname{conv}}(M'_{\varepsilon_1})).$$

As $\varepsilon_1 < 1 - \|f\|$, if $y \in \overline{\operatorname{conv}}(M'_{\varepsilon_1})$, then $f(y) \leq 1 - \varepsilon_1$ because it is true if $y \in M'_{\varepsilon_1}$, and if $y \in \overline{\operatorname{conv}}(M_{\varepsilon_1})$, $1 - \varepsilon_1 \leq f(y) \leq 1$.

Since $f(x) = 1$, this implies $x \in \overline{\operatorname{conv}}(M_{\varepsilon_1})$. Pick

$$y = \sum_{i=1}^{m} \alpha_i \pi_{p_i}(x_1) + \beta_i \pi_{p_i}(x_2) \in \operatorname{conv}(M_{\varepsilon_1})$$

such that $\|y - x\| \le \frac{\varepsilon}{2}$, with, for $1 \le i \le m$, $\alpha_i \ge 0$, $\beta_i \ge 0$, $\pi_{p_i}(x_1) \in M_{\varepsilon_1}$, $\pi_{p_i}(x_2) \in M_{\varepsilon_1}$, and $\sum_{i=1}^m (\alpha_i + \beta_i) = 1$. Then

$$z = \left(\sum_{i=1}^m \alpha_i\right) x_1 + \left(\sum_{i=1}^m \beta_i\right) x_2 \in S,$$

and

$$\|x - z\| \le \|x - y\| + \left\| \sum_{i=1}^m [\alpha_i (x_1 - \pi_{p_i}(x_1)) + \beta_i (x_2 - \pi_{p_i}(x_2)] \right\|$$

$$\le \frac{\varepsilon}{2} + \sum_{i=1}^m (\alpha_i + \beta_i) \frac{\varepsilon}{2} = \varepsilon$$

and (ii) is proved. □

We define the norm $|||\cdot|||$ on E by

$$|||x|||^2 = \frac{1}{2} j_A(x)^2 + \sum_{n \in \mathbb{N}^*} \frac{1}{2^{n+1}} f_n(x)^2.$$

Then we have:

LEMMA 3. (i) *The norm* $|||\cdot|||$ *and the canonical norm of* $c_0(\mathbb{N})$ *are equivalent.*

(ii) *For* $p \in \mathbb{N}$ *and* $x \in S$, $|||x||| = 1$ *and* $\left|\left|\left|\pi_{p+1}(x)\right|\right|\right| > \left|\left|\left|\pi_p(x)\right|\right|\right|$.

(iii) *The norm* $|||\cdot|||$ *is uniformly rotund in every direction except in the direction of* $u = \frac{1}{2}(x_2 - x_1)$.

Proof. (i) Clear.

(ii) Let $x \in S$. By Lemma 2 (ii), we have $|||x||| = 1$. Let $p \in \mathbb{N}$. As $S \subseteq E^+$, and $f_n(e_i) > 0$ for $n \in \mathbb{N}^*$ and $i \in \mathbb{N}$, we have

$$f_n(\pi_{p+1}(x)) > f_n(\pi_p(x)),$$

and, as $(e_i)_{i \in \mathbb{N}}$ is $\|\cdot\|$-monotone, by the definition of A,

$$j_A(\pi_{p+1}(x)) \ge j_A(\pi_p(x)).$$

And thus

$$\left|\left|\left|\pi_{p+1}(x)\right|\right|\right| > \left|\left|\left|\pi_p(x)\right|\right|\right|.$$

(iii) Let $\xi \notin sp(u)$ be a vector of E, $n_0 \in \mathbb{N}$ be such that $f_{n_0}(\xi) \ne 0$, $(\lambda_m)_{m \in \mathbb{N}}$ be a sequence in $\mathbb{R}$, $(y_m)_{m \in \mathbb{N}}$ and $(z_m)_{m \in \mathbb{N}}$ be two sequences in E such that $\lim |||y_m + z_m||| =$

2, and, for any $m \in \mathbb{N}$, $|||y_m||| = |||z_m||| = 1$ and $y_m - z_m = \lambda_m \xi$. By Lemma 2 (ii), we have

$$
\begin{aligned}
|||y_m + z_m|||^2 &\leq \frac{1}{2}(j_A(y_m) + j_A(z_m))^2 + \sum_{n \in \mathbb{N}^*} \frac{1}{2^{n+1}}(f_n(y_m) + f_n(z_m))^2 \\
&= j_A(y_m)^2 + j_A(z_m)^2 - \frac{1}{2}(j_A(y_m) - j_A(z_m))^2 \\
&\quad + \sum_{n \in \mathbb{N}^*} \frac{1}{2^{n+1}}[2f_n(y_m)^2 + 2f_n(z_m)^2 - f_n(y_m - z_m)^2] \\
&\leq 4 - \frac{1}{2^{n_0+1}}[f_{n_0}(y_m - z_m)]^2 \\
&= 4 - \frac{\lambda_m^2}{2^{n_0+1}}(f_{n_0}(\xi))^2;
\end{aligned}
$$

thus $\lim \lambda_m = 0$ and we have (iii). □

2. Construction of the family $\{E(\theta); \theta \in \mathcal{T}\}$

In this section, to any $\theta \in \mathcal{T}$, we associate a Banach space $E(\theta)$, isomorphic to $c_0(\mathbb{N})$, and which has a URED norm if θ is well founded, and a non-rotund norm otherwise. The construction is inspired by the construction of the James tree space ([J] or see [L-S)]).

On the space $c_{00}(T)$ of the finitely supported functions from $T = \omega^{<\omega}$ to $\mathbb{R}$, we define the norm $\|\cdot\|_T$ by

$$
\|y\|_T^2 = \sup \left(\left|\left|\left| \sum_{s \prec b} y(s) e_{|s|} \right|\right|\right|^2 + \sum_{s \in b^*} \frac{c_{|s|}}{2^{s}} y(s)^2 \right)
$$

where we take the supremum on the branches b of T, where b^* is the complement in T of $\{s; s \prec b\}$ and where, for any $n \in \mathbb{N}^*$, c_n is in(0, 1], and satisfies

$$
0 < c_n \leq (\sup_{x \in S} x(n))^{-2} \inf_{x \in S} \left(|||\pi_n(x)|||^2 - |||\pi_{n-1}(x)|||^2 \right).
$$

According to Lemma 3 (ii), and since S is compact, such $c_n's$ exist. The space $E(T)$ is the closure of $c_{00}(T)$ in this norm.

If $y \in E(T)$, and if b is a branch of T, we let

$$
b(y) = \sum_{s \prec b} y(s) e_{|s|} \in E,
$$

$$b^*(y) = \left(\sum_{s\in b^*} \frac{c_{|s|}}{2^{\bar{s}}} y(s)^2\right)^{1/2}$$

$$\|y\|_b = \left(|||b(y)|||^2 + b^*(y)^2\right)^{1/2}.$$

Then

$$\|y\|_T = \sup\{\|y\|_b \,;\, b \text{ branch of } T\}.$$

If $s \in T$, $\chi_s \in c_{00}(T)$ is the characteristic function of $\{s\}$. If $V \subseteq T$, we denote by $E(V)$ the closure of the set $\{\chi_s; s \in V\}$ in the norm $\|\cdot\|_T$.

Then we have:

THEOREM 4. *Let $\theta \in \mathcal{T}$ be an infinite tree on $\mathbb{N}$. Then $E(\theta)$ is isomorphic to $c_0(\mathbb{N})$, and:*

If θ is well founded, the norm of $E(\theta)$ is URED.

If θ is not well founded, the norm of $E(\theta)$ is not rotund.

First, we show some properties of $E(\theta)$.

LEMMA 5. *(i) The sequence $(\chi_{\underline{s}_i}; i \in \mathbb{N})$ is equivalent to the canonical basis of $c_0(\mathbb{N})$.*

(ii) Let β be a branch of T, and for any $i \in \mathbb{N}$, let $\beta_i \in \omega^{<\omega}$ such that $|\beta_i| = i$ and $\beta_i \prec \beta$. If $x \in S$, then

$$\left\|\sum_{s\prec\beta} x(|s|)\chi_s\right\|_T = \left\|\sum_{i\in\mathbb{N}} x(i)\chi_{\beta_i}\right\|_T = |||x||| = 1.$$

In other words, S provides us with a segment on the unit sphere of $E(\{s; s \prec \beta\})$.

Proof. Let $(\mu_i)_{i=0}^n \in \mathbb{R}^{<\omega}$, and i_0 be such that $\sup_{0\leq i\leq n} |\mu_i| = |\mu_{i_0}|$, and $y = \sum_{i=0}^n \mu_i \chi_{\underline{s}_i}$.

The basis $(e_i)_{i\in\mathbb{N}}$ of $(E, |||\cdot|||)$ is equivalent to the canonical basis of $c_0(\mathbb{N})$; it is unconditional, and there exists $k > 0$ and $k' > 0$ independent of y such that for all branches b such that $\underline{s}_{i_0} \prec b$,

$$\frac{1}{k}|\mu_{i_0}| \leq \left|\left|\left|\mu_{i_0} e_{|\underline{s}_{i_0}|}\right|\right|\right| \leq k' \,|||b(y)||| \leq k' \,\|y\|_T$$

and for any branch b,

$$|||b(y)||| = \left|\left|\left|\sum_{s\prec b} \mu_{\bar{s}} e_{|s|}\right|\right|\right| \leq k|\mu_{i_0}|,$$

and

$$b^*(y) \leq \sqrt{2}|\mu_{i_0}|.$$

Therefore

$$\begin{aligned} \|y\|_T &= \sup\{(|||b(y)|||^2 + b^*(y)^2)^{1/2};\, b \text{ branch of } T\} \\ &\leq \sqrt{k^2+2}|\mu_{i_0}| \end{aligned}$$

and $\{\chi_{\underline{s}_i};\, i \in \mathbb{N}\}$ is equivalent to the canonical basis of $c_0(\mathbb{N})$.

We show (ii). If $b \neq \beta$ is a branch of T, the intersection $\{s \prec b\} \cap \{s \prec \beta\}$ is $\{\beta_0, \beta_1, ..., \beta_n\}$ for some $n \in \mathbb{N}$, and then

$$\left\|\sum_{i\in\mathbb{N}} x(i)\chi_{\beta_i}\right\|_T^2 = \sup_{n\in\mathbb{N}}\left(\left|\left|\left|\sum_{i=0}^{n} x(i)e_i\right|\right|\right|^2 + \sum_{i>n}\frac{c_i}{2^{\overline{\beta}_i}}x(i)^2\right).$$

Using the definition of the $c_i's$ we obtain

$$\left\|\sum_{i\in\mathbb{N}} x(i)\chi_{\beta_i}\right\|_T^2 = \left|\left|\left|\sum_{i\in\mathbb{N}} x(i)e_i\right|\right|\right|^2 = |||x|||^2 = 1. \qquad \square$$

If $y \in E(T)$, we denote by $\sum_{i\in\mathbb{N}} y(\underline{s}_i)\chi_{\underline{s}_i}$ its decomposition in the basis $(\chi_{\underline{s}_i};\, i \in \mathbb{N})$.

Proof of Theorem 4. Using Lemma 5 (i) and the definition of $E(\theta)$, we have $E(\theta)$ isomorphic to $c_0(\mathbb{N})$.

If θ is not well founded, there exists a branch β of θ. Then Lemma 5 (ii) shows that $E(\theta)$ is not rotund.

Let now $\theta \in \mathcal{T}$ be a well founded tree. We are going to show that the norm of $E(\theta)$ is URED. If not, there exists $(\lambda_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$, $(y_n)_{n\in\mathbb{N}}$ and $(z_n)_{n\in\mathbb{N}}$ two sequences in $E(\theta)$, a vector $v \in E(\theta) \setminus \{0\}$, and $\varepsilon > 0$ such that $\lim \left\|\frac{y_n+z_n}{2}\right\|_T = 1$, and for any $n \in \mathbb{N}$, $\lambda_n > \varepsilon$, $y_n - z_n = \lambda_n v$, and $\|y_n\|_T = \|z_n\|_T = 1$. Let $(b_n)_{n\in\mathbb{N}}$ be a sequence of branches of T such that $\lim \left\|\frac{y_n+z_n}{2}\right\|_{b_n} = 1$. Since, for $n \in \mathbb{N}$,

$$\begin{aligned} \left\|\frac{y_n+z_n}{2}\right\|_{b_n} &\leq \frac{1}{2}\left(\|y_n\|_{b_n} + \|z_n\|_{b_n}\right) \\ \|y_n\|_{b_n} &\leq \|y_n\|_T = 1 \\ \|z_n\|_{b_n} &\leq \|z_n\|_T = 1, \end{aligned}$$

we have

$$\lim \|y_n\|_{b_n} = \lim \|z_n\|_{b_n} = \lim \left\|\frac{y_n+z_n}{2}\right\|_{b_n} = 1. \tag{1}$$

Then we show:

LEMMA 6. *The set* supp $v = \{t \in \theta;\ v(t) \neq 0\}$ *is finite, and there exists* $N \in \mathbb{N}$ *such that for* $n \geq N$, supp $v \subseteq \{s;\ s \prec b_n\}$.

Proof. Let $t \in$ supp v. Assume that there exists a subsequence $(b_{n_m})_{m\in\mathbb{N}}$ of $(b_n)_{n\in\mathbb{N}}$ such that $t \notin \{s;\ s \prec b_{n_m}\}$ for any m. Then,

$$\begin{aligned}
|y_{n_m}(t) - z_{n_m}(t)| &= |\lambda_{n_m} v(t)| \geq \varepsilon |v(t)| \\
\left\| \frac{y_{n_m} + z_{n_m}}{2} \right\|_{b_{n_m}}^2 &= \left\| b_{n_m}\left(\frac{y_{n_m} + z_{n_m}}{2} \right) \right\|^2 + \sum_{s\in b_{n_m}^*} \frac{c_{|s|}}{2^{\bar{s}}} \left(\frac{y_{n_m}(s) + z_{n_m}(s)}{2} \right)^2 \\
&\leq \frac{1}{2} \Bigg(\left\| b_{n_m}(y_{n_m}) \right\|^2 + \left\| b_{n_m}(z_{n_m}) \right\|^2 \\
&\qquad + \sum_{s\in b_{n_m}^*} \frac{c_{|s|}}{2^{\bar{s}}} \left(y_{n_m}(s)^2 + z_{n_m}(s)^2 \right) \Bigg) \\
&\quad - \frac{c_{|t|}}{2^{\bar{t}}} \left(\frac{y_{n_m}(t) - z_{n_m}(t)}{2} \right)^2 \\
&\leq \frac{1}{2} \left(\left\| y_{n_m} \right\|_{b_{n_m}}^2 + \left\| z_{n_m} \right\|_{b_{n_m}}^2 \right) - \frac{c_{|t|}}{2^{\bar{t}}} \left(\varepsilon \frac{v(t)}{2} \right)^2 .
\end{aligned}$$

Then passing to the limit, we obtain

$$1 \leq 1 - \frac{c_{|t|}}{2^{\bar{t}}} \left(\varepsilon \frac{v(t)}{2} \right)^2$$

and this is a contradiction.

Therefore, if $t \in$ supp v, there exists $N(t) \in \mathbb{N}$ such that $t \prec b_n$ for $n \geq N(t)$.

Moreover, if $t' \in$ supp v, and if $n \geq \sup(N(t), N(t'))$, then $t \prec b_n$ and $t' \prec b_n$. As θ is well founded, supp v is finite, and for some $N \in \mathbb{N}$, if $n \geq N$, then supp $v \subseteq \{s;\ s \prec b_n\}$. □

We now come back to the proof of Theorem 4.

Let $\zeta = \sum_{s\in \text{supp } v} v(s) e_{|s|} \in E$. It belongs to E_p for some $p \in \mathbb{N}$, therefore by Lemma 1 (ii), $\zeta \notin sp(u)$. For $n \geq N$, we have, by Lemma 6,

$$b_n(y_n) - b_n(z_n) = \lambda_n \zeta$$

and

$$b_n^*(y_n) = b_n^*(z_n) = b_n^* \left(\frac{y_n + z_n}{2} \right).$$

Using (1), we have

$$\lim[2\,\|y_n\|_{b_n}^2 + 2\,\|z_n\|_{b_n}^2 - \|y_n + z_n\|_{b_n}^2] = 0,$$

then, with [D-G-Z], II.2.3,

$$\lim[2\,|||b_n(y_n)|||^2 + 2\,|||b_n(z_n)|||^2 - |||b_n(y_n) + b_n(z_n)|||^2] = 0.$$

Since the norm $|||\cdot|||$ on E is uniformly rotund in the direction ζ (Lemma 3, (iii)), we have

$$\lim \lambda_n = 0$$

and this is a contradiction with $\lambda_n > \varepsilon$.

Consequently, the norm of $E(\theta)$ is URED, and this concludes the proof of Theorem 4. □

3. Main results

In this section, we define a standard Borel structure on the set of all equivalent norms on a Banach space Y, and by Theorem 4 we show the announced results when Y is an infinite dimensional Banach space with a basis.

Let Y be a separable Banach space. The set of the equivalent norms on Y and the set $\mathcal{N}(Y)$ of the symmetric closed bounded convex sets with nonempty interior in Y are in one to one correspondance through the map which associates a norm with its unit ball. We shall identify a norm with its unit ball.

We equip the set $\mathcal{F}(Y)$ of the closed subsets of Y with the Effros-Borel structure (see [C]) about which we recall some facts.

The Effros Borel structure on the set of closed subsets of a Polish space P is a Borel structure defined from the Borel structure induced by the Hausdorff topology on the set of closed subsets of a compactification of P. The Effros Borel structure is standard, that is to say this Borel structure is generated by a Polish topology on $\mathcal{F}(P)$. If $(V_n)_{n\in\mathbb{N}}$ is a countable base for the topology of P, the family $\{\{F \in \mathcal{F}(P);\, F \cap V_n \neq \emptyset\};\, n \in \mathbb{N}\}$ generates the Effros Borel structure on $\mathcal{F}(P)$. This Borel structure is therefore independent of the compactification of P.

Then we have:

PROPOSITION 7. *The subset $\mathcal{N}(Y)$ is a Borel subset of $\mathcal{F}(Y)$.*

By this proposition, $\mathcal{N}(Y)$ is a standard Borel space, and there exists a Polish topology on $\mathcal{N}(Y)$ which generates the Borel structure induced by the Effros Borel structure.

This proposition is shown by classical techniques (see Annex 1).

If the dimension of Y is finite, we have

$$\begin{aligned}\{F \in \mathcal{N}(Y);\ F \text{ rotund}\} &= \{F \in \mathcal{N}(Y);\ F \textit{ URED}\}\\ &= \{F \in \mathcal{N}(Y);\ F \text{ uniformly convex}\}\end{aligned}$$

and this set is easily seen to be Δ^1_1.

The main result, obtained by the completness method (see [K-L, p. 110]) is:

THEOREM 8. *Let Y be an infinite dimensional Banach space with a basis, and $\mathcal{A} \subseteq \mathcal{N}(Y)$ a Σ^1_1 set of norms on Y, including all the URED norms. Then $\mathcal{A}$ contains a norm which is not rotund.*

COROLLARY 9. *Let Y be an infinite dimensional Banach space with a basis. The set of rotund norms on Y, and the set of URED norms are true Π^1_1.*

The result for the rotund norms will be extended to any infinite dimensional separable Banach space.

Let Y be an infinite dimensional Banach space with a basis. We shall assume, without loss of generality, that Y is equipped with a monotone normalized basis $\underline{y} = (y_i)_{i \in \mathbb{N}}$.

To any $\theta \in \mathcal{T}$, we are going to associate $F_\theta \in \mathcal{N}(Y)$, so that if $\theta \in WF$, then F_θ is URED, and if $\theta \notin WF$, then F_θ is not rotund.

For $j \in \mathbb{N}$, we denote by S_j the set $\{s;\ s \preceq \underline{s}_j\}$.

Let $\theta \in \mathcal{T}$ be an infinite tree on $\mathbb{N}$. The total order on θ induced by the total order on T defines a strictly increasing bijection from $\mathbb{N}$ onto θ. We denote by $\underline{s}_{\theta_i}$ the image of i by this bijection, and by $\overline{s}^\theta$ the inverse image of $s \in \theta$. For $j \in \mathbb{N}$, S^θ_j is $S_j \cap \theta$, and for $x \in E$, $S^\theta_j(x)$ is the vector in Y given by

$$S^\theta_j(x) = \sum_{s \in S^\theta_j} x(|s|) y_{\overline{s}^\theta}.$$

Then we define two sets in $\mathcal{N}(Y)$

$$C_\theta = \overline{\operatorname{conv}}(B_Y \cup \{\pm S^\theta_j(x_1), \pm S^\theta_j(x_2);\ j \in \mathbb{N}\})$$

$$F_\theta = \left\{ z \in Y;\ \frac{1}{2} j_{C_\theta}(z)^2 + \frac{1}{2} \left\| \sum_i z(i) \chi_{\underline{s}_{\theta_i}} \right\|_T^2 \leq 1 \right\}$$

where if $z \in Y$, $z = \sum_i z(i) y_i$.

Theorem 8 follows from our next lemma.

LEMMA 10. (i) *The map* $\varphi = \mathcal{T} \to \mathcal{N}(Y)$ *defined by* $\varphi(\theta) = F_\theta$ *is* Δ_1^1.
(ii) *If* $\theta \in WF$, F_θ *is URED.*

If $\theta \notin WF$, F_θ *is not rotund.*

Assume that this lemma is true, and let $\mathcal{A} \subseteq \mathcal{N}(Y)$ be a Σ_1^1 set containing the URED norms. Then $\varphi^{-1}(\mathcal{A})$ is Σ_1^1 by (i), and contains WF by (ii). As WF is not Σ_1^1 (see [K-L]), there exists $\theta \notin WF$ such that F_θ, which is not rotund by (ii), is in $\mathcal{A}$, and Theorem 8 follows. □

Proof of Corollary 9. First,$\{F \in \mathcal{N}(Y);\ F \text{ rotund}\}$ and $\{F \in \mathcal{N}(Y);\ F\ \textit{URED}\}$ are not Σ_1^1 by Theorem 8. By classical methods, one shows that they are Π_1^1 (see Annex 2). □

Proof of Lemma 10 (*i*). First we have the following result.

Fact 11. If $z = \sum_i z(i)y_i \in sp_{\mathbf{Q}}(\underline{y})$, the map $\psi^z = \mathcal{T} \to \mathbb{R}$ defined by

$$\psi^z(\theta) = \frac{1}{2} j_{C_\theta}(z)^2 + \frac{1}{2}\left\| \sum_i z(i)\chi_{\underline{s}_{\theta_i}} \right\|_T^2$$

is Δ_1^1.

We check this fact in Appendix 3.

Let $\mathcal{O}$ be open in Y. If $\theta \in \mathcal{T}$, as $\mathring{F}_\theta$ is not empty, we have $F_\theta \cap \mathcal{O} \neq \emptyset$ if and only if there exists $z \in sp_{\mathbf{Q}}(\underline{y})$ such that $z \in F_\theta \cap \mathcal{O}$. Thus

$$\{\theta \in \mathcal{T};\ F_\theta \cap \mathcal{O} \neq \emptyset\} = \cup_{z \in sp_{\mathbf{Q}}(\underline{y}) \cap \mathcal{O}} \{\theta;\ \psi^z(\theta) \leq 1\}.$$

By Fact 11, this set is Δ_1^1, and (i) follows. □

Proof of Lemma 10 (*ii*). Let $\theta \in WF$, $(z_n)_{n\in\mathbb{N}} \subseteq Y$, $(z'_n)_{n\in\mathbb{N}} \subseteq Y$, $(\lambda_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$, and $\zeta \in Y \setminus \{0\}$ such that $(z_n)_{n\in\mathbb{N}}$ is bounded, $z'_n - z_n = \lambda_n \zeta$ for any $n \in \mathbb{N}$, and

$$\lim \left(2j_{F_\theta}(z_n)^2 + 2j_{F_\theta}(z'_n)^2 - j_{F_\theta}(z_n + z'_{n^2})^2\right) = 0.$$

Since for $z \in Y$,

$$j_{F_\theta}(z)^2 = \frac{1}{2} j_{C_\theta}(z)^2 + \frac{1}{2}\left\| \sum_i z(i)\chi_{\underline{s}_{\theta_i}} \right\|_T^2,$$

by [D-G-Z], Fact II 2.3(ii), we have

$$\lim \left(2 \left\| \sum_i z_n(i) \chi_{\underline{s}_{\theta_i}} \right\|_T^2 + 2 \left\| \sum_i z'_n(i) \chi_{\underline{s}_{\theta_i}} \right\|_T^2 - \left\| \sum_i (z_n(i) + z'_n(i)) \chi_{\underline{s}_{\theta_i}} \right\|_T^2 \right) = 0$$

and since the norm of $E(\theta)$ is URED, we conclude that

$$\lim \lambda_n = 0.$$

Thus F_θ is URED.

Let $\theta \notin WF$, b a branch of θ, and $x \in S = [x_1, x_2]$. Since $(x(i))_{i \in \mathbb{N}}$ is in $\ell_1(\mathbb{N})$ (Lemma 1 (ii)), we can define

$$x_b = \sum_{s \prec b} x(|s|) y_{\overline{s}^\theta}.$$

Then

$$j_{F_\theta}(x_b)^2 = \frac{1}{2} j_{C_\theta}(x_b)^2 + \frac{1}{2} \left\| \sum_{s \prec b} x(|s|) \chi_s \right\|_T^2 .$$

Using Lemma 5 (ii), we have

$$\left\| \sum_{s \prec b} x(|s|) \chi_s \right\|_T = 1.$$

Moreover, $j_{C_\theta}(x_b) = 1$. Indeed, clearly $x_b \in C_\theta$, and it suffices to show that $x_b \notin \overset{\circ}{C}_\theta$. Assume $x_b \in \overset{\circ}{C}_\theta$. Then

$$x_b \in \operatorname{conv}(B_Y \cup \{\pm S_j^\theta(x_1), \pm S_j^\theta(x_2);\ j \in \mathbb{N}\}).$$

As $\underline{y}$ is monotone, and $x(0) > 1$,

$$x_b \in \operatorname{conv}(\{S_j^\theta(x_1), S_j^\theta(x_2);\ j \in \mathbb{N}\})$$

and this is a contradiction, because for any $i \in \mathbb{N}, x(i) \neq 0$. Thus $x_b \notin \overset{\circ}{C}_\theta$, and $j_{C_\theta}(x_b) = 1$. Therefore if $x \in S$, then $j_{F_\theta}(x_b) = 1$, and F_θ is not rotund. □

We now extend the result on rotund norms to any separable Banach space.

THEOREM 12. *Let Y be an infinite dimensional separable Banach space, and $\mathcal{A} \subseteq \mathcal{N}(Y)$ be a Σ_1^1 set of norms on Y including all the rotund norms. Then $\mathcal{A}$ contains a norm which is not rotund.*

Moreover, the set of rotund norms on Y is a true Π_1^1 set.

Proof. There exists $\underline{y} = (y_i)_{i\in\mathbb{N}}$ a basic sequence in Y ([L-T], 1.a. 5). Let $Y' = \overline{sp}(\underline{y})$, and for $F \in \mathcal{N}(Y)$, let $r(F) = F \cap Y' \in \mathcal{N}(Y')$.

Fact 13. The map $r \colon \mathcal{N}(Y) \to \mathcal{N}(Y')$ is Δ^1_1.

Indeed, let $\mathcal{O}$ be an open set of Y. Then

$$\{F \in \mathcal{N}(Y); r(F) \cap \mathcal{O} \neq \emptyset\} = \{F; \exists\, z \in sp_{\mathbf{Q}}(\underline{y}) \cap \mathcal{O}, z \in F\}.$$

Since this set is Δ^1_1, the fact follows.

Every rotund norm on a closed subspace of Y can be extended to an equivalent rotund norm on Y ([J-Z], see [D-G-Z], II.8.2). Therefore if $\mathcal{A} \subseteq \mathcal{N}(Y)$ is Σ^1_1 and contains the rotund norms, then $r(\mathcal{A})$ is Σ^1_1, and contains the rotund norms on Y', and by Theorem 8 contains a norm which is not rotund; thus $\mathcal{A}$ contains a non-rotund norm. The end of the proof is the same as that of Corollary 9. □

COROLLARY 14. *Let Y be an infinite dimensional separable reflexive Banach space. The set of the Gateaux-differentiable norms on Y is true Π^1_1.*

Proof. As Y is reflexive, a norm $F \in \mathcal{N}(Y)$ is Gateaux-differentiable if and only if the corresponding dual norm is rotund (see [D-G-Z], II.1.6).

Fact 15. Let Y be a separable reflexive Banach space. The map $\mathcal{D} \colon \mathcal{N}(Y) \to \mathcal{N}(Y^*)$ which, to a norm F on Y, associates its dual norm F^0, is a bijective borelian map.

Indeed, since Y is reflexive, this map is bijective. Let $\mathcal{O}$ be an open set in Y^*, $\underline{y}$ be a total sequence in Y, and $\underline{y}^*$ a total sequence in Y^*. Then

$$\begin{aligned}&\{F \in \mathcal{N}(Y);\ F^0 \cap \mathcal{O} \neq \emptyset\}\\ &\quad = \{F \in \mathcal{N}(Y); \exists\, y^* \in sp_{\mathbf{Q}}(\underline{y}^*) \cap \mathcal{O}, \forall\, y \in sp_{\mathbf{Q}}(\underline{y}), |y^*(y)| \leq 1 \text{ or } y \notin F\}\end{aligned}$$

and this set is Δ^1_1. Thus the map is Δ^1_1 and the fact follows.

As $\{G \in \mathcal{N}(Y^*); G \text{ rotund}\}$ is true Π^1_1 and image under $\mathcal{D}$ of $\{F \in \mathcal{N}(Y); F$ Gateaux-differentiable$\}$, this set is true Π^1_1 as well. □

Our work leads to the following:

Problem. Let Y be an infinite dimensional separable Banach space ; if a Σ^1_1 set in $\mathcal{N}(Y)$ contains the norms which are at the same time URED and locally uniformly rotund, does it contain a non-rotund norm?

Appendix 1

Proof of Proposition 7. Let $F \in \mathcal{F}(Y)$. We have the equivalence

$$F \in \mathcal{N}(Y) \Leftrightarrow \begin{cases} \text{(i)} & \exists\, M \in \mathbb{N},\, F \subseteq M.B_Y \\ \text{(ii)} & \exists\, m \in \mathbf{Q}^{*+},\, m.B_Y \subseteq F \\ \text{(iii)} & \forall\, x, y \in F,\, \dfrac{x+y}{2} \in F \\ \text{(iv)} & \forall\, x \in F,\, -x \in F \end{cases}$$

As the relation $\subseteq$ is Δ^1_1 in $\mathcal{F}(Y)$, the conditions (i) and (ii) are Δ^1_1.

Let $(O_m)_{m\in\mathbb{N}}$ be a countable basis of open sets of Y. The closed set F satisfies (iii) if and only if

$$\forall (m,n) \in \mathbb{N}^2, \qquad \left.\begin{array}{c} O_m \cap F \neq \emptyset \\ O_n \cap F \neq \emptyset \end{array}\right\} \quad \Longrightarrow \quad \frac{1}{2}(O_m + O_n) \cap F \neq \emptyset.$$

This is clear since $\{\frac{1}{2}(O_m + O_n);\, m, n \in \mathbb{N}, x \in O_m, y \in O_n\}$ is a basis of neighbourhoods of $\frac{x+y}{2}$. Therefore, the following set is Δ^1_1:

$\{F;\, F$ verifies (iii)$\}$

$$= \cap_{(m,n)\in\mathbb{N}^2}[\{F;\, \tfrac{1}{2}(O_m + O_n) \cap F \neq \emptyset\} \cup \{F;\, O_m \cap F = \emptyset\} \cup \{F;\, O_n \cap F = \emptyset\}].$$

Similarly, F verifies (iv) if and only if

$$\forall m \in \mathbb{N},\, O_m \cap F \neq \emptyset \Rightarrow (-O_m) \cap F \neq \emptyset$$

Then the following set is Δ^1_1:

$$\{F;\, F \text{ verifies (iv)}\} = \cap_{m\in\mathbb{N}}[\{F;\, (-O_m) \cap F \neq \emptyset\} \cup \{F;\, O_m \cap F = \emptyset\}].$$

Consequently, $\mathcal{N}(Y)$ is Δ^1_1. □

Appendix 2

Let Y be a separable Banach space. The following sets are Π^1_1:

$$\mathcal{N}_R = \{F \in \mathcal{N}(Y);\, F \text{ rotund}\}$$

$$\mathcal{N}_u = \{F \in \mathcal{N}(Y);\, F\ \mathit{URED}\}.$$

Proof. We have the equivalence

$$F \notin \mathcal{N}_R \Leftrightarrow \exists\, y, z \in Y,\, j_F(y) = j_F(z) = j_F\left(\frac{y+z}{2}\right) = 1.$$

Fact. The set $\{(F, y) \in \mathcal{N}(Y) \times Y;\ j_F(y) = 1\}$ is Δ_1^1.

Indeed, $j_F(y) = 1$ if and only if for any $\lambda \in \mathbf{Q}$ such that $\lambda > 1$, we have $y \in F$ and $\lambda y \notin F$. Since $\{(F, y);\ y \in F\}$ is Δ_1^1, the fact follows.

Then the complement of $\mathcal{N}_R$ is Σ_1^1 as projection of

$$\left\{(F, y, z) \in \mathcal{N}(Y) \times Y \times Y;\ j_F(y) = j_F(z) = j_F\left(\frac{y+z}{2}\right) = 1\right\},$$

and, this set is Δ_1^1 by the above. Therefore $\mathcal{N}_R$ is Π_1^1.

We have $F \notin \mathcal{N}_u$ if and only if there exists $\xi \in Y \setminus \{0\}$, $\varepsilon \in \mathbf{Q}^{*+}$,

$$(\lambda_n)_{n\in\mathbb{N}} \in \mathbb{R}^\omega,\ (y_n)_{n\in\mathbb{N}} \in Y^\omega,\ (z_n)_{n\in\mathbb{N}} \in Y^\omega$$

such that $\lim j_F(y_n + z_n) = 2$, $(y_n)_{n\in\mathbb{N}}$ is bounded, and for any $n \in \mathbb{N}$, $\lambda_n > \varepsilon$, and

$$j_F(y_n) = j_F(z_n) = 1,\ y_n - z_n = \lambda_n \xi.$$

By classic methods, it follows that $\mathcal{N}_u$ is Π_1^1. $\square$

Appendix 3

Proof of Fact 11. Let $(\xi_j)_{j\in\mathbb{N}}$ be an enumeration of $sp_{\mathbf{Q}}(\underline{y}) \cap B_Y$.

First, if $x \in E$ and $j \in \mathbb{N}$, the map $\mathcal{T} \to Y\colon \theta \mapsto S_j^\theta(x)$ is continuous. Indeed, if $(\theta^\ell)_{\ell\in\mathbb{N}}$ is a sequence of trees in $\mathcal{T}$ which converges towards θ, there exists $L \in \mathbb{N}$ such that if $\ell \geq L$ and $i \leq j$, $\underline{s}_i \in \theta$ if and only if $\underline{s}_i \in \theta^\ell$. Then if $\ell \geq L$, we have $S_j^{\theta^\ell}(x) = S_j^\theta(x)$.

Consequently, the map

$$\psi_1 = \mathcal{T} \to (Y^5)^\omega$$

defined by

$$\psi_1(\theta) = (\xi_j, S_j^\theta(x_1), S_j^\theta(x_2), -S_j^\theta(x_1), -S_j^\theta(x_2))_{j\in\mathbb{N}}$$

is continuous.

Next, the map

$$Y^\omega \to \mathcal{F}(Y)\colon (z_n)_{n\in\mathbb{N}} \mapsto \overline{\operatorname{conv}}(\{z_n;\ n \in \mathbb{N}\})$$

is Δ_1^1. Indeed, let $\mathcal{O}$ be an open set of Y. We have

$$\{(z_n)_{n\in\mathbb{N}} \in Y^\omega;\ \overline{\operatorname{conv}}(\{z_n;\ n \in \mathbb{N}\}) \cap \mathcal{O} \neq \emptyset\}$$

$$= \left\{(z_n)_{n\in\mathbb{N}} \in Y^\omega;\ \exists\ (\lambda_i)_i \in \mathbf{Q}^{<\omega},\ \sum_i \lambda_i = 1, \lambda_i \geq 0 \text{ for any } i, \text{ and } \sum_i \lambda_i z_i \in \mathcal{O}\right\}$$

and this set is Δ_1^1. Thus the map $\psi_2 = (Y^5)^\omega \to \mathcal{F}(Y)$ defined by

$$\psi_2[((z_j^k)_{k=1}^5)_{j\in\mathbb{N}}] = \overline{\text{conv}}\{z_j^k;\ 1 \le k \le 5,\ j \in \mathbb{N}\}$$

is Δ_1^1.

If $z \in Y$, the map $\psi_3^z = \mathcal{N}(Y) \to \mathbb{R}^+$ defined by $\psi_3^z(C) = j_C(z)$ is Δ_1^1. Indeed, if (a, b) is an interval in $\mathbb{R}^+$,

$$\{C \in \mathcal{N}(Y);\ j_C(z) \in (a, b)\} = \{C \in \mathcal{N}(Y);\ z \notin a.C \text{ and } z \in b.C\}$$

and this set is Δ_1^1.

Finally, if $i \in \mathbb{N}$, the map $\mathcal{T} \to E(T)$: $\theta \mapsto \chi_{\underline{s}_{\theta_i}}$ is continuous. Indeed, if $(\theta^\ell)_{\ell\in\mathbb{N}}$ converges towards θ, there exists $L \in \mathbb{N}$ such that if $\ell \geq L$, and if $j \leq \overline{s}_{\theta_i}$, then $\underline{s}_j \in \theta$ if and only if $\underline{s}_j \in \theta^\ell$. Therefore, if $\ell \geq L$, then $\underline{s}_{\theta_i^\ell} = \underline{s}_{\theta_i}$, and $\chi_{\underline{s}_{\theta_i^\ell}} = \chi_{\underline{s}_{\theta_i}}$.

Consequently, if $z = \sum_i z(i) y_i \in sp_{\mathbf{Q}}(\underline{y})$, the map $\psi_4^z = \mathcal{T} \to \mathbb{R}$ defined by

$$\psi_4^z(\theta) = \left\| \sum_i z(i) \chi_{\underline{s}_{\theta_i}} \right\|_T$$

is continuous.

Hence for $z \in sp_{\mathbf{Q}}(\underline{y})$, we have

$$\begin{aligned} \psi^z(\theta) &= \frac{1}{2} j_{C_\theta}(z)^2 + \frac{1}{2} \left\| \sum_i z(i) \chi_{\underline{s}_{\theta_i}} \right\|_T^2 \\ &= \frac{1}{2}[\psi_3^z \circ \psi_2 \circ \psi_1(\theta)]^2 + \frac{1}{2}[\psi_4^z(\theta)]^2, \end{aligned}$$

and the map ψ^z is Δ_1^1. □

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