ILLINOIS JOURNAL OF MATHEMATICS
Volume 40, Number 2, Summer 1996

GEOMETRIC EMBEDDINGS OF OPERATOR SPACES

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1. Basic facts

We denote typically by $\mathcal{A}$ a $\mathbf{C}^*$-algebra with 1 and by G the Banach Lie group of invertible elements of $\mathcal{A}$. Sometimes we assume that $\mathcal{A}$ is represented faithfully in a Hilbert space $\mathcal{K}$ and this representation may change when needed. The general reference for this sort of thing is [15].

Throughout we use *reductive homogeneous space of operators* (or simply reductive space) for the following type of data: a $\mathbf{C}^*$-algebra $\mathcal{A}$ is given and the group G or a convenient Lie subgroup of G acts on a $\mathbf{C}^\infty$-Banach manifold M in such a way that the isotropy groups of points in M are provided with stable infinitesimal supplements (the "horizontal spaces" of the reductive space) in the sense that the adjoint action of the isotropy groups leave these supplements stable (see [8] for the finite dimensional analogue and [9] for the case considered here). The horizontal spaces provide the *canonical connection* of the reductive space. Numerous examples of reductive spaces are described in [1], [2], [4], [12].

Given reductive spaces M, M' with groups G, G', a morphism from M to M' is a smooth map $\Psi\colon M \to M'$ together with a Lie group homomorphism $\psi\colon G \to G'$ with the equivariance $\Psi(L_g\varepsilon) = L_{\psi(g)}\Psi(\varepsilon)$ (we denote by L both actions) and also with the infinitesimal condition that the tangent map of ψ preserves the horizontal spaces. The groups G and G' operate on the space $\mathrm{Hom}(M, M')$ of morhpisms from M to M' by

$$\begin{aligned} \Psi_g &= L_g\Psi, \quad \psi_g = \mathrm{Ad}_g\,\psi, \qquad \text{for } g \in G', \\ \Psi^h &= \Psi L_h, \quad \psi^h = \psi\,\mathrm{Ad}_h, \qquad \text{for } h \in G. \end{aligned}$$

It is clear that these left actions commute and that $(\Psi_g)^h = (\Psi^h)_g = \Psi_{g\psi(h)}$, $(\psi_g)^h = (\psi^h)_g = \Psi_{g\psi(h)}$.

Denote by $Q \subset \mathcal{A}$ the set of reflections in $\mathcal{A}$, i.e., the invertible elements ε of $\mathcal{A}$ that satisfy $\varepsilon = \varepsilon^{-1}$. This space is studied in detail in [4]. It is a reductive space with group G acting by inner automorphism $\Lambda_g\varepsilon = g\varepsilon g^{-1}$. The selfadjoint elements of Q form a smooth submanifold P and the polar decomposition induces a fibration $Q \xrightarrow{\pi} P$. More explicitly, if $\varepsilon = \mu\rho$ with $\mu > 0$ and ρ unitary then automatically ρ is in P and $\pi(\varepsilon) = \rho$. The fibers $Q_\rho = \pi^{-1}(\rho)$ are characterized by $Q_\rho = \{\varepsilon \in Q;\ \varepsilon\rho > 0\}$ (for

Received October 22, 1993.
1991 Mathematics Subject Classification. Primary 58B20, 58E10; Secondary 46L05.

details see [4], Sections 3 and 4). These fibers are also reductive spaces with group the unitary group U_ρ of the bilinear form $B_\rho(x, y) = \langle \rho x, y\rangle$ $(x, y \in \mathcal{K})$ induced by ρ (acting by inner automorphism also). In fact the isotropy of this action is also known (see [4], 5.1) and we can get each fiber faithfully by acting on ρ with *positive* elements of U_ρ only. In other words, $u \to u\rho u^{-1}$ is a diffeomorphism from the space of positive elements of U_ρ onto the fiber Q_ρ ("positive" means positive for the ordinary involution of $\mathcal{A}$). Of course the definition of U_ρ does not depend on the representation of $\mathcal{A}$ since it can be described alternatively as $U_\rho = \{h \in G; u^\sharp = u^{-1}\}$ where $x \to x^\sharp$ is the involution $x^\sharp = px^\star\rho$. Recall that the Finsler metric for Q_ρ is defined for $X \in T(Q_\rho)_\varepsilon$ by $\|X\|_\varepsilon = \|\mu^{-1/2}X\mu^{1/2}\|$.

Concerning the geometry of G^+ the following remarks may be useful (see [2] for details). The G-invariant connection of G^+ has covariant derivative

$$\frac{DY}{dt} = \frac{dY}{dt} - \frac{1}{2}(\dot{\gamma}\gamma^{-1}Y + Y\gamma^{-1}\dot{\gamma})$$

where $Y(t)$ is a tangent field along the curve $\gamma(t)$ in G^+. The corresponding exponential has the formula

$$\exp_a X = e^{\frac{1}{2}Xa^{-1}}ae^{\frac{1}{2}a^{-1}X}$$

for $X \in T(G^+)_a$.

The space G^+ carries also the G-invariant Finsler metric $\|X\|_a = \|a^{-\frac{1}{2}}Xa^{-\frac{1}{2}}\|$ where $\|\ \|$ is the original norm of $\mathcal{A}$ (the inclusion $G^+ \subset \mathcal{A}$ permits to identify the tangent spaces $T(G^+)_a$ with the real space of self adjoint elements X of $\mathcal{A}$).

2. Embeddings of reflections into positive operators

Fix a selfadjoint reflection ρ. Define Φ and ϕ by

$$Q_\rho \xrightarrow{\Phi} G^+, \qquad \Phi(\varepsilon) = \varepsilon\rho,$$
$$U_\rho \xrightarrow{\phi} G, \qquad \Phi(g) = \hat{g} = (g^\star)^{-1}.$$

The best way to think about Φ is that for $\varepsilon = \mu\rho$ we set $\Phi(\varepsilon) = \mu$. Then:

THEOREM 2.1. *The pair (Φ, ϕ) is an isometric morphism of reductive homogeneous spaces of operators.*

Proof. To see that Φ preserves connections recall [4], [2] that the transport forms of Q_ρ and G^+ are given respectively by

$$K^Q_\varepsilon(X) = -\frac{1}{2}\varepsilon X, \qquad K^G_a(Y) = -\frac{1}{2}a^{-1}Y$$

Then the preservation of the forms is a direct computation. The remainder of the proof is straightforward. □

This theorem gives a perfect model of the space of reflections inside the space of positive operators. The existence of such embeddings has interesting consequences.

THEOREM 2.2. *The norm of Jacobi fields along geodesics in Q_ρ is a convex function of the parameter.*

THEOREM 2.3. *If $\gamma(t)$ and $\delta(t)$ are geodesics in Q_ρ then $t \to \operatorname{dist}(\gamma(t), \delta(t))$ is a convex function of $t \in \mathbf{R}$, where* dist *denotes the geodesic distance in Q_ρ.*

For the proofs apply Theorem 1 and Theorem 2 in [14]. When spelled out these results translate into rather complicated operator inequalities which will be studied in a forthcoming paper. Similarly, using [14] again, one can obtain various convexity results for Q_ρ, e.g., that geodesic balls are convex sets.

Our next goal is to study all embeddings of Q_ρ into G^+ related to the canonical embeddings exhibited above. Denote by $\Omega(\Phi) \subset \operatorname{Hom}(Q_\rho, G^+)$ the orbit of (Φ, ϕ) by the action of G. From $(\Phi_g)^k = \Phi_{g\phi(k)}$ it follows that $\Omega(\Phi)$ decomposes as a disjoint union of U_ρ-orbits. Consequently $\Omega(\Phi) \bmod U_\rho$ corresponds to the homogeneous space G/U_ρ. This homogeneous space has a natural representation as the G-orbit of ρ in the space G^s of invertible selfadjoint elements of $\mathcal{A}$, which will be used as the moduli space for the set of orbits. In fact setting $\chi(\Phi_g, \phi_g) = L_g\rho$ we obtain a map from $\Omega(\Phi)$ into the proposed moduli space whose fibers are the U_ρ orbits contained in $\Omega(\Phi)$. We collect these results in the following theorem:

THEOREM 2.4. *For $(\Psi, \psi) \in \Omega(\Phi)$ define $\chi(\Psi, \psi) = L_{a^{-1}}(\mathrm{Ad}_a(\psi(\rho)))$ where $a > 0$ is defined by $a^2 = \Psi(\rho)$. Then χ is an analytic map from $\Omega(\Phi)$ onto $G^s_\rho = \{L_g\rho;\ g \in G\}$ and the fibers of χ are the U_ρ-orbits contained in $\Omega(\Phi)$.*

Proof. First we verify that χ as defined in the statement of the Theorem has the value $L_g\rho$ for $\Psi = \Phi_g$. In fact, $a^2 = \Psi(\rho) = \Phi_g(\rho) = (g^{-1})^\star\Phi(\rho)g^{-1} = (g^{-1})^\star g^{-1}$ by definition of Φ. Then

$$\begin{aligned} L_{a^{-1}}\,\mathrm{Ad}_a\,\psi_g(\rho) &= a(ag(\rho^{-1})^\star g^{-1}a^{-1})a \\ &= a^2 g\rho g^{-1} = (g^{-1})^\star \rho g^{-1} = L_g\rho, \end{aligned}$$

as claimed. Next, $\chi(\Psi, \psi) = \chi(\Psi', \psi')$ for $\Psi = \Phi_g$, $\Psi' = \Phi_h$ means that $L_g\rho = L_h\rho$. Then a routine calculation shows that $(g^{-1}h)^\sharp = (g^{-1}h)^{-1}$ and we are done. □

We close this section with a decomposition theorem of G^+ in terms of images of some of the embeddings Φ_g.

THEOREM 2.5. *The space G^+ decomposes as the disjoint union of images of a family of Φ_g's. More precisely, let $\mathcal{B}$ be the commutant of ρ in $\mathcal{A}$ and denote by $\mathcal{B}^+$*

the space of positive invertible elements of $\mathcal{B}$. Then the map

$$\mathcal{B}^+ \times Q_\rho \xrightarrow{\Xi} G^+, \qquad \Xi(b, \varepsilon) = \Phi_b(\varepsilon) = L_b\Phi(\varepsilon)$$

is a diffeomorphism onto G^+.

Proof. The map Φ: $Q_\rho \to G^+$ has image the set of exponentials of symmetric elements which anticommute with ρ. Then the map Ξ corresponds to $(b, c) \to L_b c$ where c stands for such an exponential and the theorem follows from Theorem 2 in [3]. □

3. Embedding of positive operators into reflections

Let $\tilde{\mathcal{A}}$ be the algebra of 2×2 matrices with entries in $\mathcal{A}$. We can make $\tilde{\mathcal{A}}$ into a $\mathbf{C}^\star$-algebra by representing $\mathcal{A}$ faithfully in a Hilbert space $\mathcal{K}$ and then making the matrices act on $\mathcal{K} \oplus \mathcal{K}$. This representation of $\tilde{\mathcal{A}}$ will be used occasionally for other purposes, so we assume it has been chosen once and for all. In this section we denote by G and $\tilde{G}$ the groups of invertible elements and by G^+ and $\tilde{G}^+$ the spaces of positive invertible elements of $\mathcal{A}$ and $\tilde{\mathcal{A}}$. Both G and $\hat{G}$ are Lie groups and their Lie algebras are $\mathcal{A}$ and $\tilde{\mathcal{A}}$, respectively.

Also, let

$$\text{(3.1.i)} \quad p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \rho = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 2p - 1 \quad \bar{p} = 1 - p$$

and

$$\text{(3.1.ii)} \quad \alpha = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 2\alpha - 1 \quad \bar{\alpha} = 1 - \alpha$$

These are all elements of $\tilde{\mathcal{A}}$. It is easy to see that

(3.1.iii) p, $\bar{p}$, α and $\bar{\alpha}$ are projections,
(3.1.iv) ρ and η are reflections,
(3.1.v) $\rho\eta = -\eta\rho$.

Finally let $g \to \hat{g} = (g^\star)^{-1} = (g^{-1})^\star$ be the contragredient map.

THEOREM 3.2. *The map*

$$f(g) = g\alpha + \hat{g}\bar{\alpha} = \frac{1}{2}\begin{pmatrix} g + \hat{g} & g - \hat{g} \\ g - \hat{g} & g + \hat{g} \end{pmatrix}$$

is a Lie group embedding of the Lie group G into the Lie group $\tilde{G}$. The tangent map f_1' of f at $g = 1$ is the Lie algebra homomorphism

$$f_1'(X) = \begin{pmatrix} Z & Y \\ Y & Z \end{pmatrix}$$

where $X = Y + Z$ is the decomposition of $X \in \mathcal{A}$ in its symmetric and antisymmetric parts $Y = \frac{1}{2}(X + X^\star)$, $Z = \frac{1}{2}(X - X^\star)$.

The proof is just a routine verification. It is also easy to verify the following:

(3.2.i) If $g = e^Z$ is a positive element of G and we write $g = e^Z$ with $Z = Z^\star$, then

$$f(g) = e^{Z_\eta} = \begin{pmatrix} \cosh(Z) & \sinh(Z) \\ \sinh(Z) & \cosh(Z) \end{pmatrix}.$$

(3.2.ii) If g is unitary then

$$f(g) = \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}.$$

(3.2.iii) $f(g^\star) = f(g)^\star$.
(3.2.iv) $f(g)\rho = \rho f(\hat{g})$.

These results follow from standard identities including $e^Z = \cosh(Z) + \sinh(Z)$.
The element

$$\rho = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

determines the involution in $\tilde{\mathcal{A}}$ given by $x^\sharp = \rho x^\star \rho$ and the element $x^\sharp$ is the adjoint of x for the bilinear form $B_\rho(\xi, \zeta) = \langle \rho\xi, \zeta \rangle$ defined on the Hilbert space $\mathcal{K} \oplus \mathcal{K}$. It satisfies

(3.2.v) $f(g^{-1}) = f(g)^\sharp$.

THEOREM 3.3. *The image of G under f consists of the elements of $\tilde{U}_\rho$ that commute with η.*

Proof. It is clear from 3.2.v that the image $f(G)$ is contained in $\tilde{U}_\rho$ and that all $f(g)$ commute with η. To see the converse, use Theorem 5.1 and remarks after 3.4 in [2] to write an arbitrary $w \in \tilde{U}_\rho$ as

$$w = \begin{pmatrix} k & b^\star \\ b & l \end{pmatrix}\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} = \begin{pmatrix} au & b^\star v \\ bu & cv \end{pmatrix}$$

where $k = \sqrt{1 + b^\star b} > 0$, $l = \sqrt{1 + bb^\star} > 0$, and u and v are unitary in $\mathcal{A}$. Suppose that w commutes with η. Writing down as matrices both terms of $w\eta = \eta w$ we get that $ku = lv$ and $b^\star u = bv$, and by uniqueness of polar decompositions this implies $k = l$ and $u = v$. It follows that $b = b^\star$. Thus w has the form

$$w = \begin{pmatrix} k & b \\ b & k \end{pmatrix}\begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}$$

with $k = \sqrt{1+b^2}$. Calculating we see that $f(g) = w$ for $g = \frac{1}{2}(b + \sqrt{1+b^2})u$ and the theorem follows. □

Consider the space $\tilde{Q}$ of all reflections in $\tilde{\mathcal{A}}$ and the subspace $\tilde{P}$ of self-adjoint elements of $\tilde{Q}$. Let $\tilde{Q}_\rho$ denote the fiber over ρ of the polar decomposition map $\tilde{Q} \xrightarrow{\tilde{\pi}} \tilde{P}$. We have:

THEOREM 3.4. *Defining* $F\colon G^+ \to \tilde{Q}_\rho$ *by* $F(a) = \rho f(a)$ *the pair* (F, f) *is an isometric morphism of reductive spaces.*

Proof. For the equivariance just calculate

$$F(L_g a) = \rho f(\hat{g} a g^{-1}) = \rho f(\hat{g}) f(a) f(g)^{-1}$$

and

$$L_{f(g)} F(a) = f(g) \rho f(a) f(g)^{-1}.$$

This means that the equality $F(L_g a) = L_{f(g)} F(a)$ follows from $\rho f(\hat{g}) = f(g)\rho$, an easy consequence of formula 3.2.iv above.

Let us prove that the pair (F, f) preserves the connections. Recall that the transport forms for G^+ and $\hat{G}$ are, respectively, $K_a(X) = -\frac{1}{2}a^{-1}X$ and $\Omega_\varepsilon(Y) = -\frac{1}{2}\varepsilon Y$. The first formula is proved in 2.3 of [2], and the second one can be obtained from the corresponding formula for projections (where the right-hand side is $[\gamma, \dot{\gamma}]$) given in [1], [4], or [11] using the change of variable $\varepsilon = 2\gamma - 1$. Thus $F^*\Omega = f_1' K$ or more specifically $(F^*\Omega)_a(X) = f_1'(K_a(X))$ for all $X \in TG_a^+$ and any $a \in G^+$, where F^* is the pull-back of forms induced by F and $f_1'\colon \mathcal{A} \to \mathcal{L}$ is the tangent map of the homomorphism $f\colon G \to \tilde{U}_\rho$ (here $\mathcal{L} \subset \tilde{\mathcal{A}}$ is the Lie algebra of $\tilde{U}_\rho$ studied in [4], Section 5).

Now, by definition,

$$\begin{aligned}(F^*\Omega)_a(X) &= \Omega_{F(a)}(TF_a(X)) = -\frac{1}{2}F(a)TF_a(X)\\ &= -\frac{1}{2}\rho(a\alpha + a^{-1}\bar{\alpha})\rho(X\alpha + a^{-1}Xa^{-1}\bar{\alpha}).\end{aligned}$$

Using $\alpha\rho = \rho\bar{\alpha}$ and $\alpha^2 = \bar{\alpha}^2 = \rho^2 = 1$ we get

$$(F^*\Omega)_a(X) = -\frac{1}{2}(a^{-1}X\alpha + Xa^{-1}\bar{\alpha}).$$

Now differentiating $f(a) = a\alpha + \hat{\alpha}\bar{\alpha}$ we get $f_1'(Z) = Z\alpha - Z^\star\bar{\alpha}$, and so

$$f_1'(K_a(X)) = f_1'\left(-\frac{1}{2}a^{-1}X\right) = -\frac{1}{2}a^{-1}X\alpha + \frac{1}{2}Xa^{-1}\bar{\alpha}$$

and we conclude that $(F^*\Omega)_a(X) = f_1'(K_a(X))$ as claimed. Concerning the isometry statement, by homogeneity it suffices to verify it at $a = 1 \in G^+$, and there it is clear by direct calculation. □

COROLLARY 3.5. *F can also be defined as* $F(L_g 1) = L_{f(g)}\rho$.

A map ϕ: $X \to Q$ where Q is the set of all reflections of a $\mathbf{C}^*$-algebra produces by pull-back a "vector bundle-with-connection" from the bundle of fixed subspaces. More precisely, and assuming that the algebra is represented in a Hilbert space $\mathcal{K}$, let $\xi \to Q$ be the bundle of fixed subspaces $\xi_\varepsilon = \{\varepsilon = 1\}$ where in general we write $\{\tau = 1\} = \{x \in \mathcal{K}; \tau x = x\}$ (and $\{\tau = -1\} = \{x \in \mathcal{K}; \tau x = -x\}$). This vector bundle has the canonical connection $D_X x = p(X(x))$ where p is the projection on the fixed space $\{\varepsilon = 1\}$, that is $p = (1+\varepsilon)/2$. On ξ there is also the canonical metric defined by the form $B_\rho(x, y) = \langle \rho x, y\rangle$.

In the case under consideration, we have constructed the map F: $G^+ \to \tilde{Q}_\rho$ and our next goal is to calculate the pull back of ξ under F (pull-backs of canonical bundles over reflection spaces are studied in the finite dimensional case in [10] for the purpose of classifying connections on vector bundles). To describe the answer more succinctly recall a construction given in [2] (section 3 under the title "The Bundle E"). Let $E = G^+ \times \mathcal{K}$ be the trivial bundle over G^+ with fiber $\mathcal{K}$ and provide it with the transport connection whose formula is

$$D_X x = X(x) + \frac{1}{2}a^{-1}Xx.$$

We can also provide E with the metric $\langle\langle x, y\rangle\rangle_a = \langle ax, y\rangle$. Then:

THEOREM 3.6. *The map* $x \to ((1 + a)x, (1 - a)x)$ *from* $\mathcal{K}$ *into the fixed space of* $\varepsilon = F(a)$ *is an isomorphism of bundles with connection and metric from E onto* $F^*(\xi)$ *where isomorphic in the case of the metrics is to be understood as "up to positive constant".*

Proof. Suppose that $x(t)$ is a section of E over the curve $a(t) \in G^+$. Then

$$\begin{aligned} &p\left(\frac{d}{dt}((1+a)x, (1-a)x)\right) \\ &\quad = \frac{1}{2}\begin{pmatrix} 1+\frac{1}{2}(a+a^{-1}) & \frac{1}{2}(a-a^{-1}) \\ \frac{1}{2}(-a+a^{-1}) & 1-\frac{1}{2}(a+a^{-1}) \end{pmatrix}\begin{pmatrix} \dot{a}x + (1+a)\dot{x} \\ -\dot{a}x + (1-a)\dot{x} \end{pmatrix} \end{aligned}$$

has the form $((1 + a)y, (1 - a)y)$ where $y = \dot{x} + \frac{1}{2}a^{-1}\dot{a}x$ as a routine calculation shows. This amounts to: the map $x \to (1+a)x, (1-a)x)$ preserves the connections of E and $F^*(\xi)$. To see that it is also isometric, calculate

$$\begin{aligned} B_\rho((1+a)x,(1-a)x) &= \langle (1+a)x,(1+a)x\rangle - \langle (1-a)x,(1-a)x\rangle \\ &= 4\langle ax,x\rangle = 4\langle x,x\rangle_a \end{aligned}$$

and so the metrics differ by a factor of 4. □

The next issue we want to take up is the description of the image of the map F. We present two characterizations, of which the first is rather simple:

3.7. *The image of F is the set of elements of $\tilde{Q}_\rho$ that anti-commute with η.*

A direct application of 3.3. gives 3.7.

The second characterization requires some preliminaries. In particular we need a description of the elements of $\tilde{Q}_\rho$ similar to the description of the elements of $\tilde{U}_\rho$ given in the previous section. Suppose that $b \in \mathcal{A}$ and define $0 < k = (1 + b^\star b)^{1/2}$, $0 < l = (1 + bb^\star)^{1/2}$. Then

$$\lambda = \lambda(b) = \begin{pmatrix} k & b^\star \\ b & l \end{pmatrix}$$

describes the arbitrary positive element in $\tilde{U}_\rho$ and the representation is unique (see [4], end of Section 3). Notice that $\lambda(-b) = \lambda(b)^{-1}$. According to 3.3 in [4], an arbitrary element $\varepsilon \in \tilde{Q}_\rho$ can then be written as $\varepsilon = \lambda\rho\lambda^{-1} = \rho\lambda^{-2}$, or

$$\varepsilon = \begin{pmatrix} 1 + 2b^\star b & -2kb^\star \\ 2bk & -(1 + 2bb^\star) \end{pmatrix}.$$

The reflection ε has fixed space the image under λ of the fixed space of ρ (which is $\mathcal{K} \oplus 0$), so the fixed subspace of ε is

$$\{\varepsilon = 1\} = \left\{ \begin{pmatrix} kx \\ bx \end{pmatrix}; x \in \mathcal{K} \right\}$$

and the reverse space of ε is

$$\{\varepsilon = -1\} = \left\{ \begin{pmatrix} b^\star y \\ ly \end{pmatrix}; y \in \mathcal{K} \right\}.$$

The elements k and l are invertible and so we can define $m = bk^{-1}$, $n = b^\star l^{-1}$. Then we can paraphrase the foregoing:

THEOREM 3.8 ([4]). *The fixed and reversed spaces for* $\varepsilon = \rho\lambda(b)^{-2}$ *are the graphs in* $\mathcal{K} \oplus \mathcal{K}$ *of the maps*

$$m = b(1 + b^{\star}b)^{-1/2}\colon \mathcal{K} = \mathcal{K} \oplus 0 \to \mathcal{K} = 0 \oplus \mathcal{K},$$

$$n = m^{\star} = b^{\star}(1 + bb^{\star})^{-1/2}\colon \mathcal{K} = 0 \oplus \mathcal{K} \to \mathcal{K} = \mathcal{K} \oplus 0.$$

In other words, $\varepsilon(x \oplus y) = x \oplus y$ *is equivalent to* $y = mx$ *and* $\varepsilon(x \oplus y) = -x \oplus y$ *is equivalent to* $x = ny$ *or* $y = m^{\star}x$. *The element* m *has* $\|m\| < 1$ *and any map* $\mathcal{K} \to \mathcal{K}$ *with norm less than* 1 *is such an* m. *The correspondence* $\varepsilon \to m = m(\varepsilon)$ *is analytic from* $\tilde{Q}_\rho$ *into the space of linear operators in* $\mathcal{K}$.

To prove that all m with $\|m\| < 1$ are obtained in this way simply define $b = m(1 - m^{\star}m)^{-1/2}$ and verify the identity. In our situation, we have:

LEMMA 3.9. *For* $a \in G^+$ *the element* m *corresponding to* $F(a)$ *is the Cayley transform* $m = (1 - a)/(1 + a)$ *of* a.

Proof. It follows from $F(a) = \rho f(a)$ that $\lambda^{-2} = f(a)$, or $\lambda^2 = f(a^{-1})$. With the previous notations this reads

$$1 + 2b^{\star}b = \frac{1}{2}(a + a^{-1}), \qquad -2bk = \frac{1}{2}(a - a^{-1}).$$

But $1 + 2b^{\star}b = 2k^2 - 1$ so the first equation gives $k^2 = \frac{1}{2}(1 + \frac{1}{2}(a + a^{-1}))$. Then from the second equation we get

$$-2bk^{-1} = \frac{\frac{1}{2}(a - a^{-1})}{\frac{1}{2}(1 + \frac{1}{2}(a + a^{-1}))}$$

and this simplifies to $bk^{-1} = (1 - a)/(1 + a)$, as claimed. □

With this the second characterization of the image of F is immediate:

THEOREM 3.10. *The image of* $F\colon G^+ \to \tilde{Q}_\rho$ *consists of the* $\varepsilon \in \tilde{Q}_\rho$ *where* $m(\varepsilon)$ *is self-adjoint.*

Together with the embedding of G^+ into $\tilde{Q}_\rho$ given by (F, f) we can consider the orbit $\Omega(F, f) \subset \mathrm{Hom}(G^+, \tilde{Q}_\rho)$ of (F, f) under the group $\tilde{U}_\rho$. This orbit decomposes as a disjoint union of G-orbits of elements of $\Omega(F, f)$. We show here that this space of G-orbits can be parametrized by a natural moduli space. The proposed moduli space is the $\tilde{U}_\rho$-orbit.

$$\mathcal{O}_\eta = \{\Lambda_h \eta = h\eta h^{-1}; h \in \tilde{U}_\rho\} \subset \tilde{Q}$$

where η is the reflection defined in 3.1.ii. Define a map $\Omega(F, f) \xrightarrow{\chi} \mathcal{O}_\eta$ by $\chi(F', f') = \tau \in \tilde{Q}$ where the fixed space of τ is the graph of $\lim_{a\to 0} m(F'(a))$ and the reverse space of τ is the graph of $\lim_{a\to\infty} m(F'(a))$. Then we have:

THEOREM 3.11. *The map χ is constant on G-orbits in $\Omega(F, f)$ and defines a bijection of the space $\Omega(F, f)/G$ of G-orbits onto $\mathcal{O}_\eta$.*

Proof. By homogeneity it suffices to show that χ describes η when the morphism is (F, f). But in this case the formula for m given in 3.9 shows the result and we are done. □

THEOREM 3.12. *The elements τ of $\mathcal{O}_\eta$ are characterized by $\tau \in \tilde{Q}$ and $\tau^\sharp = -\tau$. Therefore the set $\mathcal{O}_\eta$ is a closed analytic submanifold of $\tilde{Q}$.*

Proof. It is clear that if $\tau \in \tilde{\mathcal{O}}_\eta$ then it satisfies the announced properties since η does and the elements of $\tilde{U}_\rho$ commute with the involution $x \to x^\sharp$.

Now suppose that $\tau \in \tilde{Q}$ satisfies $\tau^\sharp = -\tau$. Then by polar decomposition $\tau = \mu\sigma = \mu^{1/2}\sigma\mu^{-1/2}$ with $\mu > 0$ and σ unitary. We conclude (see [4]) that $\sigma^2 = 1$ so that σ is an orthogonal reflection and $\mu\sigma = \sigma\mu^{-1}$. Next write $-\tau = \rho\tau^\star\rho = \rho\sigma^{-1}\mu\rho = \rho\sigma\mu\rho$ or $\mu(-\sigma) = (\rho\mu\rho)(\rho\sigma\rho)$. Thus $\mu = \rho\mu^{-1}\rho$ and $-\sigma = \rho\sigma\rho$ by uniqueness or polar decompositions. A direct calculation shows that σ has the form

$$\sigma = \begin{pmatrix} 0 & u^{-1} \\ u & 0 \end{pmatrix}$$

with $u \in \mathcal{A}$ unitary. Furthermore μ is in $\tilde{U}_\rho$ for, μ being positive, the condition $\mu^{-1} = \rho\mu\rho$ is equivalent to $\mu^\sharp = \mu^{-1}$. Then according to 3.1 in [4], τ can be written as $\tau = \mu\sigma = \mu^{1/2}\sigma\mu^{-1/2}$. But

$$\sigma = \begin{pmatrix} 0 & u^{-1} \\ u & 0 \end{pmatrix} = \nu\eta\nu^{-1} \text{ where } \nu = \begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence $\tau = s(\tau)\eta s(\tau)^{-1}$ with $s(\tau) = \mu^{1/2}\nu$ which is obviously in $\tilde{U}_\rho$ being the product of two elements of $\tilde{U}_\rho$. □

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