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# CONNECTIONS BETWEEN ADDITIVE COCYCLES AND BISHOP OPERATORS

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## 1. Introduction

**T** will denote the circle group written additively so that its elements are real numbers in [0, 1) and are added modulo 1. For any $y \in \mathbf{R}$, $\|y\|$ is the distance from $y$ to $\mathbf{Z}$ and $\{y\}$ is the fractional part of $y$, so that $\{y\} \in \mathbf{T}$ and $\{y\} \equiv y \bmod 1$. If $\alpha \in \mathbf{R}$ is irrational, then $x \to \{x + \alpha\}$ defines an ergodic transformation on **T**.

A measurable function $\phi$: $\mathbf{T} \to \mathbf{R}$ is called a *trivial $\alpha$-additive cocycle* (or just *trivial*) if there exists a measurable function $\psi$: $\mathbf{T} \to \mathbf{R}$ and $c \in \mathbf{R}$ such that

$$\phi(x) = \psi(\{x + \alpha\}) - \psi(x) + c \qquad \text{for almost every } x \in \mathbf{T}.$$

The function $\psi$ is called a *cobounding function* for $\phi$. The functional equation above has generated much interest since its solvability has connections to areas such as ergodic theory and group representations (e.g., [3], [6], [10]). Much of the research that has been done goes the way of showing that certain families of functions are not $\alpha$-additive cocycles for all $\alpha$ or for a set of $\alpha$ (e.g., [2], [8], [10], [12].) There has also been recent research in the construction of $\alpha$-additive cocycles with certain properties. For example, in [1] it is shown (via construction) that for any $\alpha$ there is a continuous trivial $\alpha$-additive cocycle whose cobounding function is non $L^1$. One commonality among most of this work is that the functions $\phi$ that have been studied are bounded. Very little is known about the case when $\phi$ is unbounded.

Long ago, E. Bishop suggested the following operator as one that might not have any closed non-trivial invariant subspaces: For $\alpha \in \mathbf{R}$ irrational, define $B_\alpha$: $L^2(\mathbf{T}) \to L^2(\mathbf{T})$ by

$$(B_\alpha f)(x) = x\, f(\{x + \alpha\}).$$

These operators have come to be known as *Bishop operators*. In 1974, A. M. Davie [4] showed that Bishop operators have closed non-trivial hyperinvariant subspaces when $\alpha$ is not a Liouville number. It is still not known if all Bishop operators have closed non-trivial invariant subspaces. His secondary result was that these operators never have eigenvalues.

An immediate connection between the study of additive cocycles and the study of Bishop operators is that Davie's second result is equivalent to: *For all $\alpha$*, $\log(x)$ *is a*

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*non-trivial $\alpha$-additive cocycle*. This result appears to be not very well known among the mathematicians who study cocycles of an irrational rotation. More importantly, the technique that Davie uses can be applied to study a large family of functions with one infinite jump.

On the other hand, there are techniques familiar to those who study cocycles of an irrational rotation that can be applied to study Bishop operators. In particular, G. W. MacDonald, who generalized Davie's work by proving results about other multipliers besides the function $m(x) = x$, asked if operators of a certain type commute with Bishop operators where the irrational is a Liouville number [11]. (If this were the case, one could conclude that all Bishop operators have closed non-trivial invariant subspaces.) MacDonald's question can be answered using familiar techniques from the study of additive cocycles.

In this article, we show these two interesting connections between the study of additive cocycles and Bishop operators. In Section two, we use Davie's technique to show that a large class of unbounded functions are not $\alpha$-additive cocycles for any $\alpha$. This family includes the functions $x^a$ for $a < 0$. And in Section three, we state and solve the problem of MacDonald.

## 2. A family of non-trivial additive cocycles

Let $\mathcal{L}(\mathbf{T})$ denote the vector space of all complex measurable functions on $\mathbf{T}$ defined almost everywhere and where we identify functions that are equal almost everywhere. Fix $\alpha \in (0, 1)$ irrational. This first lemma shows the connection between a function being a trivial $\alpha$-additive cocycle and the eigenvalues of a certain operator on $\mathcal{L}(\mathbf{T})$.

LEMMA 1. *Let $\phi \in \mathcal{L}(\mathbf{T})$ be a real-valued function. $\phi$ is a trivial $\alpha$-additive cocycle if and only if the operator $E_\phi$: $\mathcal{L}(\mathbf{T}) \to \mathcal{L}(\mathbf{T})$ defined by*

$$(E_\phi f)(x) = e^{\phi(x)} f(\{x + \alpha\})$$

*has an eigenvalue.*

*Proof.* ($\Rightarrow$) If there exists a real measurable function $\psi \in \mathcal{L}(\mathbf{T})$ and $c \in \mathbf{R}$ such that $\phi(x) = \psi(\{x + \alpha\}) - \psi(x) + c$ a.e., then

$$e^{\phi(x)} = e^{\psi(\{x+\alpha\})} e^{-\psi(x)} e^c \quad \text{a.e.}$$

So $e^{-\psi(x)}$ is an eigenfunction for the eigenvalue $e^c$.

($\Leftarrow$) If there exists $g \in \mathcal{L}(\mathbf{T})$ and $c \in \mathbf{C}$ such that

$$e^{\phi(x)} g(\{x + \alpha\}) = c\, g(x) \quad \text{a.e.,}$$

we may take absolute values and can thus assume that $c \geq 0$ and $g(x) \geq 0$ a.e. If $c = 0$, then $g(x) = 0$ a.e., therefore $c > 0$. It is clear that $\{x\colon g(x) = 0\}$ is invariant under translation by $\alpha$, and since it cannot be a measure 1, it must be of measure 0. We may take logarithms to get

$$\phi(x) + \log g(\{x+\alpha\}) = \log c + \log g(x) \text{ a.e.} \qquad \square$$

The next theorem is a generalization of the theorem of Davie already mentioned at the end of the last section [4, Theorem 2]. The proof uses the techniques developed by Davie.

THEOREM 1. *Let $\phi$: $\mathbf{T} \to \mathbf{R}$ be an increasing convex function with $\phi(0) = 0$. Then $\Phi_\phi$: $\mathcal{L}(\mathbf{T}) \to \mathcal{L}(\mathbf{T})$ defined by*

$$(\Phi_\phi f)(x) = \phi(x) f(\{x+\alpha\})$$

*has no eigenvalues.*

The proof will require the following lemma from classical analysis. The proof is very simple and thus omitted. (Drawing the graph of $\phi$ shows how to obtain a proof.)

LEMMA 2. *Let $\phi$ be an increasing convex function on $[a, b]$ with $\phi(a) = 0$. Then $\{x \in [a, b]\colon |1 - \phi(x)| > \frac{1}{2}\}$ has measure $\geq \frac{b-a}{3}$.*

*Proof of Theorem* 1. We will show that if there is a $\lambda \in \mathbf{C}$ such that $\phi(x)f(\{x + \alpha\}) = \lambda f(x)$ a.e., then $f(x) = 0$ a.e. If $\lambda = 0$, then $f(x) = 0$ a.e., so assume $\lambda \neq 0$.

Let $\{q_n\}$ be the sequence of denominators to the convergents of $\alpha$. (The reader unfamiliar with the number theoretic terminology and elementary results of continued fractions is asked to look in [5, Chapter X].) Since $\|q_n\alpha\| \to 0$, $|f(x) - f(\{x - q_n\alpha\})| \to 0$ in measure. Passing to a subsequence, we may assume

$$|f(x) - f(\{x - q_n\alpha\})| < \frac{1}{n} \tag{1}$$

on a set of measure $\geq \frac{5}{6}$. $f = \lambda^{-q_n} \Phi_\phi^{q_n} f$. That is,

$$f(x) = \lambda^{-q_n} \left( \prod_{k=0}^{q_n-1} \phi(\{x + k\alpha\}) \right) f(\{x + q_n\alpha\}) \text{ a.e.}$$

Translate both sides by $-q_n\alpha$ to get

$$f(\{x - q_n\alpha\}) = \lambda^{-q_n} \left( \prod_{k=1}^{q_n} \phi(\{x - k\alpha\}) \right) f(x) \text{ a.e.}$$

$\mathbf{T}$ can be divided into intervals on each of which $\lambda^{q_n} \prod_{k=1}^{q_n} \phi(\{x - k\alpha\})$ satisfies the hypothesis of Lemma 2. Thus

$$\begin{aligned} |f(x) - f(\{x - q_n\alpha\})| &= |f(x)| \left| 1 - \lambda^{-q_n} \prod_{k=1}^{q_n} \phi(\{x - k\alpha\}) \right| \quad \text{a.e.} \\ &\geq \frac{1}{2}|f(x)| \text{ on a set of measure } \geq \frac{1}{3}. \end{aligned}$$

Equation 1 now implies $\frac{1}{2}|f(x)| \leq \frac{1}{n}$ on a set of measure $\geq \frac{1}{6}$. Letting $n \to \infty$, we have $f(x) = 0$ on a set of measure $\geq \frac{1}{6}$. Since $\{x\colon f(x) = 0\}$ is invariant under translation by $\alpha$, it must be a measure of 1. □

COROLLARY 1. *For any $a < 0$, $x^a$ is not a trivial $\alpha$-additive cocycle.*

*Proof.* Define $\phi \in \mathcal{L}(\mathbf{T})$ by

$$\phi(x) = \begin{cases} e^{\frac{1-a}{a}x^a} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$$

Since $\phi'(x) > 0\ \forall x \neq 0$ and $\phi''(x) \geq 0\ \forall x \neq 0$ (the constant $\frac{1-a}{a}$ has been chosen so that $\phi''$ has this property), $\phi$ satisfies the hypothesis of Theorem 1. By Lemma 1, $\frac{1-a}{a}x^a$ is not a trivial $\alpha$-additive cocycle. For any $r \in \mathbf{R}$, $r \neq 0$, $rf$ is a trivial $\alpha$-additive cocycle if and only if $f$ is one; therefore $x^a$ is not a trivial $\alpha$-additive cocycle. □

*Remark* 1. Theorem 1 can be used to show that many increasing or decreasing functions with one infinite jump are not trivial $\alpha$-additive cocycles. For example, $\log(x)$ (this was Davie's original result) and $\log|\log(x)|$ are not trivial.

*Remark* 2. Theorem 1 has an obvious improvement. *Let $\phi\colon \mathbf{T} \to \mathbf{R}$ be an increasing function with $\phi(0) = 0$ and such that $\phi^n$ is convex for some $n \in \mathbf{N}$. Then $\Phi_\phi$ has no eigenvalues.* This result can be used to show that many other functions with one infinite jump are not trivial $\alpha$-additive cocycles. For example, $\log \sin x$ is not trivial. But there are functions of this type for which this method does not work. For example, we do not know if $\frac{1}{\log x - a}$ for $a > 0$ is trivial.

*Remark* 3. A result of Helson and Merrill can be used to show that *given $\alpha$ irrational, there exist many functions with two infinite jumps which are not trivial $\alpha$-additive cocycles.* Let $f\colon \mathbf{T} \to \mathbf{R}$ be a function with one infinite jump which is not a trivial $\alpha$-additive cocycle (e.g., $\log x$, $x^{-\frac{1}{2}}$, $\log \sin x$). For any irrational $\beta$, $f(\{x + \beta\}) - f(x)$ is a function with two infinite jumps. Section 2 of [7] implies the existence of many $\beta$ (the set of $\beta$s may depend on $\alpha$ and $f$) such that $f(\{x+\beta\}) - f(x)$ is not a trivial $\alpha$-additive cocycle.

## 3. Bishop operators

MacDonald showed that if $\alpha$ is not a Liouville number, then for a large collection of real valued $\phi \in L^\infty(\mathbf{T})$ with $\log|\phi| \in L^p(\mathbf{T})$, $p > 1$ ($p$ depends on $\alpha$), the operator $\Phi_{\phi,\alpha}$: $L^2(\mathbf{T}) \to L^2(\mathbf{T})$ defined by

$$(\Phi_{\phi,\alpha} f)(x) = \phi(x) f(\{x+\alpha\})$$

has a closed non-trivial hyperinvariant subspace. (For the specifics, see [11, Theorem 2.6].)

MacDonald suggested the following approach for attacking the unsolved Bishop operator problem: For $\alpha$ a Liouville number, find a $\beta$ that is not a Liuoville number and a $\phi \in L^\infty(\mathbf{T})$ covered by his theorem such that $B_\alpha$ commutes with $\Phi_{\phi,\beta}$. Since his result gives hyperinvariant subspaces for $\Phi_{\phi,\beta}$ we could conclude that $B_\alpha$ has a closed non-trivial invariant subspace.

The following theorem shows that this approach will not work.

THEOREM 2. *Let $\alpha \in \mathbf{R}$ be irrational. If there exists $\beta \in \mathbf{R}$ irrational and $\phi \in L^\infty(\mathbf{T})$ with* $\log|\phi| \in L^1(\mathbf{T})$ *such that $B_\alpha \Phi_{\phi,\beta} = \Phi_{\phi,\beta} B_\alpha$, then $\beta = \{p\alpha\}$ for some $p \in \mathbf{Z}$.*

*Remark* 4. If $\alpha$ is a Liouville number, then so is $\{p\alpha\}$ $\forall p \in \mathbf{Z}$. Thus, even if we consider $\{\phi \in L^\infty(\mathbf{T})\colon \log|\phi| \in L^1(\mathbf{T})\}$, we cannot find a $\beta$ that is not a Liouville number such that $\Phi_{\phi,\beta}$ commutes with $B_\alpha$.

The proof of Theorem 2 will require three lemmas. The first is a recent result from the theory of continued fractions, the second is an immediate consequence of the first, and the third is a lemma from classical analysis.

LEMMA 3 **(Kraaikamp and Liardet)** [9]. *Let $\alpha$ be irrational and let $\{q_n\}$ be the sequence of denominators of the convergents of $\alpha$. For each $\beta \in \mathbf{R}$ we have $\{\beta\} = \{p\alpha\}$ for some $p \in \mathbf{Z}$ if and only if $\|q_n\beta\| \le \frac{1}{4} q_n \|q_n\alpha\|$ for all large $n$.*

This lemma has an immediate consequence. (This strengthens a result in [13].)

LEMMA 4. *Let $\alpha, \beta \in \mathbf{R}$ be irrationals and let $\{q_n\}$ be the sequence of denominators to the convergents of $\alpha$. If*

$$\limsup_{n\to\infty} \frac{|1 - e^{2\pi i q_n \beta}|}{|q_n(1 - e^{2\pi i q_n \alpha})|} < \frac{1}{2\pi},$$

*then $\{\beta\} = \{p\alpha\}$ for some $p \in \mathbf{Z}$.*

*Proof.* There exists $n_0$ such that $\forall\, n \geq n_0$,

$$|1 - e^{2\pi i q_n \beta}| \leq \frac{q_n}{2\pi}|1 - e^{2\pi i q_n \alpha}|.$$

Since $4\|x\| \leq |1 - e^{2\pi i x}| \leq 2\pi\|x\| \;\forall\, x \in \mathbf{R}$, we have $4\|q_n\beta\| \leq q_n\|q_n\alpha\|$. Lemma 3 implies that $\beta = \{p\alpha\}$ for some $p \in \mathbf{Z}$. □

LEMMA 5. *Let* $(\log)\hat{}(n)$ *be the n-th Fourier coefficient of the function* $\log x$ *on* $\mathbf{T}$. *Then* $|(\log)\hat{}(n)| \geq \frac{C}{|n|}$ *for some* $C > 0$ *and all large* $|n|$.

*Proof.* $|(\log)\hat{}(n)| = \left|\int_0^1 \log x\, e^{-2\pi i n x}\, dx\right|$. We integrate by parts: $u = \log x$, $du = \frac{1}{x}dx$, $dv = e^{-2\pi i n x}\, dx$, $v = \frac{i}{2\pi i n}e^{-2\pi i n x}$.

$$|(\log)\hat{}(n)| = \frac{1}{2\pi|n|}\left|\log x\, e^{-2\pi i n x}\Big|_0^1 - \int_0^1 \frac{e^{-2\pi i n x}}{x}\, dx\right|.$$

To get the desired estimate, we need only consider the imaginary part.

$$|(\log)\hat{}(n)| \geq |\mathrm{Im}[(\log)\hat{}(n)]| = \frac{1}{2\pi|n|}\left|\log x\, \sin 2\pi n x\Big|_0^1 - \int_0^1 \frac{\sin 2\pi n x}{x}\, dx\right|.$$

Since $\lim_{x\to 0+} \log x\, \sin 2\pi n x = 0$, we have

$$|\mathrm{Im}[(\log)\hat{}(n)]| = \frac{1}{2\pi|n|}\left|\int_0^1 \frac{\sin 2\pi n x}{x}\, dx\right|.$$

But since $\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}$, it is easy to see that $\lim_{|n|\to\infty}\left|\int_0^1 \frac{\sin 2\pi n x}{x}\, dx\right| = \frac{\pi}{2}$. Therefore, there is a $C > 0$ such that for $|n|$ large enough, $|(\log)\hat{}(n)| \geq \frac{C}{2\pi|n|}$. □

*Proof of Theorem* 2. If $B_\alpha \Phi_{\phi,\beta} = \Phi_{\phi,\beta} B_\alpha$, then applying this operator to the constant function 1, we get $x\,\phi(\{x+\alpha\}) = \phi(x)\,\{x+\beta\}$ a.e. Taking absolute values, we may assume $\phi \geq 0$. And since $\log\phi \in L^1(\mathbf{T})$, we may take logarithms to get

$$\log x - \log\{x+\beta\} = \log\phi(x) - \log\phi(\{x+\alpha\}) \text{ a.e.}$$

Computing Fourier coefficients of both sides we get

$$(\log)\hat{}(n)(1 - e^{2\pi i n\beta}) = (\log\phi)\hat{}(n)(1 - e^{2\pi i n\alpha}).$$

And thus $(\log\phi)\hat{}(n) = (\log)\hat{}(n)\frac{(1-e^{2\pi i n\beta})}{(1-e^{2\pi i n\alpha})}$, for $n \neq 0$. Now we use Lemma 5 to get

$$\left|(\log\phi)\hat{}(n)\right| \geq C\left|\frac{(1 - e^{2\pi i n\beta})}{n(1 - e^{2\pi i n\alpha})}\right| \quad \text{for all large } |n|.$$

But $\log \phi \in L^1(\mathbf{T})$, and the Riemann-Lebesgue lemma implies that

$$\lim_{|n|\to\infty} |(\log \phi)\hat{}(n)| = 0,$$

and we have

$$\lim_{|n|\to\infty} \left| \frac{(1 - e^{2\pi i n\beta})}{n(1 - e^{2\pi i n\alpha})} \right| = 0.$$

Lemma 4 implies that $\beta = \{p\alpha\}$ for some $p \in \mathbf{Z}$. $\square$

*Remark* 5. In the language of cocycles, the result says that if we restrict ourselves to cobounding functions in $L^1(\mathbf{T})$ and $\{\beta\} \neq \{p\alpha\}$ for any $p \in \mathbf{Z}$, then $\log\{x + \beta\} - \log x$ is not an $\alpha$-trivial additive cocycle.

*Remark* 6. It is clear that the argument works not only for the multiplier $x$ but also for any real valued multiplier $m(x) \in L^\infty(\mathbf{T})$ such that $\log |m(x)| \in L^1(\mathbf{T})$ and such that the Fourier coefficients of $\log |m(x)|$ satisfy the same decay condition as in the hypothesis of Lemma 5.

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