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DAY POINTS FOR QUOTIENTS OF THE FOURIER ALGEBRA $A(G)$, EXTREME NONERGODICITY OF THEIR DUALS AND EXTREME NON ARENS REGULARITY

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To the Memory of Mahlon M. Day

Introduction

Let J be a closed ideal of the Fourier algebra $A = A(G)$ of the metrisable locally compact group G, with identiy e, and $F = Z(J) \subset G$ its zero set. G need not be abelian, yet the results that follow are new even if $G = R$ or T (the real line or the torus). Let $PM(G) = A(G)^*$.

Call $a \in F$ a Mahlon M. Day point of J and let $D_1(J)$ be the set of all such, if there is a sequence $u_n \in A \cap C_c(G)$ such that (i) $1 = u_n(a) = \|u_n\|$, (ii) for any neighborhood V of a there is some k such that $F \cap \operatorname{supp} u_n \subset F \cap V$ if $n \geq k$ and (iii) $\{u_n\}$ is a Sidon sequence in A/J, i.e. there is some $d > 0$ such that $\| \sum_1^n \alpha_j u_j \|_{A/J} \geq d \sum_1^n |\alpha_j|$ for all complex α_j and $n \geq 1$.

The usefulness of this concept comes from our Theorem 4. It implies that if $D_1(J) \neq \emptyset$ then $\boldsymbol{P} = (A/J)^*$ is extremely nonergodic at each $a \in D_1(J)$ and (if G is separable metric) the Banach algebra A/J is extremely non Arens regular. Namely $\boldsymbol{P}/W_{\boldsymbol{P}}(a)$ (hence $\boldsymbol{P}/WAP_{\boldsymbol{P}}$) has ℓ^∞ as a quotient and the set of topologically invariant means on $\boldsymbol{P}$ at a, $TIM_{\boldsymbol{P}}(a)$, contains the big set $\mathcal{F}$, hence card $TIM_{\boldsymbol{P}}(a) \geq 2^C$.

Hence, if we discover points in $D_1(J)$, we get big sets $TIM_{\boldsymbol{P}}(a)$. We do that in Theorems 2 and 3 and then apply the results to arbitrary G in Cor. 6,7. In Ch. III we apply the results to abelian G, i.e. to w^* closed translation invariant subspaces P of $L^\infty(\widehat{G})$ with $\sigma(P) = G \cap \overline{P} = F$, where $\overline{P} = \{\overline{f}; f \in P\}$.

A very mild application of this to second countable *abelian* G and even to $G = T$ is the following: Let $P \subset \ell^\infty(Z)$ (or $L^\infty(\widehat{G})$) be a w^* closed translation invariant space such that $\sigma(P) = G \cap \overline{P} = F$. If F contains, or is, an ultrathin symmetric set F_0 ([GMc] p. 333) (or the Cantor 1/3 set), then the set of topological invariant means on P, $TIM_P(e)$ [and in fact $TIM_P(x)$], contains the big set $\mathcal{F} = \{\varphi \in \ell^{\infty *}; 1 = (\varphi, 1) = \|\varphi\|, \varphi = 0 \text{ on } c_0\}$ (which contains $\beta N \sim N$) [for each $x \in F_0$]. Hence card $TIM_P(e) = 2^C =$ card P^*.

If however F is a perfect Helson (or compact scattered) S subset of T or R and $e \in F$ then card $TIM_P(e) = 1 =$ card $IM_P(e)$.

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This new result for $P \subset \ell^\infty(Z)$ with $\sigma(P) = F_0$ cannot be obtained by the usual methods used to prove that if $Q = \ell^\infty(Z)$ then $TIM_Q(e)$ is big. Since P *is not a pointwise subalgebra* of $\ell^\infty(Z)$, finite intersections of translates of sets $A \subset Z$ which are building blocks for elements of *TIM* on $\ell^\infty(Z)$ do not play the same role for P as they play for $\ell^\infty(Z)$ (see Paterson [Pa], Ch. 7).

Again, let $J \subset A$, $Z(J) = F$, $\boldsymbol{P} = (A/J)^*$ be as above. Let $H \subset G$ be a closed nondiscrete metrisable subgroup. We show that (the interior of $F \cap H$ in H) $\mathrm{int}_H F \subset D_1(J)$. Hence $\boldsymbol{\mathcal{F}} \subset TIM_{\boldsymbol{P}}(x)$ and card $TIM_{\boldsymbol{P}}(x) \geq 2^c$ if $x \in \mathrm{int}_H F$ (and this holds even for $\boldsymbol{P} \subset PM_p(G) = A_p(G)^*$ à la Herz [Hz].

If $G = H = F$ (thus $\boldsymbol{P} = PM(G)$), $x = e$ and G is separable metric, this is due to Ching Chou [Ch2] (for beautiful definitive results see Z. Hu [Hu] and also Lau-Paterson [LP]).

Our results also improve results of Fournier and Cowling in [FC] in showing the existence and prevalence of convolution operators on $L^2(G)$ ($L^p(G)$) with "thin" support which are far from being 'ergodic' at $a \in D_1(J)$ (a fortiori very far from being convolution by a bounded measure). They also improve and simplify results of ours in [Gr5] (see more attributions in [Gr5], p. 53).

We delineate now in more detail the results we obtain in this paper.

Restricting our results to metrisable G, in Section 1 we get:

THEOREM 2. *Let $J \subset A(G)$ be a closed ideal and $F = Z(J)$. Assume that R or T is a closed subgroup of G and $S \subset R$ (or T) is an ultrathin symmetric set such that $aSb \subset F$ for some $a, b \in G$. Then $aSb \subset D_1(J)$.*

THEOREM 3. *Let J be a closed ideal of $A = A(G)$ (or of $A = A_p(G)$ à la Herz [Hz]) with $F = Z(J)$. Let $H \subset G$ be a closed nondiscrete subgroup. Then $\mathrm{int}_{aHb} F \subset D_1(J)$ in particular $D_1(0) = G$. ($\mathrm{int}_{H_0} F$ is the interior of $F \cap H_0$ in H_0).*

In Theorem 2 we improve a result of Y. Meyer [Me] for $A(R)$ and then using theorems of Herz [Hz] lift the result to $A(G)$.

In Theorem 3, while F is not as thin as in Theorem 2, the result holds for all $A_p(G)$, $1 < p < \infty$ [Hz], where $A_2(G) = A(G)$. Methods in abelian harmonic analysis fail in this case, and a global approach is taken.

If $p \neq 2$, $A_p(G)$ is very different from $A_2(G)$. Since if G_1, G_2 are compact abelian and $A_p(G_1)^*$, $A_p(G_2)^*$ are isometric as Banach spaces then G_1, G_2 are isomorphic as topological groups by Benyamini and Lin [BL]. While $A_2(G)^*$ is isometric to $\ell^\infty(Z)$ for all infinite metric compact abelian G.

Let $A = A(G)$ [or $A_p(G)$]. If $\Phi \in A^*$ let supp Φ, be the support of Φ as an element of A^*, (see sequel and [Hz], p. 120). If $\boldsymbol{P} \subset A^*$ let $\boldsymbol{P}_c = \mathrm{ncl}\{\Phi \in \boldsymbol{P};\ \mathrm{supp}\,\Phi$ is compact$\}$ (where ncl is norm closure). If $a \in G$ let $E_{\boldsymbol{P}}(a) = \mathrm{ncl}\{\Phi \in \boldsymbol{P};\ a \notin \mathrm{supp}\,\Phi\}$; $W_{\boldsymbol{P}}(a) = C(\lambda\delta_a) + E_{\boldsymbol{P}}(a)$, where $(\lambda\delta_a, v) = v(a)$ if $v \in A$. Let $\sigma(\boldsymbol{P}) = \{x \in G;\ \lambda\delta_x \in \boldsymbol{P}\}$. Let $TIM_{\boldsymbol{P}}(a) = \{\psi \in \boldsymbol{P}^*;\ 1 = (\psi, \lambda\delta_a) = \|\psi\|,\ \psi = 0$ on $E_{\boldsymbol{P}}(a)\}$; $WAP_{\boldsymbol{P}} = \boldsymbol{P} \cap WAP$ where $\Phi \in A^*$ is in WAP iff $\{u \cdot \Phi; u \in A,\ \|u\| \leq 1\}$

is relatively weakly compact in A^*, where $(u \cdot \Phi, v) = (\Phi, uv)$ for $u, v \in A$. We prove in Section 2

THEOREM 4. *Let G be arbitrary, J a closed ideal of $A = A(G)$, or $A_p(G)$. Let $\boldsymbol{Q} \subset A^*$ be a norm closed A module such that $\boldsymbol{P}_c \subset \boldsymbol{Q} \subset \boldsymbol{P} = (A/J)^*$ and $D_1(J) \neq \emptyset$.*

Then $\boldsymbol{Q}/W_{\boldsymbol{Q}}(x)$ (a fortiori $\boldsymbol{Q}/WAP_{\boldsymbol{Q}}$ and $\boldsymbol{Q}/\mathbf{M}(F)$) has ℓ^∞ as a quotient and $TIM_{\boldsymbol{Q}}(x)$ contains $\boldsymbol{\mathcal{F}}$, (i.e., $\boldsymbol{Q}$ is ENE) for each $x \in D_1(J)$.

Consequently A/J is ENAR if G is second countable nondiscrete.

Here $\mathbf{M}(F) = \mathrm{ncl}\{\lambda\mu;\ \mu \in M(F)\}$ where $(\lambda\mu, v) = \int v\,d\mu$ for $v \in A$. The Banach algebra A/J is Arens regular if $\boldsymbol{P} = WAP_{\boldsymbol{P}}$. A/J is extremely non Arens regular (ENAR) if $\boldsymbol{P}/WAP_{\boldsymbol{P}}$ is "as big as $\boldsymbol{P}$" namely if it contains a subspace which has $\boldsymbol{P}$ as a quotient. We abbreviate the conclusion of Theorem 4 about $\boldsymbol{Q}$ writing that $\boldsymbol{Q}$ is extremely nonergodic (ENE) at each $x \in D_1(J)$.

Assume, for simplicity, in Corollaries 6 and 7 that G is metrisable.

COROLLARY 6. *Let $A = A(G)$ and $J \subset A$, $\boldsymbol{P} = (A/J)^*$, $\boldsymbol{P}_c \subset \boldsymbol{Q} \subset \boldsymbol{P}$, $\sigma(\boldsymbol{P}) = F$ be as in Theorem 4. Assume that R (or T) is a closed subgroup of G, and $S \subset R$ (or T) an ultrathin symmetric set (see Section 1) such that $aSb \subset F$, for some $a, b \in G$.*

Then $\boldsymbol{Q}$ is ENE at each $x \in aSb$. Thus A/J is ENAR if G is second countable nondiscrete.

The reader should note that even the fact that $\boldsymbol{Q} \neq W_{\boldsymbol{Q}}(x)$ is a nontrivial result. If $G = T$ and $F \subset T$ is ultrathin symmetric, it has been proved by Woodward [Wo1] that $\boldsymbol{P} \neq W_{\boldsymbol{P}}(x)$ for some $x \in F$. Corollary 6 implies that $\boldsymbol{P}/W_{\boldsymbol{P}}(x)$ has even the big nonseparable space ℓ^∞ as a quotient for each $x \in F$. Corollary 6 also improves Theorem 12 in [Gr5].

COROLLARY 7. *Let $A = A(G)$ or $A_p(G)$, $1 < p < \infty$ and $J \subset A$, $\boldsymbol{P} = (A/J)^*$, $\boldsymbol{P}_c \subset \boldsymbol{Q} \subset \boldsymbol{P}$, $\sigma(\boldsymbol{P}) = F$ be as in Theorem 4. Assume that $H \subset G$ is a closed nondiscrete subgroup, $a, b \in G$ and $\mathrm{int}_{aHb} F \neq \emptyset$.*

Then $\boldsymbol{Q}$ is ENE at each $x \in \mathrm{int}_{aHb} F$. Thus A/J is ENAR if G is second countable nondiscrete.

Corollary 7 improves a particular case of Theorem 6 in [Gr5] with a simpler proof. It (and Corollary 6) show the prevalence of convolution operators $\Phi \in \boldsymbol{P}$ on $L^p(G)$ (on $L^2(G)$) which are nonergodic at certain $x \in \sigma(\boldsymbol{P})$, i.e. such that $\Phi \notin W_{\boldsymbol{P}}(x)$ (a fortiori $\Phi \notin \mathbf{M}(F)$). (See [Gr5], p. 53.) Parts of Corollaries 6 and 7 have been improved to nonmetrisable G, H, F in [Gr6].

In Section 3 we apply the above machinery to locally compact abelian (lca) groups G. Let $\mathcal{F}$: $L^1(\widehat{G}) \to A(G)$ [$\mathcal{F}_S$: $M(\widehat{G}) \to B(G)$] denote Fourier [Stiltjes] transform. Thus $\mathcal{F}^*$: $PM(G) \to L^\infty(\widehat{G})$ is an isometry and w^*–w^* homemorphism.

If $f \in L^\infty(\widehat{G})$ let $\Sigma(f) = G \cap w^*\text{cllin}\{\overline{f}_\gamma;\ \gamma \in \widehat{G}\}$, where $f_\gamma(\chi) = f(\gamma\chi)$, G is the dual of $\widehat{G}$, and lin, w^*cl denote linear span, w^* closure, respectively.

Let $P \subset L^\infty(\widehat{G})$ be a norm closed $M(\widehat{G})$ module thus $M(\widehat{G}) * P \subset P$. This is the case iff $\boldsymbol{P} = \mathcal{F}^{*-1}P$ is a $B(G)$ module, i.e., $B(G)\cdot\boldsymbol{P} \subset \boldsymbol{P}$ where $(u\cdot\Phi, v) = (\Phi, uv)$ for $u \in B(G)$ $v \in A(G)$. Then define

$$D_{\boldsymbol{P}}(a) = \text{ncl}\,\text{lin}\{\Phi - (\chi)_{a^{-1}}\cdot\Phi;\ \chi \in \widehat{G},\ \Phi \in \boldsymbol{P}\};\ V_{\boldsymbol{P}}(a) = C(\lambda\delta_a) + D_{\boldsymbol{P}}(a).$$

$$D_P(a) = \text{ncl}\,\text{lin}\{f - a(\chi)f_\chi;\ \chi \in \widehat{G},\ f \in P\};\ V_P(a) = C\bar{a} + D_P(a)$$

$$\begin{aligned} E_P(a) &= \text{ncl}\,\text{lin}\{f - (\bar{a}h) * f;\ 0 \le h \in L^1(\widehat{G}),\ \textstyle\int h\,d\chi = 1, f \in P\};\ W_P(a) \\ &= C\bar{a} + E_P(a). \end{aligned}$$

The next paragraph shows the relevance and need of the above definitions. It should be reread before going through Section 3.

The space $E_P(a)$ is of interest in commutative harmonic analysis since $E_P(a) =$ $\text{ncl}\{f \in P;\ a \notin \Sigma(f)\}$ whenever $P \subset L^\infty(\widehat{G})$ is a norm closed $M(\widehat{G})$ submodule (Lemma 8′), and hence the reason for this definition. In this case $\mathcal{F}^* E_{\boldsymbol{P}}(a) = E_P(a)$ and $\mathcal{F}^* W_{\boldsymbol{P}}(a) = W_P(a)$, $\mathcal{F}^* D_{\boldsymbol{P}}(a) = D_P(a)$ and $\mathcal{F}^* V_{\boldsymbol{P}}(a) = V_P(a)$ (Lemma 8). It so happens then that $D_P(a) \subset E_P(a)$, $V_P(a) \subset W_P(a)$ with equality if $P \subset UC(\widehat{G})$ (UC from uniformly continuous) (see Prop. 9), a fortiori if $\sigma(P) = G \cap \overline{P}$ is compact where $\overline{P} = \{\bar{f};\ f \in P\}$. If $a \in \sigma(P)$ let

$$TIM_P(a)\ [IM_P(a)] = \{\psi \in P^*;\ 1 = (\psi, \bar{a}) = \|\psi\|,\ \psi = 0 \text{ on } E_P(a)\ [\text{on } D_P(a)]\}$$

(thus $TIM_P(a) \subset IM_P(a)$) respectively. If $a = e$, these become the set of honest to goodness topologically invariant [invariant] means on P. Also $TIM_P(a) = IM_P(a)$ if $P \subset UC(\widehat{G})$ (by Prop. 9).

In the next two corollaries let P $[Q]$ be a w^* [norm] closed $M(\widehat{G})$ submodule of $L^\infty(\widehat{G})^*$ such that $UC_P \subset Q \subset P$, where $UC_P = UC(\widehat{G}) \cap P$. Thus $P =$ $(L^1(\widehat{G})/J)^*$ for a unique closed ideal $J \subset L^1(\widehat{G})$, with $\sigma(P) = G \cap \overline{P} = \{x \in G;\ (\mathcal{F}f)(x) = 0 \text{ if } f \in J\}$.

Q is called ENE at x if $Q/W_Q(x)$ has ℓ^∞ as a quotient and $TIM_Q(x)$ contains $\boldsymbol{\mathcal{F}}$.

COROLLARY 10. *Let G be a metrisable l.c.a. group $UC_P \subset Q \subset P \subset L^\infty(\widehat{G})$ and $\sigma(P) = F$. Assume that R or T is a closed subgroup of G, $S \subset R$ (or T) an ultrathin symmetric set such that $aS \subset F$ for some $a \in G$.*

Then Q (hence P and UC_P) are ENE at each $x \in aS$.

COROLLARY 11. *Let G, P, Q, F be as in Corollary 10. Assume that H is a nondiscrete closed subgroup and $a \in G$ be such that $\text{int}_{aH} F \neq \emptyset$.*

Then Q (hence P and UC_P) are ENE at each $x \in \text{int}_{aH} F$.

If $B(\widehat{G}, F) = \mathcal{F}_S M(F)$ then ncl $B(\widehat{G}, F) \subset WAP_Q \subset V_Q(x) \subset W_Q(x)$ for all $x \in F$. A consequence of Corollary 10 [or 11] is that $Q/V_Q(x)$, Q/WAP_Q, $Q/\operatorname{ncl} B(\widehat{G}, F)$ have ℓ^∞ as a quotient and $IM_Q(x) \supset TIM_Q(x)$ both contain $\boldsymbol{\mathcal{F}}$ for all $x \in aSb$ [$x \in \operatorname{int}_{aHb} F$] respectively. Furthermore, if G is second countable, then the Banach algebra $L^1(\widehat{G})/J$ is ENAR.

1. It has been proved by J. P. Kahane that there exist continuous [smooth] curves $F \subset R^2$ [$F \subset R^n$, $n \geq 3$] which are Helson sets (see [Mc], [Mu] or [Ka 1,2,3]). Thus if $P = w^* \operatorname{cl} \operatorname{lin} F \subset L^\infty(\widehat{G})$ where $F \subset G = R^n$ [R^2], then $P = W_P(x) = V_P(x) = B(\widehat{G}, F)$ for all $x \in F$. Our Corollary 10 implies that for any line L in R^2 [R^n], $L \cap F$ cannot contain an ultrathin symmetric set.

2. Assume that G is l.c.a. metrisable, $K = \Pi_1^\infty K_n \subset G$ where K_n are finite nontrivial abelian groups. Asume that $\operatorname{int}_{aKb} F \neq \emptyset$. Then Q is ENE at each $x \in \operatorname{int}_{aKb} F$ by Corollary 11.

Additional definitions and notations

Let λ (or dx) be a fixed left Haar measure and $L^p(G)$, $1 \leq p \leq \infty$, the usual complex valued function spaces (see [HR]). Let $C(G)$, $[UC(G)]$, $WAP(G)$, $C_0(G)$, $C_c(G)$ denote the bounded [uniformly] continuous complex functions on G which are in additon weakly almost periodic, tending to 0 at ∞, have compact support, respectively.

If $f \in C(G)$ let $\operatorname{supp} f = \operatorname{cl}\{x \in G;\ f(x) \neq 0\}$ where cl denotes closure. If $F \subset G$ is closed then $M(F)$ are the complex bounded regular Borel measures on F with variation norm, thus $M(F) = C_0(F)^*$. All convolution formulas are as in [HR].

If f is a function on G, $x, y \in G$ then $f^\vee(x) = f(x^{-1})$, $f_x(y) = f(xy)$. A neighborhood (nbhd) of x is any open set $U \subset G$ containing x.

If $F, H \subset G$ then $\operatorname{int}_H F$ is the interior of $F \cap H$ in H. Thus $x \in \operatorname{int}_H F$ iff for some nbhd V (in G) of x, $x \in V \cap H \subset F \cap H$. Denote $F \sim H = \{x \in F;\ x \notin H\}$.

Let $A(G)$ denote the Fourier algebra of G, as in [Ey]. $A_p(G)$, $1 < p < \infty$, are the regular tauberian Banach algebras on G defined in [Hz]; thus $A_2(G) = A(G)$.

Let $A(G)^* = PM(G)$, the dual of $A(G)$ (denoted $VN(G)$ in [Ey] or $CV_2(G)$ in [Gr5]). If G is abelian then $A(G) = \mathcal{F}L^1(\widehat{G})$.

If $J \subset A(G) = A$ is a closed ideal let $Z(J) = \{x \in G;\ v(x) = 0 \text{ if } v \in J\}$. Equip the quotient algebra A/J with the norm $\|v\|_{A/J} = \inf\{\|v - u\|;\ u \in J\}$. If $F \subset G$ let $I_F = \{v \in A;\ v = 0 \text{ on } F\}$.

If G is a locally compact abelian group then the linear space $P \subset L^\infty(\widehat{G})$ is a $M(\widehat{G})$ [$L^1(\widehat{G})$] module iff $M(\widehat{G}) * P \subset P$ [$L^1(\widehat{G}) * P \subset P$].

Examples of norm closed $M(\widehat{G})$ modules are any $w^*(\beta)$ [norm] closed translation invariant subspace (or $L^1(\widehat{G})$ submodule) of $L^\infty(\widehat{G})$ ($C(G)$) [$UC(G)$] respectively (see [Co], p. 221).

If X is a Banach space (always over C the complex numbers) X^* denotes its dual. If $Y \subset X$ let ncl Y [lin Y] denote the norm closure [linear span] of Y in X.

The Banach spaces $c_0 \subset c \subset \ell^\infty$ over the complex field are as in [LT]. Let $c_0^\perp = \{\varphi \in \ell^{\infty *}; \varphi = 0 \text{ on } c_0 \subset \ell^\infty\}$ and $\mathcal{F} = \{\varphi \in c_0^\perp; 1 = (\varphi, 1) = \|\varphi\|\}$. $\mathcal{F}$ is a w^* compact perfect convex set such that card $\mathcal{F}$ = card $\ell^{\infty *} = 2^c$ where c is the cardinality of the reals. $X \approx Y$ denotes isomorphism of Banach spaces [LT].

The Banach algebra $(A, \| \ \|_A)$ is called (in this paper) a regular Banach algebra on (the locally compact space) X if, with the notation in [HR], (39.1), (39.11), A is a regular Banach algebra in $C_0(X)$ where X is the structure space of A.

If in addition $A \cap C_c(X)$ is norm dense in $(A, \| \ \|_A)$ then A is called a regular tauberian Banach algebra on X (which coincides with [Hz], p. 100).

For example if J is any closed ideal of $A(G)$ (or $A_p(G)$) and $F = Z(J)$ then J $[A(G)/I_F]$ is a regular [regular tauberian] Banach algebra on $G \sim F$ $[F]$ respectively ([HR], (39.15), [Hz], p. 101).

Let (to the end of this section) A be a regular Banach algebra on X and $\varphi \in A^*$. Define, supp $\varphi \subset X$ by: $x \in \operatorname{supp}\varphi$ iff for any nbhd U of x there is some $f \in A$ such that supp $f \subset U$ and $(\varphi, f) \neq 0$. supp φ is a (possibly void) closed set such that supp $(f \cdot \varphi) \subset \operatorname{supp} f \cap \operatorname{supp}\varphi$ if $f \in A$, $\varphi \in A^*$ where $(f \cdot \varphi, g) = (\varphi, fg)$ for $g \in A$, as is easily shown.

Let $\boldsymbol{P} \subset A^*$ be a closed subspace. Let $\sigma(\boldsymbol{P}) = \{x \in X; \lambda\delta_x \in \boldsymbol{P}\}$ and $\boldsymbol{P}_c$ = ncl $\{\Phi \in \boldsymbol{P};$ supp Φ is compact$\}$. If $a \in X$ let $E_{\boldsymbol{P}}(a)$ = ncl $\{\Phi \in \boldsymbol{P}, a \notin \operatorname{supp}\Phi\}$; $W_{\boldsymbol{P}}(a) = C(\lambda\delta_a) + E_{\boldsymbol{P}}(a)$; $TI_{\boldsymbol{P}}(a) = \{\psi \in A^{**}; \psi = 0 \text{ on } E_{\boldsymbol{P}}(a)\}$; $TIM_{\boldsymbol{P}}(a) = \{\psi \in TI_{\boldsymbol{P}}(a); 1 = (\psi, \lambda\delta_a) = \|\psi\|\}$ if $a \in \sigma(\boldsymbol{P})$.

Let $J \subset A$ be a closed ideal with $F = Z(J)$ ($J = \{0\}$ may occur). *In memory of M. M. Day*, see [Da], define the set $D_1(J) \subset F$ as in the introduction, with $A(G)$ replaced by A. Define $D_b(J) \subset F$ (b from "bounded") in the same way as $D_1(J)$ except that (i) is replaced by (i)′ $1 = u_n(a) \leq \sup \|u_n\|_A < \infty$.

Clearly $D_1(J) \subset D_b(J)$ and if $I \subset J$ are closed ideals in A with $F = Z(I) = Z(J)$ then $D_1(J) \subset D_1(I)$ and $D_b(J) \subset D_b(I)$ (since $\|u\|_{A/I} \geq \|u\|_{A/J}$).

$\Phi \in A^*$ is in $WAP(A^*)$ iff $\{u \cdot \Phi;\ u \in A, \|u\| \leq 1\}$ is a relatively weakly compact subset of A^*. A is Arens regular iff $A^* = WAP(A^*)$. A is ENAR iff $A^*/WAP(A^*)$ contains a closed subspace which has A^* as a quotient. Note that if A is separable and $\{x_n\}$ is dense in the unit ball of A, then t: $A^* \to \ell^\infty$ given by $(t\Phi)(n) = (\Phi, x_n)$ is an isometry, thus $A^* \subset \ell^\infty$. Hence if $A^*/WAP(A^*)$ has ℓ^∞ as a quotient then A is ENAR (since if q: $A^*/WAP(A^*) \to \ell^\infty$ is onto then $X = q^{-1}(A^*)$ has A^* as a quotient).

1. When $D_1(J)$ is nonempty

DEFINITION. The set $E \subset R$ is called symmetric (see [Me] or [GMc]) if there are $t_n > 0$ such that $t_n > \sum_{n+1}^\infty t_i$ for all n, and $E = \left\{\sum_1^\infty \varepsilon_i t_i;\ \varepsilon_i = 0 \text{ or } 1\right\}$. If in addition $\sum_1^\infty (t_{i+1}/t_i)^2 < \infty$ then E is called ultrathin symmetric.

In the next two lemmas, for closed $F \subset R$, let $A(F) = A(R)/I_F$. The following is due to Y. Meyer ([Me], p. 246).

LEMMA. *Let $E \subset R$ be ultrathin symmetric. Let $f_k \in A(E)$ be such that $\|f_k\|_{A(E)} = 1$ for $k \geq 1$ and $\|f_k\|_{A(K)} \to 0$ for each compact $K \subset E$ with $0 \notin K$. Then $\{f_k\}$ contains a subsequence which is Sidon in $A(E)$.*

We improve this as follows:

LEMMA 1. *Let $E = \left\{\sum_1^\infty \varepsilon_i t_i;\ \varepsilon_i = 0, 1\right\} \subset R$ be ultrathin symmetric and $a \in E$. Let $u_k \in A(R)$ be such that $1 = u_k(a) \leq \|u_k\|_{A(E)} \leq B < \infty$ and $\|u_k\|_{A(K)} \to 0$ for all compact $K \subset E$ with $a \notin K$. Then $\{u_k\}$ contains a subsequence which is Sidon in $A(E)$.*

Remark. This lemma also holds for sets E for which $-E = \{-x; x \in E\}$ is ultrathin symmetric.

Proof. (i) Let $a = s = \sum_1^\infty t_i$. Then $s - E = E$ and if $u'(x) = u(s-x)$ for $u \in A(R)$ and $x \in R$ then $u'' = u$ and $\|u'\| = \|u\|$, where $\|u\|$ denotes $\|u\|_{A(R)}$. Also $u \in I_E$ iff $u' \in I_E$. Let $K \subset E$ be compact and $K' = s - K$. Then $\|u\|_{A(K)} = \inf\{\|u+v\|;\ v \in I_K\} = \inf\{\|u'+v'\|;\ v \in I_K\} = \inf\{\|u'+v\|;\ v \in I_{K'}\} = \|u'\|_{A(K')}$, since $v'(x) = v(s-x) = 0$ for $x \in K'$ iff $v(y) = 0$ for $y \in s - K' = K$. In particular $\|u\|_{A(E)} = \|u'\|_{A(E)}$ since $E' = E$. If $K \subset E$ is compact and $0 \notin K$ then $\|u_k'\|_{A(K)} = \|u_k\|_{A(K')} \to 0$ since $s \notin K' = s - K$. If $v_k = (\|u_k\|_{A(E)})^{-1} u_k'$ then, since $B^{-1} \leq (\|u_k\|_{A(E)})^{-1} \leq 1$, v_k has a subsequence which is Sidon in $A(E)$ by Y. Meyer s lemma, hence so does $\{u_k\}$. This proves the case $a = s$.

(ii) Assume that $a = \sum_1^\infty t_{n_i}$ where $\{m_j\} = \{n \geq 1;\ n \notin \{n_i\}\}$ is infinite. Consider the set $a + E_0$ where $E_0 = \left\{\sum_1^\infty \varepsilon_j t_{m_j};\ \varepsilon_j = 0 \text{ or } 1\right\}$. Then E_0 is ultrathin symmetric and $a + E_0 \subset E$. Let $u'(x) = u(a+x)$ if $x \in R$; thus $\|u'\| = \|u\|$. Clearly, if $D \subset F$ then $I_F \subset I_D$ and $\|u\|_{A(F)} \geq \|u\|_{A(D)}$. Hence $\|u\|_{A(a+E_0)} = \inf\{\|u+v\|;\ v \in I_{a+E_0}\} = \inf\{\|u'+v'\|;\ v \in I_{a+E_0}\} = \inf\{\|u'+v'\|;\ v' \in I_{E_0}\} = \|u'\|_{A(E_0)}$.

And $B \geq \|u_k\|_{A(E)} \geq \|u_k\|_{A(a+E_0)} = \|u_k'\|_{A(E_0)} \geq u_k'(0) = 1$.

If now $K \subset E_0$ is compact then $u \in I_K$ iff $u'(x) = u(x+a) = 0$ for all $x \in K - a$ iff $u' \in I_{K-a}$. And $\|u\|_{A(a+K)} = \inf\{\|u+v\|;\ v \in I_{a+K}\} = \inf\{\|u'+v'\|;\ v \in I_{a+K}\} = \inf\{\|u'+v'\|;\ v' \in I_{a+K-a}\} = \|u'\|_{A(K)}$.

If $K \subset E_0$ is compact and $0 \notin K$ then $\|u_k'\|_{A(K)} = \|u_k\|_{A(a+K)} \to 0$ since $a \notin a+K$. Hence we can apply Meyer's lemma and get that some subsequence $\{u_{n_k}'\}$ is Sidon in $A(E_0)$. Thus $\{u_{n_k}\}$ is Sidon in $A(a+E_0)$ since $\|u\|_{A(a+E_0)} = \|u'\|_{A(E_0)}$, by the above. But $B \geq \|u_k\|_{A(E)} \geq \|u_k\|_{A(a+E_0)}$. Hence $\{u_{n_k}\}$ is Sidon in $A(E)$.

(iii) Assume now that $a = \sum t_{n_i}$ where $\{n;\ n \notin \{n_i\}\}$ is finite. Thus $a = \sum_1^k t_{n_i} + \sum_{N+1}^\infty t_j$ with $n_i \leq N$ for $i \leq k$. Define then the sequence $\{s_n\}$ by $s_i = t_{n_i}$ if $i \leq k$ and $s_i = t_i$ if $i \geq N+1$. Then the set $E_1 = \left\{\sum_1^\infty \varepsilon_i s_i;\ \varepsilon_i = 0 \text{ or } 1\right\}$ is an ultrathin symmetric set and $a = \sum_i^\infty s_i$. Also $B \geq \|u_k\|_{A(E)} \geq \|u_k\|_{A(E_1)} \geq u_k(a) = 1$. And if $K \subset E_1$ is compact and $a \notin K$ then $K \subset E$ is compact and $a \notin K$. Thus $\|u_k\|_{A(K)} \to 0$. Hence by case (i) there exists a subsequence $\{u_{k_j}\}$ which is Sidon in $A(E_1)$, a fortiori in $A(E)$. □

Proof of Remark. If $u'(x) = u(-x)$ for all x, then $\|u'\|_{A(R)} = \|u\|_{A(R)}$. And if $F = -E$ then $(I_E)' = I_F$, thus $\|u'\|_{A(F)} = \|u\|_{A(E)}$. Use of Lemma 1 for the sequence $\{u'_n\}$ at $-a \in F$ will imply that $\{u_n\}$ has a subsequence which is Sidon in $A(E)$.

THEOREM 2. *Let G be any locally compact group $J \subset A(G)$ be a closed ideal and $F = Z(J)$. Assume that R (or T) is a closed subgroup of G and $S \subset R$ is an ultrathin symmetric set such that $aSb \subset F$ for some $a, b \in G$.*

If F is first countable at each $x \in aSb$, a fortiori if F is metrisable then $aSb \subset D_1(J)$.

Remarks. (i) We show that if F is first countable at $x \in aSb$ then $x \in D_1(J)$.
(ii) If Lemma 1 holds for $A_p(R)$ then this theorem holds for $A_p(G)$, since only results in [Hz] are used.

Proof. Fix $s \in S$ and let V_n be open in G such that $asb \in V_n$, let cl V_n be compact and $V_n \cap F$ be a neighborhood base in F at $asb \in F$ (F is first countable at asb). Let $v_n \in A(G) = A$ be such that $v_n(asb) = 1 = \|v_n\|$ and supp $v_n \subset V_n$. If V is a nbhd of asb there is some n_0 such that $F \cap \operatorname{supp} v_n \subset V \cap F$ if $n \geq n_0$.

Let $A' = A/J$ where for $v \in A(G)$, $v' = v + J$ and $\|v'\| = \inf\{\|v + u\|;\ u \in J\}$. We show, using Lemma 1, that there is a subsequence v'_{n_j} which is Sidon in A'.

Let $r\colon A(G) \to A(R)$ be the restriction map $(rv)(x) = v(x)$ if $x \in R$. Then r is onto and $\|r\| \leq 1$ by Herz [Hz], p. 92. Now ℓ_a, r_b defined by $\ell_a u(x) = u(ax)$, $r_b u(x) = u(xb)$ are isometric isomorphisms on $A(G)$ ([Hz], p. 97) and $\ell_a r_b = r_b \ell_a$.

If $u_n = \ell_a r_b v_n$ then $ru_n(s) = v_n(asb) = 1 = \|v_n\| \geq \|r\ell_a r_b v_n\| \geq \ell_a r_b v_n(s) = 1$ hence $ru_n(s) = 1 = \|ru_n\|$.

For closed $L \subset G$ [$L \subset R$] let $I_L = \{v \in A(G);\ v = 0 \text{ on } L\}$, [$I_L^R = \{u \in A(R);\ u = 0 \text{ on } L\}$]. Let $A(L) = A(G)/I_L$, $A^R(L) = A(R)/I_L^R$ and $q\colon A(R) \to A^R(S)$ be the cannonical map (thus $\|q\| \leq 1$.)

Let $K \subset S$ be compact such that $s \notin K$. Then $asb \notin aKb \subset F$. Hence there is an n_0 such that for $n \geq n_0$, $V_n \cap aKb = \emptyset$; thus $a^{-1}V_n b^{-1} \cap K = \emptyset$ (and $asb \in V_n$). Now supp $u_n = \sup \ell_a r_b v_n \subset a^{-1}V_n b^{-1}$. Hence if $n \geq n_0$, $K \cap \operatorname{supp} ru_n \subset K \cap a^{-1}V_n b^{-1} = \emptyset$ and $\|ru_n\|_{A^R(K)} = 0$. Hence $\|ru_n\|_{A^R(K)} \to 0$ for any compact $K \subset S$ such that $s \notin K$. We also note that $qru_n(s) = 1 \geq \|qru_n\| \geq qru_n(s)$; hence $qru_n(s) = 1 = \|qru_n\|$. We now apply Lemma 1 and get that there is a subsequence u_{n_j} and some $c > 0$ such that $\|\sum_1^k \alpha_j qru_{n_j}\| \geq c\sum_1^k |\alpha_j|$ for all $k \geq 1$ and complex α_j.

Fix $v \in A(G)$ and let $u = \ell_a r_b v$. We claim that $\|v\|_{A/J} \geq \|qru\|_{A^R(S)}$. This will show that v'_{n_j} is a Sidon sequence in $A' = A/J$. One has $\|v\|_{A/J} = \inf\{\|v + w\|; w \in J\} \geq \inf\{\|\ell_a r_b(v + w)\|;\ w \in J\} = \inf\{\|u + w\|;\ w \in \ell_a r_b J\} \geq \inf\{\|u + w\|;\ w \in I_H\}$ (where $H = a^{-1}Fb^{-1}$) $\geq \inf\{\|ru + rw\|;\ w \in I_H\} \geq \inf\{\|ru + w\|;\ w \in I_{H\cap R}^R\}$ (since $rI_H \subset I_{H\cap R}^R$) $\geq \inf\{\|ru + w\|;\ w \in I_S^R\}$ (since $S \subset a^{-1}Fb^{-1} \cap R = H \cap R$) $=$

$\|qru\|_{A^R(S)}$. Hence $\left\| \sum_1 \alpha_j v'_{n_j} \right\|_{A/J} \geq \left\| \sum_1^k \alpha_j q r u_{n_j} \right\|_{A^R(S)} \geq c \sum_1^k |\alpha_j|$ for all $k \geq 1$, and complex α_j. □

COROLLARY 2′. *Theorem 2 holds for any set $S \subset R$ expressible as a union $S = \bigcup_{\alpha \in I}(x_\alpha + S_\alpha)$ where S_α or $-S_\alpha$ are ultrathin symmetric, $x_\alpha \in R$ and I is any index set. In particular it holds if S is any symmetric set.*

Proof. To make the additive and multiplicative notation consistent replace $x_\alpha + S_\alpha$ by $x_\alpha S_\alpha$ and $-S$ by S^{-1}. The proof of Theorem 2 works if S or S^{-1} are ultrathin symmetric by the remark after Lemma 1. Let now $S = \bigcup_{\alpha \in I} x_\alpha S_\alpha \subset R$ with S_α or S_α^{-1} ultrathin symmetric and $aSb \subset F$. If $s \in S$ then $asb \in ax_\beta S_\beta b \subset F$ for some β. Use of Theorem 2 with S replaced by S_β shows that $asb \in (ax_\beta)S_\beta b \subset D_1(J)$.

Let $S = \{\sum_1^\infty \varepsilon_i t_i;\ \varepsilon_i = 0, 1\}$ be symmetric where $\infty > t_m > \sum_{n+1}^\infty t_j > 0$ for all $n \geq 1$. Let $x = \Sigma t_{n_i}$ and $M = \{m \geq 1; m \notin \{n_i\}\}$. If $M = \{m_j\}$ is infinite let $s_j = t_{m_j}$. Choose $s_{j_1} = s_1$ and if s_{j_k} was chosen let $j_{k+1} > j_k$ be such that $s_{j_{k+1}} < (1/2)s_{j_k}$. then $\Sigma(s_{j_{k+1}}/s_{j_k})^2 < \infty$ and $S_x = \{\Sigma \varepsilon_k s_{j_k}, \varepsilon_k = 0, 1\}$ is ultrathin symmetric such that $x + S_x \subset S$.

If M is finite then $x = \sum_1^k t_{n_i} + \sum_{N+1}^\infty t_k$ where $n_k \leq N$. Choose then $N+1 \leq k_1 < k_2 < \cdots$ such that $\sum_j (t_{k_{j+1}}/t_{k_j})^2 < \infty$. Let $S_x = \{\sum_1^\infty \varepsilon_j t_{k_j}; \varepsilon_j = 0, 1\}$. Then S_x is ultrathin symmetric and $x - S_x \subset S$. □

THEOREM 3. *Let G be a locally compact group, $H \subset G$ a closed nondiscrete subgroup. Let $J \subset A = A_p(G)$ be a closed ideal, $F = Z(J)$ and $a, b \in G$. Let F be metrisable.*

Then $\operatorname{int}_{aHb} F \subset D_1(J)$. In particular $D_1(0) = G$ if G is metrisable, nondiscrete.

Remark. We show that for any closed F, if F is first countable at $y \in F$ and $y \in \operatorname{int}_{aHb} F$, then $y \in D_1(J)$.

Proof. Let V_n be open such that $x_0 \in V_0 \cap aHb \subset F$, cl V_0 is compact, cl $V_{n+1} \subset V_n$ for $n \geq 0$, and $V_n \cap F$ is a neighborhood base in F at x_0. Let $v_n \in A$ be such that $v_n(x_0) = 1 = \|v_n\|$ and supp $v_n \subset V_n$ for $n \geq 0$ (see [Gr3], p. 379). We show that $v'_n \in A/J$ has no weak Cauchy sequence in A/J, where for $v \in A$, we let $v' = v + J \in A/J$ with $\|v'\| = \inf\{\|v + u\|; u \in J\}$. It will follow from H. Rosenthal's theorem [Ro], p. 808, that v_n contains a subsequence v_{n_k} such that $\{v'_{n_k}\}$ is a Sidon sequence in A/J; thus $x_0 \in D_1(J)$.

Assume that $u'_k = v'_{n_k}$ is a weak Cauchy sequence in A/J and let $\boldsymbol{P} = \{\Phi \in A^*; \Phi = 0 \text{ on } J\} = (A/J)^*$. Let r: $A_p(G) \to A_p(H)$ be the *onto* restriction map; thus $rv(x) = v(x)$ if $x \in H$, $v \in A$, $\|r\| \leq 1$, and $rA_p(G) = A_p(H)$ (due to Herz [Hz]).

Let $\Phi \in A_p(H)^*$ and $w \in J$. Then

$$(\ell_a^* r_b^*[(\ell_a r_b v_0) \cdot r^*\Phi], w) = (\Phi, r\ell_a r_b(v_0 w)) = (\Phi, 0) = 0$$

since if $h \in H$ and $\ell_a r_b(v_0 w)(h) = v_0(ahb)w(ahb) \neq 0$ then $ahb \in V_o \cap aHb \subset F$ and then $w(ahb) = 0$ since $F = Z(J)$. Thus $r\ell_a r_b(v_o w) = 0 \in A_p(H)$. Hence $\ell_a^* r_b^*[(\ell_a r_b v_o) \cdot r^*\Phi] \in \boldsymbol{P}$ for all $\Phi \in A_p(H)^*$.

It follows that $(\ell_a^* r_b^*[(\ell_a r_b v_0) \cdot r^*\Phi], u_k') = (\Phi, r\ell_a r_b(v_0 u_k))$ is a Cauchy sequence of scalars for all $\Phi \in A_p(H)^*$. (Note that $(\Phi, u) = (\Phi, u')$ for $\Phi \in \boldsymbol{P}$, $u \in A$ is well defined.)

Now supp $r(\ell_a r_b v_0 u_k) \subset a^{-1}V_0 b^{-1} \cap H$ and the latter set has closure K which is compact. If follows (from the Hahn Banach theorem) that $r\ell_a r_b(v_o u_k)$ is a weak Cauchy sequence in $A_K^p(H) = \{u \in A_p(H);\ \mathrm{supp}\, u \subset K\}$. Now by a joint result of Cowling and ours [Gr5], p. 131, $A_K^p(H)$ is weak sequentially complete. Hence $r\ell_a r_b(v_0 u_k) \to w_0$ weakly in $A_K^p(H)$ (hence in $A_p(H)$) for some $w_0 \in A_p(H)$. (If $p = 2$, then $A(G)$ as a predual of a W^* algebra is weak sequentially complete, hence this result in [Gr5] is not needed.) Since $\lambda\delta_h \in A_p(H)^*$, $v_0(ahb)u_k(ahb) \to w_0(h)$, for all $h \in H$.

If $h_0 = a^{-1}x_0 b^{-1} \in a^{-1}V_0 b^{-1} \cap H$ then $v_0(ah_0 b)u_k(ah_0 b) = 1$; thus $w_0(a^{-1}x_0 b^{-1}) = 1$. If $a^{-1}x_0 b^{-1} \neq h_1 \in a^{-1}V_0 b^{-1} \cap H$, then $x_0 \neq ah_1 b \in V_0 \cap aHb \subset F$.

But $V_n \cap F$ is a base of neighborhoods in F at x_0. Thus for some k_0, $u_k(ah_1 b) = 0$ if $k \geq k_0$. Hence $v_0(ah_1 b)u_k(ah_1 b) = 0 = w_0(h_1)$ if $k \geq k_0$. But $x_0 \in V_0 \cap aHb \subset F$ and x_0 is not an isolated point of F since H is not discrete. It follows that $w_0 \in A_p(H)$ is not a continuous function, a contradiction. □

Remark. We prove in Theorem 2[3] more than stated. Namely we show that if $x \in aSb$ $[x \in \mathrm{int}_{aHb} F]$ and $v_n \in A \cap C_c(G)$ is *any* sequence satisfying (i) $v_n(x) = 1 = \|v_n\|$, and (ii) $F \cap \mathrm{supp}\, v_n = K_n$ is such that for any nbhd V of x there is some k such that $K_j \subset V$ if $j \geq k$, then v_n has a subsequence which is Sidon in A/J.

2. Extreme nonergodicity of $\boldsymbol{P} = (A/J)^*$ at any $a \in D_1(J)$

If $J \subset A(G)$ (or $A_p(G)$) a closed ideal with $F = Z(J)$ then A/I_F, [J] are regular Banach algebras on F $[G \sim F]$ respectively, hence so are $A_p(G)$, $A(G)$. This is the reason for stating Theorem 4 in terms of regular Banach algebras.

THEOREM 4. *Let $A = A(G)$ be a regular Banach algebra on the locally compact space G. Let $J \subset A$ be a closed ideal and $\boldsymbol{Q} \subset A^*$ be a norm closed A module such that $\boldsymbol{P}_c \subset \boldsymbol{Q} \subset \boldsymbol{P} = (A/J)^*$.*

If $a \in D_b(J)$ $[a \in D_1(J)]$ then $\boldsymbol{Q}/W_{\boldsymbol{Q}}(a)$ has ℓ^∞ as a quotient and $TI_{\boldsymbol{Q}}(a)$ contains $c_0^\perp$ [and $TIM_{\boldsymbol{Q}}(a)$ contains $\boldsymbol{\mathcal{F}}$].

Remarks. Specifically we show that there is an onto operator $t: \boldsymbol{P} \to \ell^\infty$ such that the into norm (and w^*-w^*) isomorphism $t^*: \ell^\infty \to \boldsymbol{P}^*$ satisfies $t^* c_0^\perp \subset TI_{\boldsymbol{P}}(a)$

$[t^* \mathcal{F} \subset TIM_{\boldsymbol{P}}(a)]$. Furthermore $\boldsymbol{Q}/W_{\boldsymbol{Q}}(a)$ also has ℓ^∞ as a quotient and if $i: \boldsymbol{Q} \to \boldsymbol{P}$ is the imbedding then i^* restricted to $TI_{\boldsymbol{P}}(a)$ is a norm (and w^*- w^*) isomorphism such that $i^* TI_{\boldsymbol{P}}(a) = TI_{\boldsymbol{Q}}(a)$ and $i^* \, TIM_{\boldsymbol{P}}(a) \subset TIM_{\boldsymbol{Q}}(a)$.

Proof. Let $F = Z(J)$ and $v_n \in A$ be the required sequence for $a \in D_b(J)$ $[a \in D_1(J)]$ (see definition). Denote for $v \in A$, $v' = v + J$ and $\|v'\| = \|v\|_{A/J}$. Let $V_n = \{x \in G; v_n(x) \neq 0\}$.

By possibly taking a subsequence again denoted by v_n we can assume that $F \cap \operatorname{cl} V_{n+1} \subset F \cap V_n$, $V_n \cap F$ is a nbhd base in F at a, and $\operatorname{cl} V_n$ is compact.

If $v \in A$, $u \in J$, and $\Phi \in \boldsymbol{P}$, $(v + u) \cdot \Phi = v \cdot \Phi$, hence $(v' \cdot \Phi, w') = (\Phi, v'w')$ if $w' \in A/J$ is well defined and $\|v' \cdot \Phi\| \leq \|v'\| \, \|\Phi\|$. Thus $\boldsymbol{P}$ is an A/J module and $\boldsymbol{P} \subset A^*$.

Define $t: \boldsymbol{P} \to \ell^\infty$ by $(t\Phi)(n) = (\Phi, v_n') = (\Phi, v_n)$. Since $\|v_n'\|$ is bounded, $t(\boldsymbol{P}) \subset \ell^\infty$ and $\|t\Phi\| = \sup |(\Phi, v_n)| \leq \|\Phi\| B$, where $B = \sup \|v_n\|$. [Hence $\|t\| \leq 1$ if $a \in D_1(J)$]. But $t\boldsymbol{P} = \ell^\infty$. Since if $b = (b_n) \in \ell^\infty$ with norm $\|b\|$ define the linear functional F_0 on $\operatorname{lin} \{v_n'; n \geq 1\} \subset A/J$ by $(F_0, \sum_1^n \alpha_i v_i') = \sum_1^n \alpha_i b_i$. then $|(F_0, \sum_1^n \alpha_i v_i')| \leq \sum_1^n |\alpha_i b_i| \leq \|b\| \sum_1^n |\alpha_i| \leq \|b\| (1/d) \| \sum_1^n \alpha_i v_i' \|$, where d is the constant for the Sidon sequence $\{v_n'\}$ in A/J. By the Hahn Banach theorem there is an extension $\Phi_0 \in \boldsymbol{P} = (A/J)^*$ of F_0. Then $(t\Phi_0)(n) = (F_0, v_n') = b_n$, thus $t\Phi_0 = b$.

We show now that $t E_{\boldsymbol{P}}(a) \subset c_0$ and $t W_{\boldsymbol{P}}(a) \subset c$.

Let $\Phi \in \boldsymbol{P}$ be such that $a \notin \operatorname{supp} \Phi$ and let U_0 be a nbhd of a, with $\operatorname{cl} U_0$ compact and such that $(\Phi, u) = 0$ if $u \in A$ and $\operatorname{supp} u \subset U_0$. Let $v_o \in A$ be such that $v_0 = 1$ on U_0 and $\operatorname{supp} v_0$ is compact (A is regular). There exists k_0 such that $V_n \cap F \subset U_0 \cap F \subset U_0$ if $n \geq k_0$. Now $K_n = \operatorname{supp}(v_0 v_n - v_n) \subset V_{n-1} \sim U_0$, since U_0 is open.

But $K_n \cap F \subset (V_{n-1} \sim U_0) \cap F = V_{n-1} \cap F \sim U_0 \cap F = \emptyset$ if $n \geq k_0 + 1$ and K_n is compact, since $\operatorname{supp} v_n$ is compact. It follows that $v_0 v_n - v_n$ is in the smallest closed ideal J_F whose zero set is F and $J_F \subset J$ (see [HR], (39.18)). Thus $v_0 v_n - v_n \in J$ and $(\Phi, v_n) = (\Phi, v_0 v_n) = 0$ if $n \geq k_0 + 1$, since $\operatorname{supp} v_0 v_n \subset U_0$. Hence $(\Phi, v_n) \to 0$. Now $\{\Phi \in \boldsymbol{P};\ a \notin \operatorname{supp} \Phi\}$ is norm dense in $E_{\boldsymbol{P}}(a)$ and $\sup \|v_n\| < \infty$. Thus $(\Phi, v_n) \to 0$ for all $\Phi \in E_{\boldsymbol{P}}(a)$ and $t E_{\boldsymbol{P}}(a) \subset c_0$. But $(t\lambda\delta_a)(n) = v_n(a) = 1$. Thus $t W_{\boldsymbol{P}}(a) \subset c$, and $W_{\boldsymbol{P}}(a) \subset t^{-1}(c)$.

Hence $\boldsymbol{P}/W_{\boldsymbol{P}}(a)$ has $\boldsymbol{P}/t^{-1}(c) \approx \ell^\infty/c$ as a quotient.

If $\phi \in \ell^{\infty *}$ is such that $\phi = 0$ on $c_0 \subset \ell^\infty$ then $t^*\phi = 0$ on $E_{\boldsymbol{P}}(a)$ since $t(E_{\boldsymbol{P}}(a)) \subset c_0$. Thus $t^* c_0^\perp \subset TI_{\boldsymbol{P}}(a)$. [If $a \in D_1(J)$ and $\phi \in \mathcal{F}$, thus $1 = \|\phi\| = \phi(1)$ and $\phi = 0$ on c_0, then $1 = (t^*\phi, \lambda\delta_\alpha) \leq \|t^*\phi\| \leq \|\phi\| = 1$. Since $t^*\phi \in TI_{\boldsymbol{P}}(a)$, $t^*(\mathcal{F}) \subset TIM_{\boldsymbol{P}}(a)$]. Now $t: \boldsymbol{P} \to \ell^\infty$ is open since t is onto, hence $t\{\Phi \in \boldsymbol{P};\ \|\Phi\| \leq 1\}$ contains a ball B_δ of radius $\delta > 0$ around 0. Thus $\|t^*\phi\| = \sup\{(\phi, t\Phi);\ \|\Phi\| \leq 1\} \geq \sup\{|(\phi, b)|;\ \|b\| \leq \delta\} = \delta\|\phi\|$. Thus $\delta\|\phi\| \leq \|t^*\phi\| \leq B|\phi\|$ for all $\phi \in \ell^{\infty *}$ and $t^*: \ell^{\infty *} \to \boldsymbol{P}^*$ is a w^*-w^* continuous norm isomorphism into such that $t^*(c_0^\perp) \subset TI_{\boldsymbol{P}}(a)$ $[t^* \mathcal{F} \subset TIM_{\boldsymbol{P}}(a)]$.

Consider the $\boldsymbol{Q}/W_{\boldsymbol{Q}}(a)$ case where $\boldsymbol{P}_c \subset \boldsymbol{Q} \subset \boldsymbol{P}$. Let $q: \ell^\infty \to \ell^\infty/c$ be the canonical map. Let $u \in A \cap C_c(G)$ be such that $u = 1$ on some nbhd U of a and $\Phi \in \boldsymbol{P}$. Let $v \in A(G)$ be such that $\operatorname{supp} v \subset U$. Then $(\Phi - u \cdot \Phi, v) = (\Phi, v - vu) = (\Phi, 0) = 0$; thus $a \notin \operatorname{supp}(\Phi - u \cdot \Phi)$ and $\Phi - u \cdot \Phi \in E_{\boldsymbol{P}}(a)$.

Hence $t(u \cdot \Phi - \Phi) \in c_0$ and $qt(u \cdot \Phi) = qt(\Phi)$. But $u \cdot \Phi \in \boldsymbol{P}_c \subset \boldsymbol{Q}$ since $\operatorname{supp} u \cdot \Phi \subset \operatorname{supp} u$. Thus $qt(\boldsymbol{P}_c) = qt(\boldsymbol{Q}) = qt(\boldsymbol{P}) = \ell^\infty/c$.

Let now r be qt restricted to $\boldsymbol{Q}$; thus $r\Phi = qt\Phi$ for $\Phi \in \boldsymbol{Q}$. Since $E_{\boldsymbol{Q}}(a) \subset E_{\boldsymbol{P}}(a)$ we have $rE_{\boldsymbol{Q}}(a) \subset qtE_{\boldsymbol{P}}(a) = \{0\}$. Now $\lambda\delta_a \in \boldsymbol{P}_c \subset \boldsymbol{Q}$ and $r\lambda\delta_a = qt(\lambda\delta_a) = 0$ since $t\lambda\delta_a = 1 \in c$. Thus $rW_{\boldsymbol{Q}}(a) = \{0\}$ and $W_{\boldsymbol{Q}}(a) \subset r^{-1}(0)$. But $r\boldsymbol{Q} = qt\boldsymbol{Q} = \ell^\infty/c$; thus $\boldsymbol{Q}/r^{-1}(0) \approx \ell^\infty/c$ (isomorphism). But ℓ^∞/c [hence $\boldsymbol{Q}/r^{-1}(0)$] contains an isometric copy Y $[Y_0]$ of ℓ^∞ (see [Sa] for ℓ/c_0 or [Gr2], p. 161 for ℓ/c). And since ℓ^∞ is injective [LT] there exists a bounded projection P_0 of $\boldsymbol{Q}/r^{-1}(0)$ onto Y_0. If $P: \boldsymbol{Q}/W_{\boldsymbol{Q}}(a) \to \boldsymbol{Q}/r^{-1}(0)$ is the canonical quotient map then P_0P maps $\boldsymbol{Q}/W_{\boldsymbol{Q}}(a)$ onto $Y_0 \approx \ell^\infty$.

Let $i: \boldsymbol{Q} \to \boldsymbol{P}$ be the inclusion map $i\Phi = \Phi$ for all $\Phi \in \boldsymbol{Q}$; thus $qti\Phi = r\Phi$ if $\Phi \in \boldsymbol{Q}$. We claim that i^* restricted to $TI_{\boldsymbol{P}}(a)$ is a w^*-w^* continuous norm isomorphism such that $i^*(TI_{\boldsymbol{P}}(a) = TI_{\boldsymbol{Q}}(a)$ and $i^*(TIM_{\boldsymbol{P}}(a)) \subset TIM_{\boldsymbol{Q}}(a)$. In fact let $u_0 \in A \cap C_c(G)$ be fixed such that $u_0 = 1$ on some open U_0 with $a \in U_0$ and $\|u_0\| = d > 0$. Let $\psi \in TI_{\boldsymbol{P}}(a)$ and $\Phi_0 \in \boldsymbol{P}$ be such that $\|\Phi_0\| = 1$ and $(\psi, \Phi_0) \geq \|\psi\| - \varepsilon$. Then $u_0 \cdot \Phi_0 \in \boldsymbol{P}_c \subset \boldsymbol{Q}$ and $\|u_0 \cdot \Phi_0\| \leq d$. Hence $(i^*\psi, u_0 \cdot \Phi_0) = (\psi, u_0 \cdot \Phi_0) = (\psi, \Phi_0)$ since $u_0 \cdot \Phi_0 - \Phi_0 \in E_{\boldsymbol{P}}(a)$. Thus $(i^*\psi, d^{-1}u_0 \cdot \Phi_0) \geq d^{-1}(\|\psi\| - \varepsilon)$ and $\|\psi\| \geq \|i^*\psi\| \geq d^{-1}\|\psi\|$, if $\psi \in TI_{\boldsymbol{P}}(a)$.

If now $\Phi \in E_{\boldsymbol{Q}}(a) \subset E_{\boldsymbol{P}}(a)$ and $\psi \in TI_{\boldsymbol{P}}(a)$ then $(i^*\psi, \Phi) = (\psi, \Phi) = 0$ since $\psi = 0$ on $E_{\boldsymbol{P}}(a)$. Thus $i^*TI_{\boldsymbol{P}}(a) \subset TI_{\boldsymbol{Q}}(a)$.

But $i^*TI_{\boldsymbol{P}}(a) = TI_{\boldsymbol{Q}}(a)$ since if $\psi \in TI_{\boldsymbol{Q}}(a)$ then $\psi_1 \in TI_{\boldsymbol{P}}(a)$ defined by $(\psi_1, \Phi) = (\psi, u_0\Phi)$ for Φ in $\boldsymbol{P}$ satisfies $i^*\psi_1 = \psi$. This holds since if $\Phi \in \boldsymbol{Q}$ then $(i^*\psi_1, \Phi) = (\psi, u_0 \cdot \Phi) = (\psi, \Phi)$, since $u_0 \cdot \Phi - \Phi \in E_{\boldsymbol{Q}}(a)$. If $\Phi \in \boldsymbol{P}$ and $a \notin \operatorname{supp} \Phi$ then $a \notin \operatorname{supp} u_0 \cdot \Phi$ and $u_0 \cdot \Phi \in E_{\boldsymbol{Q}}(a)$. Thus $(\psi_1, \Phi) = (\psi, u_0 \cdot \Phi) = 0$. Since $\psi_1 \in \boldsymbol{P}^*$, $\psi_1 = 0$ on $E_{\boldsymbol{P}}(a)$, hence $\psi_1 \in TI_{\boldsymbol{P}}(a)$.

If, in addition, $\psi \in TIM_{\boldsymbol{P}}(a)$ then $(i^*\psi, \lambda\delta_a) = (\psi, \lambda\delta_a) = 1 = \|\psi\| \geq \|i^*\psi\| \geq (i^*\psi, \lambda\delta_a) = 1$.

But $t^*: \ell^{\infty*} \to \boldsymbol{P}^*$ is a w^*-w^* continuous norm isomorphism into such that $t^*(c_0^\perp) \subset TI_{\boldsymbol{P}}(a)$ $[t^*\mathcal{F} \subset TIM_{\boldsymbol{P}}(a)]$. Thus i^*t^* restricted to $c_0^\perp$ is a w^*-w^* continuous isomorphism into $TI_{\boldsymbol{Q}}(a)$ [such that $i^*t^*(\mathcal{F}) \subset i^*\, TIM_{\boldsymbol{P}}(a) \subset TIM_{\boldsymbol{Q}}(a)$]. □

PROPOSITION 5. *Let G be a locally compact group and $A = A(G)$ the Fourier algebra of G or $A = A_p(G)$. Let $\boldsymbol{P} \subset A^*$ be a norm closed A module and $F = \sigma(\boldsymbol{P})$. Then $WAP_{\boldsymbol{P}} \subset C\lambda\delta_a + E_{\boldsymbol{P}}(a) = W_{\boldsymbol{P}}(a)$ for all $a \in G$.*

Proof. The proof involves routine arguments such as Prop. 9 and Prop. 4 of [Gr4] and is left to the reader. □

In the following, G is an arbitrary locally compact group, $J \subset A = A(G)$ is a closed ideal with $Z(J) = F$, and $\boldsymbol{Q}$ is a norm closed A submodule of $PM(G)$ such that $\boldsymbol{P}_c \subset \boldsymbol{Q} \subset \boldsymbol{P} = (A/J)^*$.

COROLLARY 6. *Assume that R (or T) is a closed subgroup of G, $S \subset R$ (or T) a symmetric set such that $aSb \subset F$ for some $a, b \in G$ and F is metrisable. Then*

(∗) *$\boldsymbol{Q}/W_{\boldsymbol{Q}}(x)$ (a fortiori $\boldsymbol{Q}/WAP_{\boldsymbol{Q}}$ and $\boldsymbol{Q}/\mathbf{M}(F)$) has ℓ^∞ as a quotient and $TIM_{\boldsymbol{Q}}(x)$ contains $\boldsymbol{\mathcal{F}}$ for all $x \in aSb$.*

Consequently A/J is ENAR if G is second countable nondiscrete.

In the next corollary, $A = A(G)$ can be replaced by $A_p(G)$. It improves part of Theorem 6 in [Gr5], with a much simpler proof.

COROLLARY 7. *Assume that H is a closed nondiscrete subgroup of G and $int_{aHb}F \neq \emptyset$ for some $a, b \in G$, where F is metrisable.*
Then (∗) holds true for all $x \in int_{aHb}F$.
Consequently A/J is ENAR if G is second countable nondiscrete.

Proof of Corollaries 6 *and* 7. If $x \in aSb$ [$x \in \mathrm{int}_{aHb}F$] then $x \in D_1(J)$ by Corollary 2′ [Theorem 3]. Hence by Theorem 4, (∗) holds for such x.
But by Prop. 5, $WAP_{\boldsymbol{Q}} \subset W_{\boldsymbol{Q}}(x)$ holds true. Taking $\boldsymbol{Q} = \boldsymbol{P}$ we get that $\boldsymbol{P}/WAP_{\boldsymbol{P}}$ has ℓ^∞ as a quotient.
If, in addition, G is second countable then A is norm separable and since $\boldsymbol{P} = (A/J)^*$, A/J is ENAR. □

Remark. (i) In Corollary 6 it is enough that the relative topology of F is first countable at each $x \in F$.
(ii) If $F \subset T$ is any perfect compact Helson set [He] then $A(F) = A(T)/I_F = C(F)$ is Arens regular as is well known (see more such F in Section 3).
(iii) If $\boldsymbol{P} \subset A^*$ is a w^* closed A module and $\sigma(\boldsymbol{P})$ contains a metrisable compact perfect set then $\boldsymbol{P}_c$ and $\boldsymbol{P}$ have ℓ^∞ as a quotient if G is amenable as discrete, even if $A = A_p(G)$ by our Theorem 2 in [Gr5].

COROLLARY 7′. *Let $A = A_p(G)$, $J \subset A$ a closed ideal such that $D_b(J) \neq \emptyset$. Then A/J is ENAR provided G is second countable.*

Question. Let $J \subset A(R)$ be a closed ideal such that $D_b(J) = \emptyset$. Is then A/J Arens regular?

3. The abelian case

Let $\mathcal{F}_S$: $M(\widehat{G}) \to B(G)$ [$\mathcal{F}$: $L^1(\widehat{G}) \to A(G)$] denote Fourier Stiltjies [Fourier] transform. Thus $\mathcal{F}_S\mu(x) = \int \overline{\chi(x)} d\mu(\chi)$ for $x \in G$, see [Ru] or [HR]. For $\mu \in M(\widehat{G})$, $g \in L^\infty(\widehat{G})$, $f \in L^1(\widehat{G})$ let $\mu^\vee(E) = \mu(E^{-1})$, $f^\vee(\chi) = f(\chi^{-1})$, $\int f\, d(g\mu) = \int fg\, d\mu$, where $E \subset \widehat{G}$ is a Borel set. $PM(G)$ is a $B(G)$ module by $(u \cdot \Phi, v) = (\Phi, uv)$. It is known that

$$(*) \qquad \mathcal{F}^*[(\mathcal{F}_S\mu)\cdot\Phi] = \mu^\vee * \mathcal{F}^*\Phi \quad \text{if } \mu \in M(\widehat{G}),\ \Phi \in PM(G).$$

To prove $(*)$ note that $(h, \mu * f) = (\mu^\vee * h, f)$ if $f \in L^1(\widehat{G})$, $h \in L^\infty(\widehat{G})$, $\mu \in M(\widehat{G})$, by Fubini's theorem (or [Pi], p. 83). Hence $(\mathcal{F}^*[(\mathcal{F}_S\mu)\cdot\Phi], f) = (\Phi, \mathcal{F}(\mu * f)) = (\mu^\vee * \mathcal{F}\Phi, f)$.

If $\boldsymbol{P} \subset PM(G)$ and $\mathcal{F}^*\boldsymbol{P} = P$ then $B(G)\cdot\boldsymbol{P} \subset \boldsymbol{P}$ iff $M(\widehat{G}) * P \subset P$ as readily follows from $(*)$. Thus $\boldsymbol{P}$ is a norm $[w^*]$ closed $B(G)$ module iff P is a norm $[w^*]$ closed $M(\widehat{G})$ module, respectively since $\mathcal{F}^*$ is an onto isometry and w^*–w homeomorphism.

DEFINITION. Let $P \subset L^\infty(\widehat{G})$ be a norm closed $M(\widehat{G})$ module, $\boldsymbol{P} = \mathcal{F}^{*-1}P$, and $a \in G$. We defined the spaces $D_P(a)$, $V_P(a)$, $D_{\boldsymbol{P}}(a)$ $V_{\boldsymbol{P}}(a)$ in the introduction.

Let $IM_P(a) = \{\psi \in \underline{P^*};\ 1 = (\psi, \overline{a}) = \|\psi\|,\ \psi = 0 \text{ on } D_P(a)\}$. Note that $\psi = 0$ on $D_P(a)$ iff $\psi(h_\chi) = \overline{a(\chi)}\psi(h)$ for all $\chi \in \hat{G}$ and $h \in P$. Let $\sigma(P) = G \cap \overline{P}$.

PROPOSITION 8. *Let $\boldsymbol{P} \subset PM(G)$ be a norm closed $B(G)$ module, $a \in G$ and $\mathcal{F}^*\boldsymbol{P} = P$. Then $\mathcal{F}^*E_{\boldsymbol{P}}(a) = E_P(a)$, $\mathcal{F}^*D_{\boldsymbol{P}}(a) = D_P(a)$, hence $\mathcal{F}^*V_{\boldsymbol{P}}(a) = V_P(a)$ and $\mathcal{F}^*W_{\boldsymbol{P}}(a) = W_P(a)$.*

Proof. If $\Phi \in \boldsymbol{P}$, $\mu \in M(\widehat{G})$, one gets from $(*)$ that

$$(**) \quad \mathcal{F}^*[(\mathcal{F}_S\mu)_{a^{-1}}\cdot\Phi] = \mathcal{F}^*[\mathcal{F}_S(a\mu)\cdot\Phi] = (a\mu)^\vee * \mathcal{F}^*\Phi = (\bar{a}\mu^\vee) * \mathcal{F}\Phi.$$

Take $\mu \in \delta_\chi$, so that $\mathcal{F}_S\delta_\chi = \overline{\chi}$, and let $h = \mathcal{F}^*\Phi$. Then, since $\delta_\chi^\vee = \delta_{\chi^{-1}}$, we get

$$\mathcal{F}[(\mathcal{F}_S\delta_\chi)_{a^{-1}}\cdot\Phi] = \mathcal{F}^*[(\bar{\chi})_{a^{-1}}\cdot\Phi] = (a\delta_\chi)^\vee * h = (a(\chi)\delta_\chi)^\vee * h = a(\chi)h_\chi.$$

Hence $\mathcal{F}^*\{\Phi - \chi_{a^{-1}}\cdot\Phi;\ \Phi \in \boldsymbol{P}, \chi \in \widehat{G}\} = \mathcal{F}^*\{\Phi - (\bar{\chi})_{a^{-1}}\cdot\Phi;\ \Phi \in \boldsymbol{P}, \chi \in \widehat{G}\} = \{h - a(\chi)h_\chi;\ h \in P, \chi \in \widehat{G}\}$. Thus $\mathcal{F}^*D_{\boldsymbol{P}}(a) = D_P(a)$; hence $\mathcal{F}^*V_{\boldsymbol{P}}(a) = V_P(a)$.

Let $F_M = \{\mu \in M(\widehat{G});\ \mu \geq 0, \mu(\widehat{G}) = 1\}$, $F_1 = F_M \cap L^1(\widehat{G}) = \{0 \leq f \in L^1(\widehat{G});\ \int f\, d\chi = 1\}$. Then $F_1^\vee = F_1$. By Prop. 1 of [Gr5],

$$(***) \qquad E_{\boldsymbol{P}}(a) = \text{ncl lin}\,\{\Phi - v_{a^{-1}}\cdot\Phi;\ \Phi \in \boldsymbol{P}, v \in S_A(e)\}$$

where $S(x) = \{u \in B(G);\ 1 = u(x) = \|u\|\}$ and $S_A(x) = S(x) \cap A(G)$, since $(S_A(e))_a = S_A(a)$ (by [Ru], (1.2.4)), or see the following lemma.

Clearly $\mathcal{F}^* F_1 = S_A(e)$ and $\mathcal{F}^*\{\Phi - v_{a^{-1}} \cdot \Phi; \Phi \in \boldsymbol{P}, v \in S_A(e)\} = \mathcal{F}^*\{\Phi - (\mathcal{F}f)_{a^{-1}} \cdot \Phi; \Phi \in \boldsymbol{P}, f \in F_1\} =$ (by $(**)$) $\{h - (\bar{a} f^{\vee}) * h; h \in P, f \in F_1\} = \{h - (\bar{a} f) * h; f \in F_1, h \in P\}$. Hence $\mathcal{F}^* E_{\boldsymbol{P}}(a) = E_P(a)$ and $\mathcal{F}^* W_{\boldsymbol{P}}(a) = W_P(a)$ since $\mathcal{F}^*$ is an isometry of $PM(G)$ onto $L^\infty(\widehat{G})$. □

We prove $(***)$ and more in the next result.

LEMMA 8′. *Let $\boldsymbol{P} \subset PM(G)$ be a norm closed $B(G)$ module. Then $E_{\boldsymbol{P}}(a) = ncl\,\{\Phi - v \cdot \Phi; \Phi \in \boldsymbol{P}, v \in S_i(a)\}$ for $i = 1, 2, 3$ where $S_1(a) = S_A(a), S_2(a) = S(a)$, $S_3(a) = \{v \in B(G); v(a) = 1\}$. In addition ncl can be replaced by ncl lin.*

Proof. Note that $S_1(a) \subset S_2(a) \subset S_3(a)$. Let $\Phi \in \boldsymbol{P}$ with $a \notin \operatorname{supp} \Phi$. Let $v \in S_1(a)$ be such that $\operatorname{supp} v \cap \operatorname{supp} \Phi = \emptyset$, thus $\operatorname{supp} v \cdot \Phi = \emptyset$. Hence $v \cdot \Phi = 0$ and $\Phi = \Phi - v \cdot \Phi$, which proves $E_{\boldsymbol{P}}(a) \subset \operatorname{ncl}\{\Phi - v \cdot \Phi; \Phi \in \boldsymbol{P}, v \in S_1(a)\}$.

Let $\Phi \in \boldsymbol{P}$ and $v_0 \in A \cap C_c(G)$ be such that $v_0 = 1$ on a nbhd V of a. Then $a \notin \Phi - v_0 \cdot \Phi$ and $\Phi - v_0 \cdot \Phi \in E_{\boldsymbol{P}}(a)$ (see Prop. 5). Thus if $u \in S_3(a)$ then $(\Phi - u \cdot \Phi) - v_0(\Phi - u \cdot \Phi) \in E_{\boldsymbol{P}}(a)$. But $v_0 \cdot (\Phi - u \cdot \Phi) = (v_0 - v_0 u) \cdot \Phi \in E_{\boldsymbol{P}}(a)$. In fact $(v_0 - u v_0)(a) = 0$ and since $\{a\}$ is a synthesis set [Hz] let $v_n \in A \cap C_c(G), n \geq 1$ be such that $v_n = 0$ on a nbhd V_n of a and $\|v_n - (v_0 - u v_0)\| \to 0$. But then $a \notin \operatorname{supp} v_n \cdot \Phi$ and $v_n \cdot \Phi \in E_{\boldsymbol{P}}(a)$. Thus $\|v_n \cdot \Phi - (v_0 - u v_0) \cdot \Phi\| \to 0$, hence $(v_0 - v_0 u) \cdot \Phi \in E_{\boldsymbol{P}}(a)$ and $\Phi - u \cdot \Phi \in E_{\boldsymbol{P}}(a)$. Hence $E_{\boldsymbol{P}}(a) \supset \operatorname{ncl}\{\Phi - v \cdot \Phi; \Phi \in \boldsymbol{P}, v \in S_3(a)\}$.

Now $\{\Phi \in \boldsymbol{P}; a \notin \operatorname{supp} \Phi\}$ (hence $E_{\boldsymbol{P}}(a)$) is a linear space, from the definiton of support. □

PROPOSITION 9. *Let $P \subset L^\infty(\widehat{G})$ be a norm closed $M(\widehat{G})$ module, $a \in \sigma(P)$. Then $D_P(a) \subset E_P(a), V_P(a) \subset W_P(a)$ and $TIM_P(a) \subset IM_P(a)$.*

If $P \subset UC(\widehat{G})$ then $D_P(a) = E_P(a)$, $V_P(a) = W_P(a)$, and $IM_P(a) = TIM_P(a)$.

Proof. If $x \in G$ then $u \to u_x$ is an isometric homomorphism of $B(G)$ onto $B(G)$ which maps $A(G)$ onto $A(G)$, see [Ru], (1.2.4) and (1.3.3). Also $S(x)S_A(x) \subset S_A(x)$. If $u \in S(e)$, $v \in S_A(e)$, $\Phi \in \boldsymbol{P}$, then $\Phi - u_{a^{-1}} \cdot \Phi = \Phi - (uv)_{a^{-1}} \cdot \Phi + v_{a^{-1}} \cdot (u_{a^{-1}} \cdot \Phi) - (u_{a^{-1}} \cdot \Phi) \in E_{\boldsymbol{P}}(a)$ by Lemma 8′ and since $\boldsymbol{P}$ is a $B(G)$ module. It follows that $E_{\boldsymbol{P}}(a) = \operatorname{ncl\,lin}\{\Phi - v_{a^{-1}} \cdot \Phi; v \in S_A(e), \Phi \in \boldsymbol{P}\} = \operatorname{ncl\,lin}\{\Phi - u_{a^{-1}} \cdot \Phi; u \in S(e), \Phi \in \boldsymbol{P}\} \supset \operatorname{ncl\,lin}\{\Phi - \chi_{a^{-1}} \cdot \Phi; \chi \in \widehat{G}, \Phi \in \boldsymbol{P}\} = D_{\boldsymbol{P}}(a)$. And by Proposition 8, $D_P(a) \subset E_P(a)$. If $a \in \sigma(P)$, thus $\bar{a} \in P$ then, $TIM_P(a) = \{\psi \in P^*; 1 = (\psi, \bar{a}) = \|\psi\|, \psi = 0 \text{ on } E_P(a)\} \subset IM_P(a) = \{\psi \in P^*; 1 = (\psi, \bar{a}) = \|\psi\|, \psi = 0 \text{ on } D_P(a)\}$.

Assume in addition that $P \subset UC(\widehat{G})$. Clearly $\boldsymbol{P} = \mathcal{F}^{*-1}P \subset \mathcal{F}^{*-1}UC(\widehat{G}) = (PM(G))_c$. Let $\Phi \in \boldsymbol{P}$. Then $\Phi = v_0 \cdot \Phi_0$ for some $v_0 \in A(G)$, and $\Phi_0 \in PM(G)$. Let $u_0 \in S_A(e)$. We show that $\Phi - (u_0)_{a^{-1}} \cdot \Phi \in D_{\boldsymbol{P}}(a)$; hence by $(***)$, $E_{\boldsymbol{P}}(a) = D_{\boldsymbol{P}}(a)$. Let u_α be a net in $Co\{\chi; \chi \in \widehat{G}\} \subset S(e)$ (where Co denotes convex hull) such that $u_\alpha \to u_0$ in the w^* topology of $B(G)$ (u_0 is continuous and positive definite). Then, by a theorem of Leinert and ours [GrL], $\|(u_\alpha - u_0)v\| \to 0$ for all $v \in A(G)$.

But $\Phi - \chi_{a^{-1}} \cdot \Phi \in D_{\boldsymbol{P}}(a)$, hence $\Phi - (u_\alpha)_{a^{-1}} \cdot \Phi \in D_{\boldsymbol{P}}(a)$. Thus $\|(\Phi - (u_\alpha)_{a^{-1}} \cdot \Phi) - (\Phi - (u_0)_{a^{-1}} \cdot \Phi)\| \le \|((u_\alpha)_{a^{-1}} - (u_0)_{a^{-1}})v_0\| \, \|\Phi_0\| \to 0$, since $\|((u_\alpha)_{a^{-1}} - (u_0)_{a^{-1}})v_0\| = \|(u_\alpha - u_0)(v_0)_{a^{-1}}\| \to 0$. Hence $\Phi - (u_0)_{a^{-1}} \cdot \Phi \in D_{\boldsymbol{P}}(a)$ since $D_{\boldsymbol{P}}(a)$ is norm closed. Thus $TIM_P(a) = IM_P(a)$ if $\bar{a} \in P$. □

COROLLARY 10. *Let G be a locally compact abelian group, P $[Q]$ a w^* [norm] closed $M(\widehat{G})$ submodule of $L^\infty(\widehat{G})$ such that $UC_P(\widehat{G}) \subset Q \subset P$ and $F = \sigma(P) = G \cap \overline{P}$, $a \in G$.*

Assume that R (or T) is a closed subgroup of G, $S \subset R$ (or T) a symetric set such that $aS \subset F$ and F be metrisable.

(i) Then $Q/W_Q(x)$ (a fortiori $Q/V_Q(x)$, Q/WAP_Q and $Q/ncl\ B(\widehat{G}, F)$) has ℓ^∞ as a quotient and both $TIM_Q(x)$ and $IM_Q(x)$ contain $\boldsymbol{\mathcal{F}}$, for all $x \in aS$.

(ii) If G is second countable nondiscrete then $L^1(\widehat{G})/(P)_0$ is ENAR.

Remark. $B(\widehat{G}, F) = \{\mathcal{F}_S\mu;\ \mu \in M(F)\}$ and $(P)_0 = \{f \in L^1(\widehat{G});\ (g, f) = 0 \text{ if } g \in P\}$.

COROLLARY 11. *Let G, P, Q be as above and assume that $H \subset G$ is a closed nondiscrete subgroup such that $int_{aH} F \ne \emptyset$ and F is metrisable.*

Then [(i)] and (ii) of Corollary 10 hold [for each $x \in int_{aH} F$].

Proof of Corollaries 10 *and* 11. Let $\boldsymbol{Q} = \mathcal{F}^{*-1}Q$. By Proposition 8, $W_{\boldsymbol{P}}(x) = \mathcal{F}^{*-1}W_P(x)$. Let $x \in aS$ $[x \in \mathrm{int}_{aH} F]$ respectively. By Corollaries 6 and 7, $\boldsymbol{Q}/W_{\boldsymbol{Q}}(x)$ has ℓ^∞ as a quotient and $TIM_{\boldsymbol{Q}}(x)$ contains $\boldsymbol{\mathcal{F}}$. Since $\mathcal{F}^*\colon \boldsymbol{Q} \to Q$ is an isometry onto and $\mathcal{F}^* W_{\boldsymbol{Q}}(x) = W_Q(x)$ we get that $Q/W_Q(x)$ (and, since $V_Q(x) \subset W_Q(x)$ by Prop. 9, $Q/V_Q(x)$) has ℓ^∞ as a quotient and $TIM_Q(x)$ (and, since $IM_P(x) \supset TIM_Q(x)$ by Prop. 9, $IM_Q(x)$) contains $\boldsymbol{\mathcal{F}}$.

If $\mu \in M(G)$, $f \in L^1(\widehat{G})$, then $(\mathcal{F}^*\lambda\mu, f) = \iint f(\chi)\overline{\chi(y)}\,d\chi\,d\mu(y) = (\mathcal{F}_S\mu, f)$; hence $\mathcal{F}^*\lambda\mu = \mathcal{F}_S\mu$ in $L^\infty(\widehat{G})$. Thus $\mathcal{F}^*\lambda M(F) = B(\widehat{G}, F)$. But by [Gr5], Prop. 3, $\mathbf{M}(F) = \mathrm{ncl}\ \lambda M(F) \subset W_{\boldsymbol{Q}}(x)$. Thus $\mathcal{F}^*\mathbf{M}(F) = \mathrm{ncl}\ B(\widehat{G}, F) \subset W_Q(x)$. Thus $Q/\mathrm{ncl}\ B(\widehat{G}, F)$ has ℓ^∞ as a quotient. Furthermore by Prop. 5, $WAP_{\boldsymbol{Q}} \subset W_{\boldsymbol{Q}}(x)$ and since it is known that $\mathcal{F}^*WAP_{\boldsymbol{Q}} = WAP_Q$ we get that Q/WAP_Q has ℓ^∞ as a quotient. This proves (i). Part (ii) is proved as in Corollary 6 or 7. □

DEFINITION. Let G be a separable metric l.c.a. group. The closed $F \subset G$ is an ENE set if for each w^* [norm] closed $M(\widehat{G})$ module P $[Q]$ of $L^\infty(\widehat{G})$ with $\sigma(P) = G \cap \overline{P} = F$ and $UC_P \subset Q \subset P$, Q is ENE at each $x \in F$ (i.e., $Q/W_Q(x)$ has ℓ^∞ as a quotient) and $TIM_Q(x)$ contains $\boldsymbol{\mathcal{F}}$.

Let $\bar{\alpha} = (\alpha_1, \dots, \alpha_n)$, $\overline{\beta} = (\beta_1, \dots, \beta_n)$ be in R^n. If $S = \{t\bar{\alpha};\ t \in S'\}$ where $S' \subset R$ is ultrathin symmetric then S is called an ultrathin symmetric set in R^n. Since $R \approx \{t\bar{\alpha};\ t \in R\}$, $S + \overline{\beta}$ is ENE in R^n (Corollary 10). Any closed F which is a union of translates of sets S_α where S_α or $-S_\alpha$ are ultrathin symmetric in R^n, is ENE

(Corollaries 2′ and 10). A fortiori any closed $F \subset R^n$ which is a union of nontrivial convex subsets of R^n is ENE.

And yet any Kahane curve in R^n $n \geq 2$ is not ENE (at any point on it). If $n > 2k$ there exists a k dimensional manifold $F \subset R^n = G$ which is a Helson set. Thus if $P = w^*$ cl lin $F \subset L^\infty(\widehat{G})$ then $P = W_P(x) = V_P(x) = B(\widehat{G}, F)$ for all $x \in F$ (see [Mc], [Mu]).

Problem. Characterize closed ENE subsets of R^n (of any l.c.a. group G).

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