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p-BOREL PRINCIPAL IDEALS

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Introduction

This paper is an attempt to better understand the homological structure of p-Borel ideals.

Let K be an infinite field, and I a homogeneous ideal in the polynomial ring $R = K[x_1, \dots, x_n]$. By a theorem of Galligo [9] (see also [5]) the generic initial ideal of I is fixed under the action of the upper triangular matrices in $GL(n)$. This is the reason why one calls a monomial ideal Borel-fixed if it is a generic initial ideal.

There is a combinatorial description of Borel-fixed ideals. Provided the characteristic of the field K is zero, a monomial ideal I is Borel-fixed if and only if it is strongly stable, that is, whenever $u \in I$ is a monomial, and x_i divides u, then $(x_j/x_i)u \in I$ for all $j < i$.

Strongly stable ideals have been studied extensively. Actually these ideals share most of their nice properties with the larger class of stable ideals. Recall that a monomial ideal I is said to be stable if for all monomials $u \in I$ and all $j < m(u)$ one has that $(x_j/x_{m(u)})u \in I$. Here $m(u)$ is the maximal integer i for which x_i divides u.

Stable ideals were introduced by Eliahou and Kervaire [8]. In their paper they describe explicitly the minimal free resolution of these ideals. In [2] we, and independently Peeva [13], compute the Koszul homology of stable ideals. This result is used in [2] to give an alternative description of the Eliahou-Kervaire resolution, and in [13] it is shown that this resolution admits a multiplicative structure. The Eliahou-Kervaire resolution also plays a crucial role in a theorem by Bigatti [3] and Hulett [10] which asserts that among all ideals with a given Hilbert function the lexsegment ideals have maximal Betti-numbers.

It is worth mentioning that a similar theory has been developed [1] for squarefree ideals. In particular the resolution of the so-called squarefree stable ideals is known; see [1] and [6].

If the field K is of characteristic $p > 0$, Borel-fixed ideals can also be nicely described in combinatorial terms as shown by Pardue in his thesis [12]: write $x_i^l \parallel u$ to express that x_i^l divides u but x_i^{l+1} does not, and for non-negative integers k and l with p-adic expansion $k = \sum_i k_i p^i$ and $l = \sum_i l_i p^i$, set $k \leq_p l$ if $k_i \leq l_i$ for all i. Then a monomial ideal I is Borel-fixed if and only if it satisfies the following condition: if u is a monomial in I and $x_j^l \parallel u$, then $(x_i/x_j)^k u \in I$ for all $i < j$, and all $k \leq_p l$.

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Pardue calls a monomial ideal satisfying this combinatorial condition p-Borel regardless of the characteristic of K. It is pretty obvious that p-Borel ideals have a much richer structure than the corresponding stable ideals, and of course are considerably more difficult to treat. At present not too much is known about their structure. For example one does not know the regularity of these ideals, let alone their resolution.

Among the p-Borel ideals the principal ones are the most simple. Let u be a monomial; then $\langle u\rangle$ denotes the smallest p-Borel ideal which contains u. The ideal $\langle u\rangle$ is called p-Borel principal with Borel generator u. In his thesis Pardue conjectures a formula for the regularity of a p-Borel principal ideal, and proves his conjecture in the case that at most two variables (in successive order) divide u. As one of our main results in this paper we show in Section 3 that Pardue's formula is indeed a lower bound for the regularity of a p-Borel principal ideal. We prove this by exhibiting certain Koszul cycles which we discover in Section 1 of this paper where we succeed in computing the Koszul homology of a p-Borel principal Cohen-Macaulay ideal. It is noted by Pardue [12] that a p-Borel principal ideal $\langle u\rangle$ is Cohen-Macaulay if and only if the Borel generator is of the form $u = x_i^a$. In Section 2 we give the explicit minimal free resolution of p-Borel principal Cohen-Macaulay ideals.

Pardue's and our results can only be the begin in the study of p-Borel ideals. From our point of view the most challenging tasks to be accomplished in this theory are the following: (i) prove Pardue's conjecture concerning the regularity of p-Borel principal ideals, (ii) compute the Koszul homology of these ideals, or even better their resolution, and (iii) give bounds for the regularity of general p-Borel ideals.

1. The Koszul homology of Cohen-Macaulay p-Borel principal ideals

In this section we describe a basis for the cycles of the simplest possible nonstandard Borel principal ideals. Before describing the details we recall some basic facts from Pardue's thesis [12].

As already mentioned in the introduction one has the following combinatorial description of Borel-fixed ideals in positive characterics.

PROPOSITION 1.1 (Pardue). *Let K be a field of characteristic p, $I \subset K[x_1,\dots,x_n]$ a monomial ideal. Then I is Borel-fixed if and only if the following holds: if u is a monomial in I and $x_j^l \parallel u$, then $(x_i/x_j)^k u \in I$ for all $i < j$, and all $k \leq_p l$.*

Any monomial ideal satisfying the conditions of 1.1 is called *p-Borel*, no matter what the characteristic of K is.

Let $u \in K[x_1,\dots,x_n]$ be a monomial. The smallest p-Borel ideal containing u will be called a *p-Borel principal ideal*, and denoted $\langle u\rangle$.

Let I be a monomial ideal. We denote by $G(I)$ the uniquely determined minimal set of monomial generators of I. The above combinatorial condition which describes Borel-fixed ideals needs to be checked only for the generators of I. Indeed one has:

LEMMA 1.2. *Let $I \subset K[x_1, \dots, x_n]$ be a monomial ideal. Then the following conditions are equivalent*:

(*a*) *I is Borel-fixed*;
(*b*) *if $u \in G(I)$ and $x_j^l \parallel u$, then $(x_i/x_j)^k u \in I$ for all $i < j$ and all $k \leq_p l$.*

Proof. We only need to prove the implication (b) $\Rightarrow$ (a): our hypothesis implies that $\langle u \rangle \subset I$ for all $u \in G(I)$, and we want to show that $\langle w \rangle \subset I$ for all monomials $w \in I$.

Thus we pick a monomial $w \in I$. Then there exist $u \in G(I)$ and a monomial v such that $w = vu$. It suffices to show that $\langle w \rangle \subset \langle u \rangle$ since by assumption $\langle u \rangle \subset I$.

Suppose that for any monomial u and any x_j we can show that $\langle x_j u \rangle \subset \langle u \rangle$. Then by induction on the degree of v one concludes that $\langle w \rangle \subset \langle u \rangle$.

So let us choose a monomial $z \in \langle x_j u \rangle$ Then $z = (x_t/x_s)^k x_j u$ for some k, s and t with $t < s$ and $k \leq_p l$ where $x_s^l \parallel x_j u$.

If $s \neq j$, then $x_s^l \parallel u$. Hence $(x_t/x_s)^k u \in \langle u \rangle$, and this implies that $z \in \langle u \rangle$.

If $s = j$, then $x_j^{l-1} \parallel u$. Thus if $k \leq_p l-1$, then $(x_t/x_j)^k u \in \langle u \rangle$, and so $z \in \langle u \rangle$. Otherwise $k \not\leq_p l-1$, but still $k \leq_p l$. Let $l = \sum_{i=a}^b l_i p^i$, $l_a \neq 0$, be the p-adic expansion of l. Then $l-1$ has the p-adic expansion

$$l - 1 = (p-1) + (p-1)p + \cdots + (p-1)p^{a-1} + (l_a - 1)p^a + \sum_{i=a+1}^{b} l_i p^i.$$

Therefore, since $k \leq_p l$, we have $k_i = 0$ for $i < a$, and $k_i \leq l_i$ for $i \geq a$. Since $k \not\leq_p l-1$, we must have that $k_i = 0$ for $i < a$, $k_a = l_a$, and $k_i \leq l_i$ for $i > a$. It follows that $k-1$ has the p-adic expansion

$$k - 1 = (p-1) + (p-1)p + \cdots + (p-1)p^{a-1} + (l_a - 1)p^a + \sum_{i=a+1}^{b} k_i p^i.$$

This implies that $k - 1 \leq_p l - 1$, and hence we have

$$z = (x_t/x_j)x_j(x_t/x_j)^{k-1}u = x_t(x_t/x_j)^{k-1}u,$$

so that $z \in \langle u \rangle$. $\square$

PROPOSITION 1.3 (Pardue). *Let $u = \prod_{i=1}^n x_i^{\mu_i}$, and let $\mu_i = \sum_j \mu_{ij} p^j$ for $i = 1, \dots, n$ be the p-adic expansion of the exponents of u. Then*

$$\langle u \rangle = \prod_{i=1}^{n} \prod_{j} ((x_1, \dots, x_i)^{\mu_{ij}})^{[p^j]}.$$

In particular, $\langle u \rangle = \prod_{i=1}^n \langle x_i^{\mu_i} \rangle$.

The goal of this section is to describe the Koszul homology of a p-Borel principal ideal $\langle u\rangle$ when $u = x_i^{\mu_i}$ which is the case exactly when $\langle u\rangle$ is Cohen-Macaulay; see [12].

Without loss of generality we may assume that $i = n$, and we set $\mu = \mu_n$. Then

$$\langle x_n^{\mu}\rangle = \prod_{i=0}^{m}(\mathfrak{m}^{a_i})^{[p^i]}$$

where $\mathfrak{m} = (x_1, \dots, x_n)$, and the a_i are the coefficients in the p-adic expansion $\mu = \sum_{i=0}^{m} a_i p^i$ of μ.

If A and B are subsets of the polynomial ring we set $AB = \{ab\colon a \in A, b \in B\}$, and $A^k = \{a^k\colon a \in A\}$ for any integer $k > 0$. With this notation we have the following lemma whose simple proof we leave to the reader.

LEMMA 1.4. $G(\prod_{i=0}^{m}(\mathfrak{m}^{a_i})^{[p^i]}) = \prod_{i=0}^{m} G(\mathfrak{m}^{a_i})^{p^i}$

Next we compare the Koszul cycles of a monomial ideal I and its Frobenius-power $I^{[p]}$: for $\sigma \subset \{1, \dots, n\}$, $\sigma = \{j_1, \dots, j_i\}$, $j_1 < \dots < j_i$ we set $e_\sigma = e_{j_1} \wedge \dots \wedge e_{j_i}$ where $e_1, \dots, e_n$ is a basis of $K_1(\mathbf{x}; R/I)$. Let $c = \sum_{|\sigma|=i} c_\sigma e_\sigma \in K_i(\mathbf{x}; R/I)$ be an arbitrary element. Then we set

$$c^p = \sum_{|\sigma|=i} c_\sigma^p x_\sigma^{p-1} e_\sigma \qquad \text{where } x_\sigma^{p-1} = x_{j_1}^{p-1} \cdots x_{j_i}^{p-1}.$$

Note that for any $r \in R$ one has $(rc)^p = r^p c^p$. Furthermore it is easy to see that if c is a cycle in $K_i(\mathbf{x}; R/I)$, then c^p is a cycle in $K_i(\mathbf{x}; R/I^{[p]})$. More precisely we have:

LEMMA 1.5. *Let $z_1, \dots, z_r$ be cycles in $K_i(\mathbf{x}; R/I)$ whose homology classes form a basis of $H_i(\mathbf{x}; R/I)$. Then the homology classes of the cycles $z_1^p, \dots, z_r^p$ form a basis of $H_i(\mathbf{x}; R/I^{[p]})$.*

Proof. We may assume that $K = \mathbb{Z}/p\mathbb{Z}$. Then the Frobenius homomorphism acts trivially on K, and the map

$$\varphi\colon H_i(\mathbf{x}; R/I) \to H_i(\mathbf{x}; R/I^{[p]}), \qquad [z] \mapsto [z^p]$$

is K-linear. Since the Frobenius is a flat endomorphism of R (see [11]), R/I and $R/I^{[p]}$ have the same Betti-numbers, and so

$$\dim_K H_i(\mathbf{x}; R/I) = \beta_i(R/I) = \beta_i(R/I^{[p]}) = \dim_K H_i(\mathbf{x}; I^{[p]}).$$

Hence it suffices to show that φ is surjective. So let $w \in K_i(\mathbf{x}; R/I^{(p)})$ be a cycle whose homology class is not zero, and which is homogeneous in the $\mathbb{Z}^n$-grading. The

$\mathbb{Z}^n$-degree of w corresponds to a $\mathbb{Z}^n$-shift in the resolution of $R/I^{[p]}$. Any $\mathbb{Z}^n$-shift of $R/I^{[p]}$ is of the form pa, where a is a $\mathbb{Z}^n$-shift of R/I. We write

$$w = \sum_{|\sigma|=i} \lambda_\sigma u_\sigma e_\sigma$$

where for all σ in the sum, u_σ is a monomial and $\lambda_\sigma \in K$. Then for all σ with $\lambda_\sigma \neq 0$ we have

$$\deg_{\mathbb{Z}^n} u_\sigma + a_\sigma = pa$$

for some $a = (a_1, \ldots, a_n)$, $a_i \in \mathbb{Z}$, $a_i \geq 0$. Here $a_\sigma = (c_1, \ldots, c_n)$ with $c_i = 1$ for $i \in \sigma$, and $c_i = 0$ for $i \notin \sigma$.

It follows from the equation that $a_i \neq 0$ for $i \in \sigma$. Hence $b_\sigma = a - a_\sigma \in \mathbb{N}^n$, and we have

$$\deg_{\mathbb{Z}^n} u_\sigma = pb_\sigma + (p-1)a_\sigma.$$

Thus $u_\sigma = v_\sigma^p x_\sigma^{p-1}$, where v_σ is a monomial of $\mathbb{Z}^n$-degree b_σ. This implies that $w = z^p$ where $z = \sum_{|\sigma|=i} \lambda_\sigma v_\sigma e_\sigma$. Hence it remains to show that z is a cycle in $K_i(\mathbf{x}; R/I)$.

By assumption we have

$$\begin{aligned} \partial w = \partial z^p &= \sum_{|\tau|=i-1} \left(\sum_{\sigma \supset \tau, |\sigma|=i} (-1)^{\alpha(\sigma, \sigma\setminus\tau)} \lambda_\sigma v_\sigma^p x_\sigma^{p-1} x_{\sigma\setminus\tau} \right) e_\tau \\ &= \sum_{|\tau|=i-1} \left(\sum_{\sigma \supset \tau, |\sigma|=i} (-1)^{\alpha(\sigma, \sigma\setminus\tau)} \lambda_\sigma (x^{p(b_\sigma + a_\sigma)}/x_\tau) \right) e_\tau = 0 \end{aligned}$$

in $K_{i-1}(\mathbf{x}; R/I^{[p]})$, where for $\rho \subseteq \{1, \ldots, n\}$, and $1 \leq t \leq n$ we set $\alpha(\rho, t) = |\{s \in \rho \colon s < t\}|$.

We want to show that

$$\begin{aligned} \partial z &= \sum_{|\tau|=i-1} \left(\sum_{\sigma \supset \tau, |\sigma|=i} (-1)^{\alpha(\sigma, \sigma\setminus\tau)} \lambda_\sigma v_\sigma x_{\sigma\setminus\tau} \right) e_\tau \\ &= \sum_{|\tau|=i-1} \left(\sum_{\sigma \supset \tau, |\sigma|=i} (-1)^{\alpha(\sigma, \sigma\setminus\tau)} \lambda_\sigma (x^{b_\sigma + a_\sigma}/x_\tau) \right) e_\tau = 0 \end{aligned}$$

in $K_{i-1}(\mathbf{x}; R/I)$.

This will follow once we can show the following: if $x^{pc}/x_\tau \in I^{[p]}$ for some $c \in \mathbb{N}^n$, then $x^c/x_\tau \in I$.

So suppose that $x^{pc}/x_\tau \in I^{[p]}$. Then there exist a monomial $u \in I$, and a monomial v such that $x^{pc} = u^p v x_\tau$. Therefore $(x^c/u)^p = v x_\tau$, and so $u | x^c$. Hence $x^c = wu$ for some monomial w. It follows that $x^{pc} = w^p u^p = u^p v x_\tau$. Therefore $w^p = v x_\tau$, which implies that x_τ divides w. So $w = w' x_\tau$ for some monomial w', and this finally implies that $x^c/x_\tau = w'u \in I$. $\square$

Let u be a monomial, and $\sigma \subset \{1, \dots, n\}$ a subset. We set $m(u) = \max\{i\colon x_i | u\}$, $m(\sigma) = \max\{i\colon i \in \sigma\}$, $u' = u/x_m(u)$, and $f(\sigma; u) = u' e_\sigma \wedge e_{m(u)}$.

Now we are ready to formulate and to prove the main result of this section.

THEOREM 1.6. *Let $L = \langle x_n^a \rangle$ be a p-Borel principal ideal, and let $\sum_j a_j p^j$ be the p-adic expansion of a. Then for all i, $0 \leq i \leq m$, the elements*

$$\prod_{j>i} u_j^{p^j} f(\sigma; u_i)^{p^i}$$

with $u_j \in G(\mathfrak{m}^{a_j})$ for $j \geq i$, and $|\sigma| = k-1$, $m(\sigma) < m(u_i)$ are cycles in $K_k(\mathbf{x}; R/L)$ whose homology classes for $k \geq 2$ form a basis of $H_k(\mathbf{x}; R/L)$.

We call $\prod_{j>i} u_j^{p^j} f(\sigma; u_i)^{p^i}$ a *cycle of type i*. Note that the homology classes of the cycles of type zero with $\sigma = \emptyset$ form a basis of $H_1(\mathbf{x}; R/L)$.

The following simple example demonstrates the theorem: let $R = K[x_1, x_2]$, $p = 2$, $a = 5$. Then $5 = 1 + 0 \cdot 2 + 1 \cdot 4$, so that $\langle x_2^5 \rangle = (x_1, x_2)(x_1^4, x_2^4) = (x_1^5, x_1^4 x_2, x_1 x_2^4, x_2^5)$. By the theorem, $H_2(\mathbf{x}; R/\langle x_2^5 \rangle)$ is generated by $[x_1^3 x_2^3 e_1 e_2]$, $[x_1^4 e_1 e_2]$ and $[x_2^4 e_1 e_2]$. Here

$$x_1^3 x_2^3 e_1 e_2 = f(\{1\}; x_2)^4 \text{ is of type 2,}$$

while

$$x_1^4 e_1 e_2 = x_1^4 f(\{1\}; x_2) \text{ and } x_2^4 e_1 e_2 = x_2^4 f(\{1\}; x_2) \text{ are of type 0.}$$

Proof of Theorem 1.6. Let $l = \min\{i\colon a_i \neq 0\}$, and set

$$b = a/p^l = a_l + a_{l+1} p + \dots + a_m p^{m-l}.$$

Then $L = \langle x_n^b \rangle^{[p^l]}$. Applying Lemma 1.5 we may assume that $l = 0$, and so

$$a = a_0 + a_1 p + \dots + a_m p^m, \quad \text{with } a_0 \neq 0.$$

It follows that $\langle x_n^a \rangle = J I^{[p]}$ where $J = \mathfrak{m}^{a_0}$ and $I = \langle x_n^b \rangle$ with $b = a_1 + a_2 p + \dots + a_m p^{m-1}$.

Associated with the exact sequence

$$0 \to I^{[p]}/J I^{[p]} \to R/J I^{[p]} \to R/I^{[p]} \to 0$$

we have the long exact sequence of Koszul homology

$$\dots \to H_i(\mathbf{x}; R/J I^{[p]}) \to H_i(\mathbf{x}; R/I^{[p]}) \xrightarrow{\delta_i} H_{i-1}(I^{[p]}/J I^{[p]}) \to \dots$$

We claim that $\delta_i = 0$ for $i \geq 2$. Indeed, let $w \in Z_i(\mathbf{x}; R/I^{[p]})$; then, by Lemma 1.5,

$w = z^p + I^{[p]}K_i(\mathbf{x}; R)$ where $z = \sum_{|\sigma|=i} u_\sigma e_\sigma \in K_i(\mathbf{x}; R)$, and where

$$\partial z = \sum_{|\tau|=i-1} \left(\sum_{\sigma \supset \tau, |\sigma|=i} (-1)^{\alpha(\sigma,\sigma\setminus\tau)} x_{\sigma\setminus\tau} u_\sigma \right) e_\tau$$

belongs to $IK_{i-1}(\mathbf{x}; R)$.

Since $z^p = \sum_{|\sigma|=i} u_\sigma^p x_\sigma^{p-1} e_\sigma$, it follows that

$$\partial z^p = \sum_{|\tau|=i-1} \left(\sum_{\sigma \supset \tau, |\sigma|=i} (-1)^{\alpha(\sigma,\sigma\setminus\tau)} x_{\sigma\setminus\tau} u_\sigma^p x_\sigma^{p-1} \right) e_\tau$$

belongs to $JI^{[p]}K_{i-1}(\mathbf{x}; R)$. Hence, since $\delta_i w = \partial z^p + JI^{[p]}K_{i-1}(\mathbf{x}; R)$, the homology class of $\delta_i w$ is zero.

As a consequence, for $k \geq 2$ we have the exact sequences

$$0 \to H_k(\mathbf{x}; I^{[p]}/JI^{[p]}) \to H_k(\mathbf{x}; R/JI^{[p]}) \to H_k(\mathbf{x}; R/I^{[p]}) \to 0.$$

It follows that a basis of $H_k(\mathbf{x}; R/JI^{[p]})$ is formed by a basis of $H_k(\mathbf{x}; R/I^{[p]})$ and a basis of $H_k(\mathbf{x}; I^{[p]}/JI^{[p]})$.

Arguing by induction on m we may assume that we know a basis of $H_k(\mathbf{x}; R/I)$. The induction begin is guaranteed since for $m = 0$, I is stable, and a basis is known from [2] or [13]. Thus, by Lemma 1.5, we know a basis of $H_k(\mathbf{x}; R/I^{[p]})$, and hence we conclude that the homology classes of the elements $\prod_{j>i} u_j^{p^j} f(\sigma; u_i)^{p^i}$, $i \geq 1$, with $u_j \in G(\mathfrak{m}^{a_j})$ for $j \geq i$, and $m(\sigma) < m(u_i)$ form a basis of $H_k(\mathbf{x}; R/I^{[p]})$. Thus it remains to show that the elements of a basis of $H_k(\mathbf{x}; I^{[p]}/JI[p])$ are mapped to the homology classes of the elements $\prod_{j>0} u_j^{p^j} f(\sigma; u_0)$ with $u_j \in G(\mathfrak{m}^{a_j})$ for $j \geq 0$, and $m(\sigma) < m(u_0)$.

By Lemma 1.4, these are exactly the elements $[vf(\sigma; u)]$, $v \in G(I)$, $u \in G(\mathfrak{m}^{a_0})$. We choose a minimal presentation

$$R^s \xrightarrow{(\alpha_{ij})} R^r \to I \to 0.$$

Then

$$R^s \xrightarrow{(\alpha_{ij}^p)} R^r \to I^{[p]} \to 0$$

is a minimal presentation of $I^{[p]}$. Therefore, since all $\alpha_{ij}^p \in \mathfrak{m}^{a_0}$, it follows that

$$I^{[p]}/JI^{[p]} \cong \bigoplus_{v \in G(I)} (R/J)b_v$$

is a free R/J-module with basis, say, b_v, $v \in G(I)$. Hence $H_k(\mathbf{x}; I^{[p]}/JI^{[p]})$ is isomorphic to $\bigoplus_{v\in G(I)} H_k(\mathbf{x}; R/J)b_v$.

Since J is a stable ideal, we know from [2] (or [13]) that $[f(\sigma; u)]$ with $u \in G(J)$, $|\sigma| = k-1$, $m(\sigma) < m(u)$ is a basis of $H_k(\mathbf{x}; R/J)$. Finally we see that the homomorphism given by the composition

$$\bigoplus_{v\in G(I)} H_k(\mathbf{x}; R/J)b_v \xrightarrow{\sim} H_k(\mathbf{x}; I^{[p]}/JI^{[p]}) \to H_k(R/JI^{[p]})$$

maps the element $[f(\sigma; u)]b_v$ to $[vf(\sigma; u)]$, as desired. □

2. The resolution of a p-Borel principal Cohen-Macaulay ideal

In this section we compute the resolution of the p-Borel ideals studied in the previous section. Knowing the Koszul homology we use the technique developed in [2] to compute the differentials in the resolution.

Let $L = \langle x_n^a \rangle$ be a p-Borel principal ideal in the polynomial ring $R = K[x_1, \dots, x_n]$. We set $G_k = R \otimes_K H_k(\mathbf{x}; R/L)$ for all $k \geq 0$. Then L has a free resolution of the form

$$\cdots \to G_2 \xrightarrow{d_2} G_1 \xrightarrow{d_1} G_0 \to 0.$$

It is clear that, according to 1.6, the elements $1 \otimes [vf(\sigma; u)^{p^i}]$, which for simplicity we simply denote by $[vf(\sigma; u)^{p^i}]$, form a basis of G_k. Here $v = \prod_{j>i} u_j^{p^j}$, $u = u_i$, $u_j \in G(\mathfrak{m}^{a_j})$ for $j \geq i$, $|\sigma| = k-1$, and $m(\sigma) < m(u)$. As in the previous section we set $u' = u/x_m$, $m = m(u)$.

THEOREM 2.1. *The maps d_k in the resolution of $R/\langle x_n^a \rangle$ are given by the following formulas*:

$$d_1([vf(\emptyset; u)]) = vu,$$

$$\begin{aligned} d_2([vf(t; u)^{p^i}]) &= x_m^{p^i - \sum_{q=0}^{i-1} a_q p^q} \left[v(x_t u')^{p^i} \prod_{q=1}^{i-1} x_m^{a_q p^q} f(\emptyset; x_m^{a_0}) \right] \\ &\quad - x_t^{p^i - \sum_{q=0}^{i-1} a_q p^q} \left[vu^{p^i} \prod_{q=1}^{i-1} x_t^{a_q p^q} f(\emptyset; x_t^{a_0}) \right], \end{aligned}$$

and

$$\begin{aligned} d_k([vf(\sigma; u)^{p^i}]) &= \sum_{t\in\sigma} (-1)^{\alpha(\sigma,t)+k-1} (x_t^{p^i} [vf(\sigma \setminus t; u)^{p^i}] \\ &\quad - x_m^{p^i} [vf(\sigma \setminus t; x_t u')^{p^i}] - x_m s(v(x_t u')^{p^i}; \sigma \setminus t; m)) \\ &\quad - x_{m(\sigma)} s(vu^{p^i}; \sigma \setminus m(\sigma); m(\sigma)) \end{aligned}$$

where for $\rho \subset \sigma$, $w = \prod_{j\geq i} w_j^{p^j}$, $w_j \in G(\mathfrak{m}^{a_j})$, *and* r, $1 \leq r \leq n$, *we set*

$$s(w;\rho;r) = \sum_{\mu=0}^{i-1} \sum x_r^{\psi_\mu} \prod_{t\in\rho} x_t^{\varphi_{t\mu}} \left[w \prod_{q=\mu+1}^{i-1} \left(\prod_{t\in\rho\cup r} x_t^{j_{qt}} \right)^{p^q} f(\rho; \prod_{t\in\rho\cup r} x_t^{j_{\mu t}})^{p^\mu} \right]$$

where the second sum is taken over all $j_{qt} \geq 0$ *such that* $\sum_{t\in\rho\cup r} j_{qt} = a_q$ *for* $\mu \leq q \leq i-1$, $\varphi_{t\mu} = p^i - \sum_{q=\mu}^{i-1} j_{qt}p^q - p^\mu$ *for* $t \in \rho$, $\psi_\mu = p^i - \sum_{q=\mu}^{i-1} j_{qr}p^q - 1$, *and we set* $f(\rho; v) = 0$ *if* $m(\rho) \geq m(v)$.

To prove the theorem we need the following lemma.

LEMMA 2.2. *Let* $\tilde{v} = \prod_{q\geq i} v_q^{p^q}$ *where* $v_q \in G(\mathfrak{m}^{a_q})$, $\tau \subset \sigma$, $|\tau| = j$, $j \geq 1$, *and let* $r > m(\tau)$. *Assume that the formula for* d_{j+1} *in* 2.1 *is true. Then*

$$\begin{aligned} d_{j+1}(s(\tilde{v};\tau;r)) &= \sum_{t\in\tau} (-1)^{\alpha(\tau,t)+j} x_t^{p^i} s(\tilde{v};\tau\setminus t;r) \\ &\quad + x_r^{p^i-1} x_{m(\tau)} s(\tilde{v};\tau\setminus m(\tau);m(\tau)). \end{aligned}$$

Proof. Set $w_{\mu+1} = \tilde{v} \prod_{q=\mu+1}^{i-1} \prod_{t\in\rho\cup r} x_t^{j_{qt}p^q}$, and $m' = m(\tau)$. Since $s(\tilde{v};\tau;r) \neq 0$ only if $j_{\mu r} \geq 1$, we obtain, for $0 \leq \mu \leq i-1$,

$$d_{j+1}\left(\sum_{j_{\mu r}\geq 1} x_r^{\psi_\mu} \prod_{t\in\tau} x_t^{\varphi_{t\mu}} \left[w_{\mu+1} f(\tau; \prod_{t\in\tau\cup r} x_t^{j_{\mu t}})^{p^\mu} \right] \right) = \tilde{y}_\mu - \sum_{t\in\tau} (-1)^{\alpha(\tau,t)+j} b_{t\mu} - c_\mu$$

where

$$\begin{aligned} \tilde{y}_\mu &= \sum_{t\in\tau} (-1)^{\alpha(\tau,t)+j} \left(\sum_{j_{\mu r}\geq 1} x_r^{\psi_\mu} \prod_{s\in\tau} x_s^{\varphi_{s\mu}} \left(x_t^{p^\mu} \left[w_{\mu+1} f(\tau\setminus t; \prod_{s\in\tau\cup r} x_s^{j_{\mu s}})^{p^\mu} \right] \right.\right. \\ &\quad \left.\left. - x_r^{p^\mu} \left[w_{\mu+1} f(\tau\setminus t; x_t^{j_{\mu t}+1} x_r^{j_{\mu r}-1} \prod_{s\in\tau\setminus t} x_s^{j_{\mu s}})^{p^\mu} \right] \right) \right); \end{aligned}$$

$$b_{t\mu} = \sum_{j_{\mu r}\geq 1} x_r^{\psi_\mu+1} \prod_{s\in\tau} x_s^{\varphi_{s\mu}} s\left(w_{\mu+1} \left(x_t^{j_{\mu t}+1} x_r^{j_{\mu r}-1} \prod_{s\in\tau\setminus t} x_s^{j_{\mu s}} \right)^{p^\mu}; \tau\setminus t; r \right);$$

$$c_\mu = \sum_{j_{\mu r}\geq 1} x_r^{\psi_\mu} x_{m'}^{\varphi_{m'\mu}+1} \prod_{s\in\tau\setminus m'} x_s^{\varphi_{s\mu}} s(w_\mu; \tau\setminus m'; m').$$

Considering the summands in $\tilde{y}_\mu$ for a fixed $t \in \tau$, we see that they cancel two by two, and only the summands for $j_{\mu t} = 0$ or $j_{\mu r} = 0$ are left. But if $t \neq m'$, then $f(\tau \setminus t; \prod_{s\in\tau} x_s^{j_{\mu s}})^{p^\mu} = 0$; therefore $\tilde{y}_\mu = y_\mu + h_\mu$ where

$$\begin{aligned} y_\mu = &\sum_{t\in\tau}(-1)^{\alpha(\tau,t)+j}\sum_{j_{\mu r}\geq 1} x_t^{p^i - \sum_{q=\mu+1}^{i-1} j_{qt}p^q} \prod_{s\in\tau\setminus t} x_s^{\varphi_{s\mu}} x_r^{\psi_\mu} \\ &\times \left[w_{\mu+1} f(\tau \setminus t; \prod_{s\in\tau\setminus t\cup r} x_s^{j_{\mu s}})^{p^\mu} \right]; \end{aligned}$$

$$\begin{aligned} h_\mu = &\sum_{j_{\mu m'}\geq 1} x_{m'}^{p^i - \sum_{q=\mu}^{i-1} j_{qm'}p^q} x_r^{p^i - \sum_{q=\mu+1}^{i-1} j_{qr}p^q - 1} \prod_{s\in\tau\setminus m'} x_s^{\varphi_{s\mu}} \\ &\times \left[w_{\mu+1} f(\tau \setminus m'; \prod_{s\in\tau} x_s^{j_{\mu s}})^{p^\mu} \right]. \end{aligned}$$

Set $y = \sum_{\nu=0}^{i-1} y_\nu$ and $h = \sum_{\nu=0}^{i-1} h_\nu$. Changing the summation indices, one obtains

$$b_{t\mu} = \sum_{j_{\mu r}\geq 1, j_{\mu r}\geq 0} x_t^{\varphi_{\mu t}+p^\mu} x_r^{\varphi_{\mu r}} \prod_{s\in\tau\setminus t} x_s^{\varphi_{s\mu}} s(w_\mu; \tau \setminus t; r).$$

Fix $t \in \tau, 0 \leq \mu \leq i-1, 0 \leq \nu < \mu$, and consider an arbitrary summand of $b_{t\mu}$. Computing the powers of x_s, $s \in \tau \cup r$, one sees that each summand of $b_{t\mu}$ appears in y_ν taking $j_{qt} = 0$ for $\nu + 1 \leq q \leq \mu - 1$ and $j_{\mu t} \geq 1$. Therefore

$$\begin{aligned} y - &\sum_{\mu=0}^{i-1}\sum_{t\in\tau}(-1)^{\alpha(\tau,t)+j} b_{t\mu} \\ &= \sum_{t\in\tau}(-1)^{\alpha(\tau,t)+j} x_t^{p^i} \sum_{\nu=0}^{i-1}\sum_{j_{\nu r}\geq 1} x_r^{\psi_\nu} \prod_{s\in\tau\setminus t} x_s^{\varphi_{s\nu}} \\ &\quad\times \left[\tilde{v} \prod_{q=\nu+1}^{i-1} \prod_{s\in\tau\setminus t\cup r} x_s^{j_{qs}p^q} f(\tau \setminus t; \prod_{s\in\tau\setminus t\cup r} x_s^{j_{\nu s}})^{p^\nu} \right] \\ &= \sum_{t\in\tau}(-1)^{\alpha(\tau,t)+j} x_t^{p^i} s(\tilde{v}; \tau \setminus t; r). \end{aligned}$$

On the other hand, from the definition of $s(w_\mu; \tau \setminus m'; m')$ it follows that each summand of c_μ appears in h_ν taking $j_{qr} = 0$ for $\nu + 1 \leq q \leq \mu - 1$ and $j_{\mu r} \geq 1$. Hence $h - \sum_{\mu=0}^{i-1} c_\mu = x_r^{p^i - 1} x_{m'} s(\tilde{v}; \tau \setminus m'; m')$, and this completes the proof of the lemma. □

Proof of Theorem 2.1. According to [2] we have to find a sequence of elements $g_j \in K_{k-j}(\mathbf{x}; G_j)$, $0 \leq j \leq k-1$, satisfying $d_0(g_0) = v(f(\sigma; u))^{p^i}$, $\partial_{k-j}(g_j) = d_{j+1}(g_{j+1})$ for $0 \leq j \leq k-2$. Here ∂ denotes the differential of the Koszul complex $K(\mathbf{x}; G_j)$, and for short we write d_j for the map $K(\mathbf{x}; G_j) \to K(\mathbf{x}; G_{j-1})$. With this notation the desired differential is given by the formula $d_k([vf(\sigma; u)^{p^i}]) = \partial_1(g_{k-1})$.

Let

$$g_0 = v(u')^{p^i} x_m^{p^i-1} x_\sigma^{p^i-1} e_\sigma \wedge e_m,$$

$$\begin{aligned} g_1 &= x_m^{p^i - \sum_{q=0}^{i-1} a_q p^q - 1} \sum_{t\in\sigma} (-1)^{\alpha(\sigma,t)} x_{\sigma\setminus t}^{p^i-1} \left[v(x_t u')^{p^i} \prod_{q=1}^{i-1} x_m^{a_q p^q} f(\emptyset; x_m^{a_0}) \right] e_{\sigma\setminus t} \wedge e_m \\ &\quad + (-1)^{k-1} x_{m(\sigma)}^{p^i - \sum_{q=0}^{i-1} a_q p^q - 1} x_{\sigma\setminus m(\sigma)}^{p^i-1} \left[vu^{p^i} \prod_{q=1}^{i-1} x_{m(\sigma)}^{a_q p^q} f(\emptyset; x_{m(\sigma)}^{a_0}) \right] e_\sigma. \end{aligned}$$

Note here that $x_t u' \in G(\mathfrak{m}^{a_i})$ for each $t \in \sigma$. It is easy to verify that $\partial_k(g_0) = d_1(g_1)$; therefore we obtain the formula for d_2. For $j \geq 2$ let

$$g_j = (-1)^{\varepsilon_{kj}} \left(\sum_{\rho\subset\sigma, |\rho|=j-1} (-1)^{\beta(\rho)} b_\rho e_{\sigma\setminus\rho} + (-1)^{k-j} \sum_{\tau\subseteq\sigma, |\tau|=j} (-1)^{\beta(\tau)} c_\tau e_{\sigma\setminus\tau} \wedge e_m \right),$$

where $\varepsilon_{kj} = k + \frac{(j+1)(j+2)}{2}$, $\beta(\rho) = \sum_{t\in\rho} \alpha(\sigma, t)$ and

$$b_\rho = x_{\sigma\setminus\rho}^{p^i-1}[vf(\rho; u)^{p^i}] + x_{\sigma\setminus\rho\setminus m(\sigma)}^{p^i-1} s(vu^{p^i}; \rho; m(\sigma));$$

$$c_\tau = x_{\sigma\setminus\tau}^{p^i-1} \left(\sum_{t\in\tau} (-1)^{\alpha(\tau,t)} (x_m^{p^i-1}[vf(\tau \setminus t; x_t u')^{p^i}] + s(v(x_t u')^{p^i}; \tau \setminus t; m)) \right).$$

Proceeding by induction on k, we can assume that the formulas are true for $j+1 < k$ and we will verify that $\partial_{k-j}(g_j) = d_{j+1}(g_{j+1})$. We have

$$\begin{aligned} \partial_{k-j}(g_j) &= (-1)^{\varepsilon_{kj}} \left(\sum_{\tau\subset\sigma, |\tau|=j} (-1)^{\beta(\tau)} \left(\sum_{t\in\tau} (-1)^{\alpha(\tau,t)} x_t b_{\tau\setminus t} - x_m c_\tau \right) e_{\sigma\setminus\tau} \right. \\ &\quad \left. + (-1)^{k-j} \sum_{\gamma\subseteq\sigma, |\gamma|=j+1} (-1)^{\beta(\gamma)} \left(\sum_{s\in\gamma} (-1)^{\alpha(\gamma,s)} x_s c_{\gamma\setminus s} \right) e_{\sigma\setminus\gamma} \wedge e_m \right). \end{aligned}$$

Fix $\tau \subset \sigma$, $|\tau| = j$ and set $b = \sum_{t\in\tau} (-1)^{\alpha(\tau,t)} x_t b_{\tau\setminus t} - x_m c_\tau$. If $m(\sigma) \in \tau$, then $s(vu^{p^i}; \tau\setminus t; m(\sigma)) \neq 0$ only if $t = m(\sigma)$; hence $b = (-1)^j x_{\sigma\setminus\tau}^{p^i-1} d_{j+1}([vf(\tau; u)^{p^i}])$. So, assume $r = m(\sigma) \notin \tau$. Then

$$\begin{aligned} b = (-1)^j x_{\sigma\setminus\tau}^{p^i-1} d_{j+1}([vf(\tau; u)^{p^i}]) \; &+ \; (-1)^j x_{\sigma\setminus\tau}^{p^i-1} x_{m'} s(vu^{p^i}; \tau \setminus m'; m') \\ &+ \; x_{\sigma\setminus\tau\setminus r}^{p^i-1} \sum_{t\in\tau} (-1)^{\alpha(\tau,t)} x_t^{p^i} s(vu^{p^i}; \tau \setminus t; r) \end{aligned}$$

where $m' = m(\tau)$. By 2.2 we have

$$d_{j+1}(s(vu^{p^i};\tau;r)) = \sum_{t\in\tau}(-1)^{\alpha(\tau,t)+j}x_t^{p^i}s(vu^{p^i};\tau\backslash t;r)+x_r^{p^i-1}x_{m'}s(vu^{p^i};\tau\backslash m';m').$$

Hence we obtain $b = (-1)^j d_{j+1}(b_\tau)$.

Now fix $\gamma \subseteq \sigma$, $|\gamma| = j+1$ and consider $c = \sum_{s\in\gamma}(-1)^{\alpha(\gamma,s)}x_s c_{\gamma\backslash s}$. We will show that $c = (-1)^{j+1}d_{j+1}(c_\gamma)$. For $\rho \subseteq \sigma$ and $w \in G(\mathfrak{m}^{a_i})$, set

$$A(\rho, w) = \{t \in \rho\colon\ m(\rho\setminus t) < m(x_t w')\}.$$

Then $c = x_{\sigma\backslash\gamma}^{p^i-1}(h_1+h_2)$ where

$$h_1 = \sum_{s\in\gamma}\sum_{t\in\gamma\backslash s}(-1)^{\alpha(\gamma,s)+\alpha(\gamma\backslash s,t)}x_s^{p^i}s(v(x_t u')^{p^i};\gamma\setminus\{t,s\};m);$$

$$h_2 = x_m^{p^i-1}\sum_{s\in\gamma}\ \sum_{t\in A(\gamma\backslash s,u)}(-1)^{\alpha(\gamma,s)+\alpha(\gamma\backslash s,t)}x_s^{p^i}[vf(\gamma\setminus\{t,s\};x_t u')^{p^i}].$$

On the other hand, $c_\gamma = x_{\sigma\backslash\gamma}^{p^i-1}(c_1+c_2)$ where

$$c_1 = x_m^{p^i-1}\sum_{t\in A(\gamma,u)}(-1)^{\alpha(\gamma,t)}[vf(\gamma\setminus t;x_t u')^{p^i}];$$

$$c_2 = \sum_{t\in\gamma}(-1)^{\alpha(\gamma,t)}s(v(x_t u')^{p^i};\gamma\setminus t;m).$$

Then we have $(-1)^{j+1}d_{j+1}(c_1) = x_m^{p^i-1}(y+s_1+s_2+s_3)$ where

$$y = \sum_{t\in A(\gamma,u)}\sum_{s\in\gamma\backslash t}(-1)^{\alpha(\gamma,t)+\alpha(\gamma\backslash t,s)+1}x_s^{p^i}[vf(\gamma\setminus\{t,s\};x_t u')^{p^i}];$$

$$s_1 = \sum_{t\in A(\gamma,u)}\sum_{s\in\gamma\backslash t}(-1)^{\alpha(\gamma,t)+\alpha(\gamma\backslash t,s)}x_{m(x_t u')}s(v(x_s(x_t u')')^{p^i};\gamma\setminus\{t,s\};m(x_t u'));$$

$$s_2 = \sum_{t\in A(\gamma,u)}\ \sum_{s\in A(\gamma\backslash t,x_t u')}(-1)^{\alpha(\gamma,t)+\alpha(\gamma\backslash t,s)}x_{m(x_t u')}^{p^i}[vf(\gamma\setminus\{t,s\};x_s(x_t u')')^{p^i}];$$

$$s_3 = (-1)^j\sum_{t\in A(\gamma,u)}(-1)^{\alpha(\gamma,t)}x_{m(\gamma\backslash t)}s(v(x_t u')^{p^i};\gamma\setminus t\setminus m(\gamma\setminus t);m(\gamma\setminus t)).$$

By Lemma 2.2 applied to each $s(v(x_t u')^{p^i}; \gamma \setminus t; m)$, we obtain $(-1)^{j+1}d_{j+1}(c_2) = h + s_3'$ where

$$h = \sum_{t\in\gamma}\sum_{s\in\gamma\setminus t}(-1)^{\alpha(\gamma,t)+\alpha(\gamma\setminus t,s)+1}x_s^{p^i}s(v(x_t u')^{p^i}; \gamma \setminus \{t, s\}; m);$$

$$s_3' = (-1)^{j+1}x_m^{p^i-1}\sum_{t\in\gamma}(-1)^{\alpha(\gamma,t)}x_{m(\gamma\setminus t)}s(v(x_t u')^{p^i}; \gamma \setminus t \setminus m(\gamma \setminus t); m(\gamma \setminus t)).$$

Now we have $x_m^{p^i-1}s_3 + s_3' = (-1)^{j+1}x_m^{p^i-1}s_4$ where

$$s_4 = \sum_{t\notin A(\gamma,u)}(-1)^{\alpha(\gamma,t)}x_{m(\gamma\setminus t)}s(v(x_t u')^{p^i}; \gamma \setminus t \setminus m(\gamma \setminus t); m(\gamma \setminus t)).$$

Since $\alpha(\gamma, s) + \alpha(\gamma \setminus s, t) \equiv \alpha(\gamma, t) + \alpha(\gamma \setminus t, s) + 1 \pmod 2$, we obtain $h_1 = h$. Moreover, each summand of $x_m^{p^i-1}y$ appears in h_2, because if $t \in A(\gamma, u)$, then $m(\gamma \setminus \{t, s\}) \le m(\gamma \setminus t) < m(x_t u')$, so that $t \in A(\gamma \setminus s, u)$. Therefore in h_2 remain the summands for which $t \notin A(\gamma, u)$ and $t \in A(\gamma \setminus s, u)$. But then $m(\gamma) = s$, $\alpha(\gamma, s) = j$, hence we have to show that

$$\begin{aligned} x_m^{p^i-1}x_{m(\gamma)}^{p^i}(-1)^j \sum_{t\in A(\gamma\setminus m(\gamma),u),\,t\notin A(\gamma,u)} &(-1)^{\alpha(\gamma,t)}[vf(\gamma \setminus \{t, m(\gamma)\}; x_t u')^{p^i}] \\ &= x_m^{p^i-1}(s_1 + s_2 + (-1)^{j+1}s_4). \end{aligned}$$

We now show that $s_1 + (-1)^{j+1}s_4 = 0$. Let $t \in A(\gamma, u)$ and $s \in \gamma \setminus t$. First assume that $s \in A(\gamma, u)$. Then we may assume $t < s$. If $m(x_s u') > s$, then $l = m(x_t u') = m(x_s u')$, which implies that the summand $x_l s(v((x_t x_s u')/x_l)^{p^i}; \gamma \setminus \{t, s\}; l)$ occurs in s_1 with coefficients $+1$ and -1. If $m(x_s u') = s$, then since $m(\gamma \setminus s) < s$, we have $m(\gamma) = s$, so that we obtain the contradiction $s = m(\gamma\setminus t) < m(x_t u') \le m(x_s u') = s$.

Now let $s \notin A(\gamma, u)$. Assuming $t < s$, one obtains a contradiction; therefore $s < t$. If $m(x_t u') > t$, then we obtain again a contradiction; hence $m(x_t u') = t$ and $m(\gamma) = t = m(\gamma \setminus s)$. Since $\alpha(\gamma, m(\gamma)) = j$, we get $s_1 + (-1)^{j+1}s_4 = 0$.

Consider s_2, and fix $t \in A(\gamma, u)$ and $s \in A(\gamma \setminus t, x_t u')$. First assume that $t < s$. Then the case $m(x_s u') = s$ is not possible, so that $l = m(x_t u') = m(x_s u') > s$. Therefore $t \in A(\gamma \setminus s, x_s u')$ and $s \in A(\gamma, u)$. Hence the summand $x_l^{p^i}[vf(\gamma \setminus \{t, s\}; (x_t x_s u')/x_l)^{p^i}]$ occurs in s_2 with coefficients $+1$ and -1.

Finally, let $t > s$. If $m(x_t u') > t$, then we show as above that the summands cancel. Let $m(x_t u') = t$. Then $m(\gamma) = t$ and since $s \in A(\gamma \setminus t, x_t u')$, one obtains $s \in A(\gamma \setminus t, u)$. Moreover $m(\gamma \setminus s) = m(\gamma) = t = m(x_t u') \ge m(x_s u')$; hence $s \notin A(\gamma, u)$, and this completes the proof of the theorem.

3. Bounds for the regularity

In this section we study the regularity of a Borel principal ideal $\langle x^\mu\rangle$, where $x^\mu = \prod_{k=1}^n x_k^{\mu_k}$. Recall that for a graded ideal I the regularity is defined as follows:

for each $i > 0$ let a_i be the largest integer for which $H_i(\mathbf{x}; R/I)_{a_i} \neq 0$. Then the regularity $reg(I)$ of I is $\max_i\{a_i - i + 1\}$.

According to [12] one has $reg\langle x^\mu \rangle = \mu_1 + reg\langle \frac{x^\mu}{x_1^{\mu_1}} \rangle$; therefore we can assume that x_1 does not divide x^μ.

Denote by $\lfloor * \rfloor$ the greatest integer function. For $1 \leq k \leq n$ and $j \geq 0$ define

$$d_{kj}(\mu) = \sum_{i=1}^{k} \left\lfloor \frac{\mu_i}{p^j} \right\rfloor.$$

For every k such that $\mu_k \neq 0$, let $s_k = \lfloor \log_p \mu_k \rfloor$. Set

$$D_k = d_{ks_k}(\mu) p^{s_k} + (k-1)(p^{s_k} - 1).$$

CONJECTURE 3.1 (Pardue). *If x_1 does not divide x^μ, then*

$$reg\langle x^\mu \rangle = \max_{k:\ \mu_k \neq 0} \{D_k\}.$$

We show that the conjectured formula is a lower bound for the regularity.

THEOREM 3.2. *Let $u = \prod_{k=2}^n x_k^{\mu_k}$ and let $\mu_k = \sum_{j=0}^{s_k} \mu_{kj} p^j$ where $0 \leq \mu_{kj} \leq p-1$. If $\max_{k:\ \mu_k \neq 0}\{D_k\} = D_l$, then the elements*

$$\prod_{j \geq s_l + 1} u_j^{p^j} f(\sigma, u_{s_l})^{p^{s_l}}$$

are cycles in $K_l(\mathbf{x}; R/\langle u \rangle)$ whose homology classes are non trivial in $H_l(\mathbf{x}; R/\langle u \rangle)$. Here, for $j \geq s_l$, $u_j \in G(\prod_{k \geq 2}(x_1, \ldots, x_k)^{\mu_{kj}})$, $\max(u_{s_l}) = l$, and $\sigma = \{1, \ldots, l-1\}$.

From the theorem and 3.5 below, we obtain the following:

COROLLARY 3.3. *With the notation of 3.2 the regularity of $\langle u \rangle$ is bounded below by*

$$\max_{k:\ \mu_k \neq 0} \{D_k\}.$$

Before proving the theorem we introduce some more notation and prove a few technical lemmata: Let $m = \max_{k:\ \mu_k \neq 0}\{s_k\}$, $n_j = \max\{k\colon \mu_{kj} \neq 0\}$, $0 \leq j \leq m$ and set

$$\mathfrak{m}_k = (x_1, \ldots, x_k), \quad I_j = \prod_{k=2}^{n_j} \mathfrak{m}_k^{\mu_{kj}}, \quad I = \prod_{j=1}^{m} I_j^{[p^{j-1}]}.$$

Let $l_j = \max\{k\colon s_k = j\}$, and set $r_j = D_{l_j}$. Then

$$r_j = \sum_{i=j}^{m} \left(\sum_{k=2}^{l_j} \mu_{ki} \right) p^i + (l_j - 1)(p^j - 1),$$

and therefore $r_j > D_k$ for each k such that $s_k = j$ and $k < l_j$. Hence

$$\max\{D_k\} = \max\{r_t\}.$$

Finally for $j \geq 1$ let

$$\tilde{r}_{j-1} = \sum_{i=j}^{m} \left(\sum_{k=2}^{l_j} \mu_{ki} \right) p^{i-1} + (l_j - 1)(p^{j-1} - 1).$$

LEMMA 3.4. *The following rules hold*:

(*i*) *If* $s < t$ *and* $l_s < l_t$, *then* $r_s < r_t$.
(*ii*) *If* $\max\{r_j\} = r_s$, *then* $n_j \leq l_s - 1$ *for* $s + 1 \leq j \leq m$ *and* $n_s = l_s$.
(*iii*) *Let* $\max\{r_j\} = r_t$ *and* $\max\{\tilde{r}_j\} = \tilde{r}_{s-1}$. *If* $r_t \neq r_s$, *then* $t < s$ *and* $l_t > l_s$.
(*iv*) *If* $\max\{r_j\} = r_s$, *then* $r_s \geq \deg u$.

Proof. (i) From the definition of the integers r_j it follows that

$$r_t - r_s = (l_t - 1)(p^t - 1) - (l_s - 1)(p^s - 1) - \sum_{j=s}^{t-1} \sum_{k=2}^{l_s} \mu_{kj} p^j + \sum_{j=t}^{m} \sum_{k=l_s+1}^{l_t} \mu_{kj} p^j.$$

Since $l_s < l_t$, and since each $\mu_{kj} \leq p - 1$, one obtains

$$r_t - r_s > (l_s - 1)(p^t - p^s) - (l_s - 1)(p - 1) \sum_{j=s}^{t-1} p^j = 0.$$

(ii) Assume $\mu_{kj} \neq 0$ for some $k > l_s$ and $s \leq j \leq m$. Then $s_k \geq s$ but since $k > l_s$, $s_k > s$ and $l_{s_k} > l_s$. Applying (i), we obtain the contradiction $r_{s_k} > r_s$. Therefore $n_j \leq l_s$ for $s \leq j \leq m$. As $\mu_{l_s j} = 0$ for $j \geq s + 1$ and $\mu_{l_s s} \neq 0$, one has $n_j \leq l_s - 1$ for $j \geq s + 1$ and $n_s = l_s$.

(iii) Since for $j \geq 1$ one has $r_j = p\tilde{r}_{j-1} + (l_j - 1)(p - 1)$, we obtain

$$\begin{aligned} r_t &= p\tilde{r}_{t-1} + (l_t - 1)(p - 1) \\ &\leq p\tilde{r}_{s-1} + (l_t - 1)(p - 1) \\ &= r_s - (l_s - 1)(p - 1) + (l_t - 1)(p - 1) = r_s + (l_t - l_s)(p - 1). \end{aligned}$$

By assumption $r_t > r_s$, therefore $l_t > l_s$. From (ii) applied to the ideal I and to the $\tilde{r}_j$ it follows that $\mu_{kj} = 0$ for $k > l_s$ and $j \geq s$. Since $\mu_{l_t t} \neq 0$, one has $t < s$.

(iv) There exists a j such that $l_j = n$. Then $r_j - \deg u = (n - 1)(p^j - 1) - \sum_{k=2}^{n} \sum_{i=0}^{j-1} \mu_{ki} p^i \geq 0$, because each $\mu_{ki} \leq p - 1$. As $r_s \geq r_j$, we obtain the desired inequality. $\square$

COROLLARY 3.5. *Let* $\max\{r_j\} = r_s$. *Then for the element*

$$z = \prod_{j>s}^{m} u_j^{p^j} f(\sigma, u_s)^{p^s}$$

with $u_j \in G(I_j)$ *for* $j \geq s$, $\max(u_s) = l_s$ *and* $\sigma = \{1, \ldots, l_s - 1\}$ *we have* $\deg z - (l_s - 1) = r_s$.

This follows from 3.4(ii).

LEMMA 3.6. *Let* $J = \prod_{k=2}^{h} \mathfrak{m}_k^{\nu_k}$ *with* $0 \leq \nu_k < p$ *and let* L *be an ideal generated by monomials of degree* d. *Let* $\tau = \{1, \ldots, l\}$ *and* $q \geq 1$. *If* $x_{\tau\setminus i}^{p^{q-1}-1} \in L$ *for each* $i \in \tau$ *and if* $(p^q - 1)(l - 1) \geq \sum_{k=2}^{h} \nu_k + pd$, *then* $x_{\tau\setminus i}^{p^q-1} \in JL^{[p]}$ *for each* $i \in \tau$.

Proof. Since $x_{\tau\setminus i}^{p^q-p} \in L^{[p]}$, one has $x_{\tau\setminus i}^{p^q-p} = v^p \prod_{j\in\tau\setminus i} x_j^{pq_j}$ with $v \in G(L)$ and $q_j \geq 0$. Moreover $p\sum_{j\in\tau\setminus i} q_j + (l-1)(p-1) \geq \sum_{k=2}^{h} \nu_k$ and it remains to show that $\prod_{j\in\tau\setminus i} x_j^{pq_j+p-1} \in J$. If $h \leq l$, then one sees easily that $x_{\tau\setminus i}^{p-1} \in J$. Assume $h > l$. Then $x_{\tau\setminus i}^{p-1} = g_1 g_2$ where $g_1 \in G(\prod_{k=2}^{l} \mathfrak{m}_k^{\nu_k})$ and $p\sum_{j\in\tau\setminus i} q_j + \deg g_2 \geq \sum_{k=l+1}^{h} \nu_k$. Therefore $\prod_{j\in\tau\setminus i} x_j^{pq_j} g_2 \in \mathfrak{m}_{l+1}^{\sum_{k=l+1}^{h} \nu_k} \subseteq \prod_{k=l+1}^{h} \mathfrak{m}_k^{\nu_k}$. □

Proof of Theorem 3.2. We will prove the theorem by induction on m. If $m = 0$, then $\max\{D_k\} = D_n$, and since $\langle u\rangle$ is a stable ideal, by [2] we have that $[f(\{1, \ldots, n-1\}; u_0)]$ is a basis element of $H_n(\mathbf{x}; R/\langle u\rangle)$.

Let $m > 0$. Let $\max\{r_j\} = r_t$ and $\max\{\tilde{r}_j\} = \tilde{r}_{s-1}$. Assume first $r_t = r_s$. Then by the induction hypothesis the element

$$z = \prod_{j\geq s+1} u_j^{p^{j-1}} f(\sigma, u_s)^{p^{s-1}}$$

is a cycle in $K_{l_s}(\mathbf{x}; R/I)$. From 1.5 we know that z^p is a cycle in $K_{l_s}(\mathbf{x}; R/I^{[p]})$. Consider the exact sequence

$$\cdots \to H_{l_s}(\mathbf{x}; R/\langle u\rangle) \to H_{l_s}(\mathbf{x}; R/I^{[p]}) \xrightarrow{\delta} \cdots.$$

We will show that $\delta[z^p] = 0$. We have

$$\partial z^p = \sum_{i\in\tau} (-1)^{\alpha(\tau,i)} \prod_{j\geq s+1} u_j^{p^j} (u_s' x_i)^{p^s} x_{\tau\setminus i}^{p^s-1} e_{\tau\setminus\{i\}}$$

where $\tau = \{1, \ldots, l_s\}$. By the induction hypothesis $x_{\tau\setminus i}^{p^{s-1}-1} \in \prod_{j=1}^{s-1} I_j^{p^{j-1}}$ for each $i \in \tau$. By 3.4(iv), $r_s - \deg u = (l_s - 1)(p^s - 1) - \sum_{k=2}^{n} \sum_{j=0}^{s-1} \mu_{kj} p^j \geq 0$. Now

Lemma 3.6 implies that $x_{\tau\setminus i}^{p^s-1} \in \prod_{j=0}^{s-1} I_j^{p^j}$ for each $i \in \tau$. Moreover, since the ideal I_s is stable, $u_s' x_i \in I_s$ for each $i \in \tau$. Hence $\delta[z^p] = 0$, and so z^p may be viewed as a cycle in $K_{l_s}(\mathbf{x}; R/\langle u \rangle)$.

Now let $r_t \neq r_s$. Then by 3.4(iii), $t < s$ and $l_t > l_s$, and by 3.4(ii) we have $n_j \leq l_t - 1$ for $t+1 \leq j \leq m$ and $n_t = l_t$. Therefore for the ideal $L = \prod_{j=t+1}^{m} I_j^{[p^{j-t-1}]}$ one has $G(L) \subset K[x_1, \dots, x_{l_t - 1}]$, so that $H_{l_t}(\mathbf{x}; R/L) = 0$, and we obtain the exact sequence

$$0 \to H_{l_t}(\mathbf{x}; L^{[p]}/I_t L^{[p]}) \to H_{l_t}(\mathbf{x}; R/I_t L^{[p]}) \to H_{l_t}(\mathbf{x}; R/L^{[p]}) = 0.$$

Consider the element

$$z = \prod_{j=t+1}^{m} u_j^{p^{j-t}} f(\sigma, u_t) \in K_{l_t}(\mathbf{x}; L^{[p]}/I_t L^{[p]})$$

with $\sigma = \{1, \dots, l_t - 1\}$, $u_j \in G(I_j)$ for $t \leq j \leq m$ and $\max(u_t) = l_t$. We have

$$\partial z = \sum_{i \in \tau} (-1)^{\alpha(\tau, i)} \prod_{j=t+1}^{m} u_j^{p^{j-t}} u_t' x_i x_{\tau\setminus i} e_{\tau\setminus\{i\}}$$

where $\tau = \{1, \dots, l_t\}$. Since I_t is stable, $u_t' x_i \in I_t$; therefore z is a cycle in $K_{l_t}(\mathbf{x}; R/I_t L^{[p]})$. We will show that z^{p^t} may be viewed as a cycle in $K_{l_t}(\mathbf{x}; R/\langle u \rangle)$. If $t = 0$, this is already proved, so assume $t > 0$.

By 1.5, z^p is a cycle in $K_{l_t}(\mathbf{x}; R/I_t^{[p]} L^{[p^2]})$. For $j = t-1, \dots, 0$, consider the exact sequences

$$\cdots \to H_{l_t}\left(\mathbf{x}; R/I_j \prod_{i=j+1}^{m} I_i^{[p^{i-j}]}\right) \to H_{l_t}\left(\mathbf{x}; R/\prod_{i=j+1}^{m} I_i^{[p^{i-j}]}\right) \xrightarrow{\delta} \cdots.$$

If $n_j \leq l_t$ for $0 \leq j \leq t-1$, then, by 3.6, $x_{\tau\setminus i}^{p-1} \in I_j$ for $i \in \tau$ and for $j = t-1, \dots, 0$. Therefore, as above, we obtain recursively that $\delta[z^{p^{t-j}}] = 0$ for $j = t-1, \dots, 0$, and we conclude that z^{p^t} is a cycle in $K_{l_t}(\mathbf{x}; R/\langle u \rangle)$.

Now assume $n_j \leq l_t$ for $q < j \leq t$ and $n_q > l_t$ for some q, $0 \leq q < t$. Then $s_{n_q} = q$ and $n_q = l_q$. Set $d_j = \sum_{k=2}^{n_j} \mu_{kj}$ for $0 \leq j \leq m$. Then since $l_q > l_t \geq n_j$ for $q < j \leq m$ we have $r_q = \sum_{j=q}^{m} d_j p^j + (l_q - 1)(p^q - 1)$ and $r_t = \sum_{j=t}^{m} d_j p^j + (l_t - 1)(p^t - 1)$. Therefore $r_t - r_q = (l_t - 1)(p^t - 1) - (l_q - 1)(p^q - 1) - \sum_{j=q}^{t-1} d_j p^j$ and since by assumption r_t is the maximum we obtain

$$\begin{aligned} (1) \qquad \sum_{j=q}^{t-1} d_j p^j &\leq (l_t - 1)(p^t - 1) - (l_q - 1)(p^q - 1) \\ &< (l_t - 1)(p^t - p^q) \\ &= p^q (l_t - 1)(p^{t-q} - 1). \end{aligned}$$

Hence

$$\sum_{j=q}^{t-1} d_j p^{j-q} < (l_t - 1)(p^{t-q} - 1).$$

From this inequality and from $d_j \le (l_t - 1)(p-1)$ for $q < j \le t-1$, by 3.6 it follows that $x_{\tau\setminus i}^{p^{t-q}-1} \in \prod_{j=q}^{t-1} I_j^{[p^{j-q}]}$ for $i \in \tau$. This implies that $z^{p^{t-q}}$ is a cycle in $K_{l_t}(\mathbf{x}; R/\prod_{j=q}^{m} I_j^{[p^{j-q}]})$. If $q = 0$, the theorem is proved , so let $q > 0$. Assume again that $n_j \le l_t$ for $q' < j < q$ and $n_{q'} > l_t$ for some $0 \le q' < q$. If $n_{q'} > l_q$, then $r_{q'}$ is defined and, as above, $z^{p^{t-q'}}$ is a cycle in $K_{l_t}(\mathbf{x}; R/\prod_{j=q'}^{m} I_j^{[p^{j-q'}]})$. It remains to consider the case $n_{q'} \le l_q$. Since $x_{\tau\setminus i}^{p^{t-q}-1} \in \prod_{j=q}^{t-1} I_j^{[p^{j-q}]}$ for $i \in \tau$ and $x_{\tau\setminus i}^{p-1} \in I_j$ for $q' < j < q$, we again have $x_{\tau\setminus i}^{p^{t-q'}-p} = x_{\tau\setminus i}^{(p^{t-q}-1)p^{q-q'}} x_{\tau\setminus i}^{p^{q-q'}-p} \in \prod_{j=q'+1}^{t-1} I_j^{[p^{j-q'}]}$ for $i \in \tau$. Moreover, using inequality (1), we get

$$\begin{aligned}
&d_{q'}p^{q'} + \sum_{j=q'+1}^{q-1} d_j p^j + \sum_{j=q}^{t-1} d_j p^j \\
&\le p^{q'}(p-1)(l_q - 1) + (l_t - 1)(p-1)\sum_{j=q'+1}^{q-1} p^j + (l_t - 1)(p^t - 1) - (l_q - 1)(p^q - 1) \\
&= -(l_q - 1)(p^q - p^{q'+1} + p^{q'} - 1) + (l_t - 1)(p^t - 1) + (l_t - 1)(p^q - p^{q'+1}) \\
&< (l_t - 1)(p^t - p^{q'});
\end{aligned}$$

therefore

$$\sum_{j=q'}^{t-1} d_j p^{j-q'} < (l_t - 1)(p^{t-q'} - 1).$$

Now applying 3.6 we have $x_{\tau\setminus i}^{p^{t-q'}-1} \in \prod_{j=q'}^{t-1} I_j^{[p^{j-q'}]}$. Therefore $z^{p^{t-q'}}$ is a cycle in $K_{l_t}(\mathbf{x}; R/\prod_{j=q'}^{m} I_j^{[p^{j-q'}]})$. Proceeding in this way we get the desired result. □

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