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ADDENDUM TO OUR PAPER "CONFORMAL MOTION OF CONTACT MANIFOLDS WITH CHARACTERISTIC VECTOR FIELD IN THE k-NULLITY DISTRIBUTION"

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In [4], Okumura proved that if a Sasakian manifold M of dimension > 3, admits a non-isometric conformal motion v, then v is special concircular and hence, if, in addition, M is complete and connected, then it is isometric to a unit sphere. The last part of this result follows from Obata's theorem [3]: A complete connected Riemannian manifold (M, g) of dimension > 1, admits a non-trivial solution ρ of partial differential equations $\nabla\nabla\rho = -c^2\rho g$ (for $c =$ a constant > 0), if and only if M is isometric to a Euclidean sphere of radius $1/c$. Recently, Sharma and Blair [5] extended Okumura's result to dimension 3 assuming constant scalar curvature and proved the following: Let v be a non-isometric conformal motion on a 3-dimensional Sasakian manifold. If the scalar curvature of M is constant, then M is of constant curvature and v is special concircular. Generalizing this result we prove:

THEOREM. *Let v be a non-isometric conformal motion on a 3-dimensional Sasakian manifold M such that v leaves the scalar curvature of M invariant. Then M is of constant curvature 1 and v is special concircular. Hence, if, in addition, M is complete and connected, then M is isometric to a unit sphere.*

COROLLARY. *Among all complete and simply connected 3-dimensional Sasakian manifolds only the unit 3-sphere admits a non-isometric conformal motion that leaves the scalar curvature invariant.*

For a $(2n+1)$-dimensional contact metric manifold $M(\eta, \xi, \phi, g)$ we know [1] that

$$\eta(\xi) = 1,\ (d\eta)(X, Y) = g(X, \phi Y),\ \eta(X) = g(X, \xi),\ \phi^2 = -I + \eta \otimes \xi, \tag{1}$$

$$\phi\xi = 0, \eta \circ \phi = 0,\ g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \text{rank } \phi = 2n. \tag{2}$$

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A contact metric manifold is said to be K-contact if ξ is Killing. For a K-contact manifold,

$$\nabla_X \xi = -\phi X. \tag{3}$$

$$Q\xi = 2n\xi. \tag{4}$$

A Sasakian (normal contact metric) manifold is a contact metric manifold satisfying either one of the following:

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y. \tag{5}$$

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X. \tag{6}$$

A Sasakian manifold is K-contact. A 3-dimensional contact manifold is Sasakian. The Ricci tensor of a 3-dimensional Sasakian manifold [2] is given by

$$S(X, Y) = \frac{1}{2}\{(r - 2)g(X, Y) + (6 - r)\eta(X)\eta(Y)\}, \tag{7}$$

where r denotes the scalar curvature.

A vector field ν on a Riemannian manifold (M, g) is a conformal motion if there is a smooth scalar function ρ on M such that

$$\pounds_\nu g = 2\rho g. \tag{8}$$

If ρ is constant, ν is homothetic, and for $\rho = 0$, ν is Killing. We say that a conformal motion is non-isometric if it is not Killing on any open neighborhood in M. A conformal motion ν defined by (8) satisfies the following (see [5]):

$$(\pounds_\nu S)(X, Y) = -(m - 2)(\nabla_X d\rho)Y + (\Delta\rho)g(X, Y), \tag{9}$$

$$\pounds_\nu r = -2\rho r + 2(m - 1)\Delta\rho, \tag{10}$$

where m is the dimension of M and $\Delta = -\operatorname{div}(D)$, D being the gradient operator. A conformal motion is called an infinitesimal special concircular transformation if the associated function ρ satisfies $\nabla\nabla\rho = (-c_1\rho + c_2)g$ for some constants c_1 and c_2.

In order to prove the theorem we need this result:

LEMMA. *A homothetic vector field on a K-contact manifold is Killing.*

Proof. As ν is homothetic ($\pounds_\nu g = cg$ for a constant c), $\pounds_\nu S = 0$. Writing equation (4) as $S(\xi, X) = 2n\, g(\xi, X)$ and Lie-differentiating it along ν we get

$$S([\nu, \xi], X) = 2ncg(\xi, X) + 2ng([\nu, \xi], X).$$

Substituting ξ for X and using (4) yields $c = 0$, proving the lemma.

Proof of the theorem. Since ξ is Killing, $\xi r = 0$ and hence $\pounds_\xi dr = d\pounds_\xi r = 0$ and $\pounds_\xi Dr = 0$. Thus

$$\nabla_\xi Dr = -\phi Dr \tag{11}$$

From (8) and the fact that ξ is unit it follows that

$$(\pounds_v \eta)\xi = -\eta(\pounds_v \xi) = \rho. \tag{12}$$

By hypothesis, $vr = 0$ and hence $\pounds_v dr = 0$. From (10) we also have

$$2\Delta\rho = r\rho. \tag{13}$$

Lie-differentiating (7) along v and using (9) and (13), we have

$$g(\nabla_X D\rho, Y) = \frac{1}{2}[(4-r)\rho g(X, Y)+(r-6)\{(\pounds_v\eta)(X)\eta(Y)+(\pounds_v\eta)(Y)\eta(X)\}]. \tag{14}$$

Substituting ξ for Y and using (12) we get

$$\frac{1}{2}(r-6)(\pounds_v\eta)X = \rho\eta(X) + g(\nabla_\xi D\rho, X). \tag{15}$$

The equation (15) transforms (14) into

$$\nabla_Y D\rho = \frac{1}{2}(4-r)\rho Y + \eta(Y)(2\rho\xi + \nabla_\xi D\rho) + g(\nabla_\xi D\rho, Y)\xi. \tag{16}$$

Substituting $Y = \xi$ in (16) and taking inner product with ξ, we have

$$g(\nabla_\xi D\rho, \xi) = \frac{1}{2}\rho(r-8). \tag{17}$$

Through (16) we compute $R(X, Y)D\rho = (\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})D\rho$ and contract it as $g(R(e_i, Y)D\rho, e_i)$ with respect to an orthonormal basis (e_i) and obtain

$$\begin{aligned} S(Y, D\rho) = (r-6)Y\rho + \rho Yr \ &+\ 3g(\nabla_\xi D\rho, \phi Y) + (2\xi\rho + \operatorname{div}\nabla_\xi D\rho)\eta(Y) \\ &-\ 2g(\nabla_Y\nabla_\xi D\rho, \xi) + g(\nabla_\xi\nabla_\xi D\rho, Y). \end{aligned} \tag{18}$$

Replacing Y by ϕY and using (7) gives

$$\begin{aligned} \frac{1}{2}(2-r)g(Y, \phi D\rho) \ =\ & (r-6)g(\phi Y, D\rho) + \rho g(\phi Y, Dr) - 3g(\nabla_\xi D\rho, Y) \\ & + 3\eta(Y)g(\nabla_\xi D\rho, \xi) \\ & - 2g(\nabla_{\phi Y}\nabla_\xi D\rho, \xi) + g(\nabla_\xi\nabla_{\phi Y} D\rho, \xi) \\ & - g(\nabla_\xi D\rho, \phi\nabla_\xi Y), \end{aligned} \tag{19}$$

where we used the equation

$$g(\nabla_\xi\nabla_\xi D\rho, \phi Y) = g(\nabla_\xi\nabla_{\phi Y} D\rho, \xi) - g(\nabla_\xi D\rho, \nabla_\xi\phi Y),$$

that can be obtained by differentiating the symmetry identity $g(\nabla_\xi D\rho, \phi Y) = g(\nabla_{\phi Y} D\rho, \xi)$ (this follows from Poincare lemma: $d^2 = 0$), along ξ. We now use (16) and (3) to rearrange the last three terms of (19) as

$$\begin{aligned}
&g(R(\xi, \phi Y)D\rho + \nabla_{[\xi,\phi Y]} D\rho, \xi) - g(\nabla_\xi D\rho, \nabla_\xi \phi Y) - g(\nabla_{\phi Y}\nabla_\xi D\rho, \xi)\\
&\quad = -g(Y, \phi D\rho) + g(\nabla_\xi D\rho, \phi^2 Y) - g(\nabla_{\phi Y}\nabla_\xi D\rho, \xi)(\text{using}(3.12))\\
&\quad = -g(Y, \phi D\rho) + g(\nabla_\xi D\rho, \phi^2 Y) - (\phi Y)g(\nabla_\xi D\rho, \xi) + g(\nabla_\xi D\rho, \nabla_{\phi Y}\xi)\\
&\quad = -g(\phi D\rho, Y) + \frac{1}{2}(8 - r)(\phi Y)\rho - \frac{1}{2}\rho(\phi Y)r,
\end{aligned}$$

Consequently, (19) reduces to

$$\frac{1}{6}\rho(\phi Y)r = g(\nabla_\xi D\rho, Y) + \frac{1}{2}\rho(8 - r)\eta(Y),$$

and therefore, we obtain

$$\nabla_\xi D\rho = -\frac{1}{6}\rho\phi Dr + \frac{1}{2}\rho(r - 8)\xi. \tag{20}$$

Next, differentiating (17) along Y gives

$$g(\nabla_Y \nabla_\xi D\rho, \xi) = g(\nabla_\xi D\rho, \phi Y) + \frac{1}{2}\{(r - 8)Y\rho + \rho Yr\}. \tag{21}$$

Further, the divergence term in (18) is

$$\operatorname{div}\left\{-(\rho/6)\phi Dr + \frac{1}{2}\rho(r - 8)\xi\right\} = \frac{1}{2}(r - 8)\xi\rho - (1/6)g(\phi Dr, D\rho),$$

because (e_i) can be taken as a ϕ-adapted base $(e, \phi e, \xi)$ and hence

$$-(\nabla\nabla r)(e_i, \phi e_i) = g(\phi\nabla_e Dr, e) + g(\phi\nabla_{\phi e} Dr, \phi e) = 0.$$

Thus (18) assumes the form

$$S(Y, D\rho) = 2Y\rho - \frac{1}{3}\rho Yr - \frac{1}{6}\xi\rho g(Y, \phi Dr) + \eta(Y)\left\{(r - 6)\xi\rho - \frac{1}{6}g(\phi Dr, D\rho)\right\}.$$

Use of (7) in the above equation gives

$$\begin{aligned}
\frac{1}{2}(r - 6)Y\rho + \frac{1}{3}\rho Yr \;=\; &\left\{\frac{3}{2}(r - 6)\xi\rho - \frac{1}{6}g(\phi Dr, D\rho)\right\}\eta(Y)\\
&- \frac{1}{6}\xi\rho g(\phi Dr, Y).
\end{aligned} \tag{22}$$

Substituting $Y = \xi$ gives

$$(r - 6)\xi\rho = \frac{1}{6}g(\phi Dr, D\rho). \tag{23}$$

If $r = 6$ on M, then (7) shows that M is Einstein, and being 3-dimensional, is of constant curvature 1. Now let $r \neq 6$ in some neighborhood $N(p)$ of a point p in M. Substituting $Y = \phi Dr$ in (22) and using (23) yields

$$(\xi\rho)(|Dr|^2 + 18(r-6)^2) = 0.$$

As $r \neq 6$, $\xi\rho = 0$, on $N(p)$. Differentiating it along ξ we have $g(\xi, \nabla_\xi D\rho) = 0$ and hence, from (17) we obtain $(r-8)\rho = 0$. But $\rho \neq 0$ in any open neighborhood, by hypothesis, and so, $r = 8$ on $N(p)$. Then (22) reduces to $Y\rho = 0$; i.e., $\rho =$ constant, and hence by Lemma 2, $\rho = 0$ on $N(p)$. This again contradicts our hypothesis. Hence M is of constant curvature 1, and as $r = 6$, (14) reduces to $\nabla\nabla\rho = -\rho g$; i.e., v is special concircular. The rest of the theorem follows from Obata's theorem.

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