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ON THE EMBEDDING OF A COMMUTATIVE RING IN A LOCAL RING

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ABSTRACT. Let R be a commutative ring with identity. We consider conditions in order that there exists an embedding of R in a local ring. This leads naturally to an examination of conditions in order that a quasilocal ring $(R, \mathbf{m})$ be dominated by a local ring. This, in turn, leads to a study of extensions of the residue field of a quasilocal ring.

1. Introduction

Let R be a commutative ring with identity. In this paper we consider conditions for the existence of an embedding of R in a local ring. This leads naturally to an examination of conditions for a quasilocal ring $(R, \mathbf{m})$ to be dominated by a local ring. This, in turn, leads to a study of extensions of the residue field of a quasilocal ring. We prove several results concerning domination of a quasilocal ring by a local ring including the result that a zero-dimensional quasilocal ring that is embeddable in a Noetherian ring is dominated by an Artinian local ring.

All rings considered in this paper are assumed to be commutative and unitary. If R is a subring of a ring S, we assume that the unity of S is contained in R, and hence is the unity of R.

If R is a ring with a unique maximal ideal $\mathbf{m}$, we say that R is a *quasilocal ring*. We frequently write $(R, \mathbf{m})$ to indicate that R is quasilocal with maximal ideal $\mathbf{m}$. If R is also Noetherian, then we say that R is a *local ring*. It is well known that a quasilocal ring has 0 and 1 as its unique idempotent elements. Thus a ring having a nontrivial idempotent element cannot be embedded in a quasilocal ring. We began this work by considering the following four questions:

(1) Under what conditions is a ring R a subring of a quasilocal ring?
(2) Under what conditions is a Noetherian ring R a subring of a quasilocal ring?
(3) Under what conditions is a Noetherian ring R a subring of a local ring?
(4) Under what conditions is a ring R a subring of a local ring?

In general, a ring R is a subring of a quasilocal ring if and only if the set of zero-divisors of R is contained in a prime ideal $P \in \operatorname{Spec} R$. Thus there exists an

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embedding of R into a quasilocal ring if and only if there exists $P \in \operatorname{Spec} R$ such that the canonical map of R into R_P is an injection. This gives what we regard as a satisfactory answer to (1).

Since the localization of a Noetherian ring at a prime ideal is a local ring, if a Noetherian ring R is a subring of a quasilocal ring, then R is a subring of a local ring. Thus (2) and (3) are equivalent and hold precisely if the zero-divisors of R are contained in some $P \in \operatorname{Spec} R$. This gives what we regard as a satisfactory answer to (2) and (3). Question (4) is more subtle. We note first:

Remark 1.1. It is possible for a ring to be a subring of a Noetherian ring and also a subring of a quasilocal ring and yet not be a subring of a local ring. In fact, it is possible for a quasilocal ring to be a subring of a Noetherian ring and yet not a subring of a local ring as we show in the following example.

Example 1.2. Let $(R, \mathbf{m})$ be a one-dimensional quasilocal reduced ring with a finite number $n > 1$ of minimal primes and with idempotent maximal ideal. Then R is not dominated[1] by a local ring, but the total quotient ring of R is a product of n fields. Hence R is a subring of a Noetherian ring. If $(S, \mathbf{n})$ is a local ring containing R, then since $\mathbf{m}$ is nonzero and idempotent, $\mathbf{n}$ must lie over a minimal prime of R. But this means there exists a minimal prime of R that is not contracted from S. Hence R is not a subring of S.

To obtain a specific one-dimensional quasilocal reduced ring $(R, \mathbf{m})$ with two minimal prime ideals and with idempotent maximal ideal, let x, y, z be indeterminates over a field F. Let a, b be rationally independent positive real numbers. Define a rank-one nondiscrete valuation domain W on the field $F(x, y)$ such that $F \subsetneq W$ by defining x to have W-value a and y to have W-value b. Then $W = F + Q$, where Q is the maximal ideal of W. Define $R = W[z]/(z(z-x))$. Then R is reduced with two minimal primes, the images of the prime ideals (z) and $(z-x)$ of the polynomial ring $W[z]$, while the image $\mathbf{m}$ of the maximal ideal (Q, z) of $W[z]$ is the unique maximal ideal of R. Moreover, the fact that Q is idempotent in W implies that $\mathbf{m}$ is idempotent in R.

Remark 1.3. Example 1.2 may also be modified to assume only that $(R, \mathbf{m})$ is not dominated by a local ring, rather than the stronger property that $\mathbf{m} = \mathbf{m}^2 \neq (0)$. It is shown in [GH4] that every local ring is dominated by a one-dimensional local ring.

Remark 1.4. In order that a ring R be a subring of a local ring, it is necessary that there exists $P \in \operatorname{Spec} R$ having the properties that (i) the canonical map of R to R_P is an injection, and (ii) the quasilocal ring R_P is dominated by a local ring.

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[1] We say that a quasilocal ring $(T, \mathbf{p})$ is *dominated by* a quasilocal ring $(S, \mathbf{n})$ if T is a subring of S and $\mathbf{p} \subseteq \mathbf{n}$.

Thus in considering Question (4) and conditions for a ring to be a subring of a local ring, we are led to the problem of determining conditions in order that a quasilocal ring be dominated by a local ring. This question has a rich history.

Discussion 1.5. Some necessary conditions for a quasilocal ring $(R, \mathbf{m})$ to be dominated by a local ring are:

(i) the powers of $\mathbf{m}$ intersect in (0);
(ii) the ideal (0) in R is a finite intersection of strongly primary ideals,[2] and hence R has only finitely many minimal prime ideals;
(iii) the universally contracted[3] ideals of R satisfy a.c.c.;
(iv) every universally contracted ideal of R is a finite intersection of strongly primary ideals.

It is natural to ask about other conditions that are necessary for a quasilocal ring $(R, \mathbf{m})$ to be dominated by a local ring. Jeanne Wald Kerr in [K] presents a construction which establishes existence of a zero-dimensional quasilocal ring $(R, \mathbf{m})$ with $\mathbf{m}^3 = (0)$ such that R satisfies a.c.c. on annihilator ideals but such that there is no bound on the lengths of chains of annihilator ideals of R, so R is not a subring of a Artinian ring and therefore also not a subring of a Noetherian ring. We consider a case of the Kerr construction in (1.6) and show that it yields a ring which also satisfies a.c.c. on universally contracted ideals.

The universally contracted ideals of a ring R are a subset of the set of weakly annihilated ideals of R, where an ideal I of R is *weakly annihilated* if it satisfies the following equivalent conditions [GM], [HL2].

(i) For every $a \in I$ and $b \notin I$, there exists $c \in R$ so that $ac = 0$ while $bc \neq 0$.
(ii) $\operatorname{Ann}(\operatorname{Ann}(a))$ is contained in I for each $a \in I$.
(iii) The ideal I is the union of annihilator ideals of R.

Example 1.6 (*Kerr*). Let $X = \{X_{ij} : i \in \mathbb{Z}^+, 1 \leq j \leq i\}$ be a countably infinite set of indeterminates over an infinite field K. Let M denote the maximal ideal of $K[X]$ generated by X, and let $I = M^3 + (\{X_{ij}X_{ik} : i \in \mathbb{Z}^+ \text{ and } j \neq k\})$. Then $R = K[X]/I$ is a form of the Kerr example. Let x_{ij} denote the image of X_{ij} in R and for $n \in \mathbb{Z}^+$, let $S_n = \{x_{nj}\}_{j=1}^n$. It is clear that $\mathbf{m} = M/I$ is the unique maximal ideal of R, and that R, $\mathbf{m}$, $\mathbf{m}^2$ and (0) are annihilator ideals of R. Kerr shows that the only other ideals of R that are annihilator ideals are the ideals of the form $(S) + \mathbf{m}^2$, where for some $n > 1$, S is a nonempty proper subset of S_n.

[2] An ideal is *strongly primary* if it is primary and contains a power of its radical.

[3] An ideal I of a ring S is said to be *universally contracted* if I is contracted from every extension ring of S.

We show:

(1.6.1) Each weakly annihilated ideal of R is an annihilator ideal. Hence universally contracted ideals of R are annihilator ideals, so R satisfies a.c.c. on universally contracted ideals. Indeed, each weakly annihilated ideal of R is of the form $\mathrm{Ann}(f)$ for some $f \in R$.

Proof. Let J be a nonzero proper weakly annihilated ideal of R. If a is a nonzero element of J, then $\mathrm{Ann}(a) \subseteq \mathbf{m}$ so $J \supseteq \mathrm{Ann}(\mathrm{Ann}(a)) \supseteq \mathrm{Ann}(\mathbf{m}) = \mathbf{m}^2$. Modulo $\mathbf{m}^2$, a is congruent to a unique K-linear combination $h_a = \sum a_{ij}x_{ij}$ of the x_{ij}, with only finitely many of the coefficients a_{ij} nonzero. By the *support* of a, denoted $\mathrm{supp}(a)$, we mean $\{x_{ij} : a_{ij} \neq 0\}$; if $a \notin \mathbf{m}^2$, then a and h_a have the same annihilator. Kerr shows that $\mathrm{Ann}(a) > \mathbf{m}^2$ only if $\mathrm{supp}(a)$ is a nonempty proper subset of S_n for some $n > 1$, and in this case $\mathrm{Ann}(a) = (S_n - \mathrm{supp}(a)) + \mathbf{m}^2$. If J contains an element a with $\mathrm{Ann}(a) = \mathbf{m}^2$, then, since J is weakly annihilated, $\mathbf{m} = \mathrm{Ann}(\mathbf{m}^2) \subseteq J$, and hence $J = \mathbf{m}$, an annihilator ideal. If J contains no such element a, then since $\mathrm{supp}(a + b) = \mathrm{supp}(a) \cup \mathrm{supp}(b)$ if $\mathrm{supp}(a)$ and $\mathrm{supp}(b)$ are disjoint, there exists an integer n such that $\mathrm{supp}(a) \subseteq S_n$ for each $a \in I - \mathbf{m}^2$. Let $Y = \cup\{\mathrm{supp}(a) : a \in I - \mathbf{m}^2\}$. Then Y is a nonempty subset of S_n. Since the field K is infinite, if $a, b \in I - \mathbf{m}^2$, there exists $k \in K$ such that $\mathrm{supp}(a + kb) = \mathrm{supp}(a) \cup \mathrm{supp}(b)$. It follows that $Y = \mathrm{supp}(f)$ for some $f \in I$, and since $\mathrm{Ann}(f) > \mathbf{m}^2$ by assumption, we conclude that Y is a proper subset of S_n. Since J is a weakly annihilated ideal, $\mathrm{Ann}(\mathrm{Ann}(f)) = \mathrm{Ann}((S_n - Y) + \mathbf{m}^2) = (Y) + \mathbf{m}^2 \subseteq J$. But since $\mathrm{supp}(a) \subseteq Y$ for each $a \in I - \mathbf{m}^2$, the reverse inclusion also holds. Therefore $J = (Y) + \mathbf{m}^2$ as we wished to show. □

Thus the four necessary conditions listed in (1.5) in order that a quasilocal ring $(R, \mathbf{m})$ be dominated by a local ring are not sufficient for this to occur, even in dimension zero. It seems natural to ask:

Question 1.7. Suppose $(R, \mathbf{m})$ is a zero-dimensional quasilocal ring having the property that there exists a positive integer n such that every chain of universally contracted ideals of R has length at most n. Does it follow that R is dominated by a local ring?

The condition of (1.7) implies, in particular, that R satisfies a.c.c. on annihilator ideals. Therefore $\mathrm{Ann}(\mathbf{m}) = \mathrm{Ann}(I)$ for some finitely generated ideal $I \subseteq \mathbf{m}$. It follows that $\mathrm{Ann}(\mathbf{m}^j) = \mathrm{Ann}(I^j)$ for each positive integer j. Hence some power of $\mathbf{m}$ is (0). Therefore the annihilators of distinct nonzero powers of $\mathbf{m}$ are distinct annihilator ideals. Hence if R satisfies the condition of (1.7), then $\mathbf{m}^n = (0)$. In particular, R satisfies the four necessary conditions of (1.5). We do not know the answer to (1.7) even in the special case where $(0) \subsetneq ((0) : \mathbf{m}) \subsetneq \mathbf{m} \subsetneq R$ are the only annihilator ideals of R.

(1.8) The question of domination of a quasilocal ring $(R, \mathbf{m})$ by a local ring subdivides into the case where R is an integral domain and the case where R fails to be an integral domain. In [AH1], a quasilocal domain that is dominated by a local ring is defined to be *sublocal.* It is easy to see (cf. [AH1, page 862]) that a quasilocal domain R is sublocal if and only if R is dominated by a rank-one discrete valuation domain (DVR) of its quotient field. It is noted in [AH1, Theorem 2.1] that a normal sublocal domain has behavior with respect to integral unramified extensions that fails in general for normal quasilocal domains. In general, the localization R_P of a sublocal domain $(R, \mathbf{m})$ at a prime ideal P may fail to be sublocal. In view of [GH1, (5.6)], Example 5.8 of [GH1] illustrates this statement.

(1.9) We recall that a ring R is called an *N-ring* if for each ideal I of R, there exists a Noetherian extension ring $S(I)$ of R such that I is contracted from $S(I)$; equivalently, R is an N-ring if, for each ideal I of R, the ring R/I is a subring of a Noetherian ring [GH3]. It is shown in [HL1, Theorem 2.3] that R is an N-ring if and only if for each ideal I of R, the annihilator ideals of R/I satisfy the a.c.c. Thus a sufficient condition for a ring R to be a subring of a Noetherian ring is that for every ideal I of R, the annihilator ideals of R/I satisfy a.c.c., and a ring R with this property is a subring of a local ring if and only if the zero-divisors of R are contained in some $P \in \mathrm{Spec}(R)$.

2. Extensions of the residue field

(2.1) In [EGA, $\mathrm{O_{III}}$ (10.3.1)], Grothendieck proves the following useful result. Suppose $(A, \mathbf{m})$ is a local ring with residue field k and K is an extension field of k. Then there exists a local extension ring B of A such that $\mathbf{m}B$ is the maximal ideal of B, B is flat as an A-module, and $B/\mathbf{m}B$ is k-isomorphic to K.

If the extension field K/k of (2.1) is not algebraic, one can reduce to the algebraic case by passing from A to $A(X)$, where X is a set of indeterminates over A in one-to-one correspondence with a transcendence basis for K/k, and where $A(X)$ denotes the localization of $A[X]$ at the prime ideal $\mathbf{m}A[X]$. In general, $A(X)$ is flat over A. Moreover, the passage from A to $A(X)$ preserves the Noetherian property and if A is Noetherian, then $\dim A = \dim A(X)$ [GH2].

In the case of an algebraic extension of residue fields, variations of the Grothendieck construction are of interest to us.

Construction 2.2. Suppose $(R, \mathbf{m})$ is a quasilocal ring with residue field $R/\mathbf{m} = k$, and $E = k(\{y_a\}_{a \in A})$ is an algebraic extension field of k. There exists a quasilocal extension ring S of R such that:

(1) S is an integral extension of R.
(2) S is a free R-module.
(3) $\mathbf{m}S$ is the maximal ideal of S.
(4) $S/\mathbf{m}S$ is isomorphic to E as an R-algebra. Moreover, there exists a free basis for S as an R-module that maps bijectively under the canonical map of S onto E to a basis for E over k.

Proof. Let $X = \{x_a\}_{a\in A}$ be a family of indeterminates over R and assume that A is well-ordered under a relation $<$. For $a \in A$, there is an R-algebra epimorphism ϕ_a: $R[\{x_b : b < a\}] \to k(\{y_b : b < a\})$ that first reduces coefficients modulo $\mathbf{m}$ and then substitutes y_b for x_b. This map ϕ_a extends to an R-algebra epimorphism ϕ_a^*: $R[\{x_b : b < a\}][x_a] \to k(\{y_b : b < a\})[x_a]$ mapping x_a to x_a. For each $a \in A$, we wish to choose a polynomial $f_a \in R[\{x_b : b < a\}][x_a]$ so that f_a is monic of positive degree in x_a and so that $\phi_a^*(f_a)$ is the minimal polynomial for y_a over $k(\{y_b : b < a\})$.

If a_1 is the first element of A, we let $f_{a_1}(x_{a_1})$ be any monic preimage in $R[x_{a_1}]$ of the minimal polynomial of a_1 over k. If polynomials f_b as described above have been chosen in $R[X]$ for all $b < a$, we consider the minimal polynomial $g_a(x_a) = x_a^m + u_{m-1}x_a^{m-1} + \cdots + u_1x_a + u_0$ for y_a over $k(\{y_b : b < a\})$; here $u_i \in k(\{y_b : b < a\})$ for each i. Hence $u_i = \phi_a(s_i)$ for some $s_i \in R[\{x_b : b < a\}]$. Let $f_a = x_a^m + s_{m-1}x_a^{m-1} + \cdots + s_1x_a + s_0$. By choice of the polynomials s_i, $\phi_a^*(f_a)$ is the minimal polynomial of y_a over $k(\{y_b : b < a\})$. By induction, this establishes the existence of $\{f_a\}_{a\in A}$.

For $a \in A$, let I_a be the ideal of $R[\{x_b : b \le a\}]$ generated by $\{f_b : b \le a\}$. We define $I = \cup_{a\in A} I_a$. Then $R[\{x_b : b \le a\}]/(\mathbf{m}, I_a) \cong k(\{y_b : b \le a\})$ and $R[X]/(\mathbf{m}, I) \cong E$. We prove that $I \cap R = (0)$. Since $I \cap R = \cup_{a\in A}(I_a \cap R)$, it suffices to show $I_a \cap R = (0)$ for each $a \in A$. If $r \in I_a \cap R$, then

$$r = g_1 f_{b_1} + g_2 f_{b_2} + \cdots + g_n f_{b_n},$$

where the $g_i \in R[\{x_b : b \le a\}]$ and $b_1 < b_2 \cdots < b_n \le a$. The g_i involve only finitely many of the x_b's, so by increasing n and taking zero as the coefficient of the corresponding f_{b_n}'s, it suffices to show

$$(f_{b_1}, f_{b_2}, \ldots, f_{b_n})R[x_{b_1}, \ldots, x_{b_n}] \cap R = (0).$$

This last statement is seen by induction by going modulo the principal ideal (f_{b_1}) to reduce to fewer variables. In more detail, one uses Lemma 2.3 stated below.

We conclude that R is embedded in $S := R[X]/I$. Moreover, if z_a denotes the image of x_a in S, then S is integral over R and free as an R-module with free basis the set of all monomials of the form $\prod_{a\in A} z_a^{i_a}$, where $0 \le i_a \le m_a - 1$, m_a is the degree of y_a over $k(\{y_b : b < a\})$, and only finitely many integers i_a are nonzero.

Since $R[X]/(\mathbf{m}, I)$ is isomorphic as an R-algebra to E, it follows that $\mathbf{m}S$ is the unique maximal ideal of S, so S is quasilocal with maximal ideal $\mathbf{m}S$. Moreover, the free basis $\{\prod_{a\in A} z_a^{i_a}\}$ maps bijectively onto $\{\prod_{a\in A} y_a^{i_a}\}$, which is a basis for E/k. □

LEMMA 2.3. *Suppose $x_1, \ldots, x_n$ are indeterminates over a ring T and for $1 \le i \le n$, $f_i \in T[x_1, \ldots, x_{i-1}][x_i]$ is monic in x_i of positive degree as a polynomial over $T[x_1, \ldots, x_{i-1}]$. If $J = (f_1, \ldots, f_n)$, then $J \cap T = (0)$.*

Proof. If $n = 1$, the assertion is clear. Assume that $n > 1$ and that the assertion holds for $n-1$. Let $U = T[x_1]/f_1T[x_1]$ and let $W = T[x_1, \dots, x_n]/f_1T[x_1, \dots, x_n]$. Then T canonically injects into U, W is a polynomial ring in $n - 1$ variables over U, and by our inductive hypothesis applied to the ring U, the image of J in W intersects U in (0). Therefore $J \cap T = (0)$. □

Discussion 2.4. (1) If the quasilocal ring $(R, \mathbf{m})$ of (2.2) is Artinian, then the quasilocal extension ring S produced by the construction is also Artinian since S is then 0-dimensional with finitely generated maximal ideal. Since for X a set of indeterminates over R, R Artinian implies $R(X)$ is Artinian, then even if E/k is not algebraic, the construction of (2.1) preserves the Artinian property.

(2) If $(R, \mathbf{m})$ is Noetherian, then the proof of Lemme 10.3.1.3 of [EGA, p. 21] shows that the constructed ring S is again Noetherian.

(3) If $(R, \mathbf{m})$ is a chained ring, [4] then the quasilocal extension ring S produced by the construction of (2.2) is such that each element of S is an associate of an element of R. Hence each principal ideal of S is the extension of its contraction to R, and it follows that the ideals of S are in one-to-one inclusion-preserving correspondence with the ideals of R with respect to the operations of contraction and extension. To see this assertion about principal ideals of R and S, we use the free basis $\{\prod_{a\in A} z_a^{i_a}\}$ for S as an R-module constructed in the proof of (2.2). Given $s \in S$, let $r_1, \dots, r_n$ denote the elements of R that are nonzero coefficients in the expression for s in terms of this free basis, say $s = r_1M_1 + \cdots + r_nM_n$, where the M_i are monomials in the free basis $\{\prod_{a\in A} z_a^{i_a}\}$. Since R is a chained ring, one of the r_i, say $r_1 := r$, generates $(r_1, \dots, r_n)R$. For $i > 1$ there exists $t_i \in R$ such that $r_i = t_ir$. Hence $s = r(M_1 + t_2M_2 + \cdots + t_nM_n)$. Since the images of the monomials M_i in the residue field of S are linearly independent over k, the element $t := M_1 + t_2M_2 + \cdots + t_nM_n$ has a nonzero image in the residue field of S. Hence t is a unit of S and $s = tr$. It follows that if P is the prime ideal of S consisting of the zero divisors of S, then $P = \mathbf{p}S$, where $\mathbf{p}$ is the prime ideal of zero divisors of R.

As an immediate consequence of (3), we have:

(4) If $(R, \mathbf{m})$ is a valuation domain, then the quasilocal extension ring S produced by the construction of (2.2) is also a valuation domain. Moreover the ideals of R and S are in one-to-one correspondence with respect to the inclusion map of R to S . In particular, if R is a DVR, then the ring S produced by the construction of (2.2) is also a DVR.

(5) If $(R, \mathbf{m})$ is an integral domain, then (2.2) can be used to establish the existence of a quasilocal domain T such that T is an integral extension of R, $\mathbf{m}T$ is the maximal ideal of T and $T/\mathbf{m}T$ is isomorphic as an R-algebra to E. Indeed, for S as given by (2.2), since S is a free R-module the going-down theorem holds for $R \to S$ [M1, Theorem 4, page 33], so each minimal prime $\mathbf{p}$ of S has the property that $\mathbf{p} \cap R = (0)$. Hence $T = S/\mathbf{p}$, for $\mathbf{p}$ a minimal prime of S, has the stated property.

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[4] A ring is a *chained ring* if the ideals of the ring are linearly ordered with respect to inclusion.

(6) Even if the extension field E/k is not algebraic, if $(R, \mathbf{m})$ is a local domain, then going modulo a minimal prime $\mathbf{p}$ of the local ring S given by (2.1) gives a local extension domain $S/\mathbf{p}$ of R such that $S/\mathbf{p}$ has residue field E.

The ring S constructed in (2.2) is determined by a set $\{f_a\}_{a \in A}$ of polynomials over R. In §5 we consider the way structure properties of S are affected by different choices of the set $\{f_a\}$.

In the case where $(R, \mathbf{m})$ is a quasilocal domain, we present an alternative approach to that of part (6) of (2.4) to show that, in the setting of (2.2), it is possible to construct a quasilocal integral extension domain S of R such that $S/\mathbf{m}S$ is isomorphic as an R-algebra to E. We first discuss the case of a simple extension.

Remark 2.5. Suppose $(R, \mathbf{m})$ is a quasilocal domain with residue field $R/\mathbf{m} = k$, and $E = k(\alpha)$ is a simple algebraic extension field of k. Let Ω be an algebraic closure of the quotient field K of R and let $f \in R[x]$ be a monic preimage of the minimal polynomial $\overline{f}$ of α over k. Let β be a root of f in Ω. The polynomial $f(x)$ is irreducible in $R[x]$ since its image in $(R/\mathbf{m})[x]$ is irreducible, but unless R is assumed to be integrally closed, we cannot expect f to be irreducible in $K[x]$. Let I denote the contraction to $R[x]$ of the K-algebra homomorphism of $K[x]$ onto $K[\beta]$ that maps x to β. Then $R[\beta]$ is integral over R, and $\mathbf{m}R[\beta]$ is maximal in $R[\beta]$ since $R[\beta]/\mathbf{m}R[\beta] \cong R[x]/(\mathbf{m}, I)$. Now $f(x) \in I$ and $R[x]/(\mathbf{m}, f(x)) \cong k[x]/(\overline{f}(x)) \cong E$, so $(\mathbf{m}[x], f)$ is maximal in $R[x]$, and hence $(\mathbf{m}[x], f(x)) = \mathbf{m}[x] + I$. We conclude that $R[\beta]/\mathbf{m}R[\beta] \cong E$, so $R[\beta]$ is a quasilocal domain with maximal ideal $\mathbf{m}R[\beta]$, residue field E, and $R[\beta]$ dominates R. Moreover, if R is local, then $R[\beta]$ is local.

RESULT 2.6. *Suppose $(R, \mathbf{m})$ is a quasilocal domain with residue field $R/\mathbf{m} = k$ and E/k is an algebraic field extension. Then there exists a quasilocal domain $(S, \mathbf{n})$ such that S is integral over R, $\mathbf{m}S = \mathbf{n}$, and $S/\mathbf{m}S$ is isomorphic as an R-algebra to E.*

Proof. Let Ω be an algebraic closure of the quotient field K of R and consider the set $\mathcal{S} = \{(R_a, \mathbf{m}_a, \phi_a)\}$, where $(R_a, \mathbf{m}_a)$ is a quasilocal integral extension of R, $R_a \subseteq \Omega$, $\mathbf{m}_a = \mathbf{m}R_a$, and ϕ_a: $R_a \to E$ is an R-algebra homomorphism with $\ker(\phi_a) = \mathbf{m}_a$. Define a relation $\preceq$ on $\mathcal{S}$ by $(R_a, \mathbf{m}_a, \phi_a) \preceq (R_b, \mathbf{m}_b, \phi_b)$ if R_b dominates R_a and ϕ_b restricts to ϕ_a on R_a. It is straightforward to show that $\preceq$ is a partial order on $\mathcal{S}$ and that $\mathcal{S}$ is inductive under $\preceq$. Let $(S, \mathbf{n}, \phi)$ be a maximal element of $\mathcal{S}$. Then ϕ is surjective, for if not, take an element $y \in E - \phi(S)$; y is algebraic over $k = R/\mathbf{m} \subseteq \phi(S)$, so y is algebraic over $\phi(S)$. Remark 2.5, with S playing the role of R and y the role of α, implies the existence of $\beta \in \Omega$ and ϕ^*: $S[\beta] \to \phi(S)[y]$ such that ϕ^* extends ϕ, $\phi^*(\beta) = y$ and $\ker(\phi^*) = \mathbf{n}S[\beta]$. This contradicts the maximality of ϕ. We conclude that ϕ is surjective. □

Result 2.6 yields an alternate proof to part of statement (4) of (2.4). We remark that Corollary 2.7 is known; see, for example, [M1, Thm. 83, p. 266].

COROLLARY 2.7. *Suppose* $(V, \mathbf{m})$ *is a DVR with residue field* $V/\mathbf{m} = k$ *and* E *is an extension field of* k. *There exists a DVR* $(W, \mathbf{n})$ *dominating* V *such that* $\mathbf{m}W = \mathbf{n}$ *and* $W/\mathbf{m}W$ *is isomorphic as a* V*-algebra to* E.

Proof. For any set X of indeterminates over V, the ring $V(X)$ is a DVR with maximal ideal $\mathbf{m}V(X)$ and residue field isomorphic as a V-algebra to E. Hence in proving (2.7) we may asume without loss of generality that E/k is algebraic. In that case Result 2.6 shows that there exists a quasilocal domain $(W, \mathbf{n})$ such that W is integral over V, $\mathbf{n} = \mathbf{m}W$, and $W/\mathbf{m}W \cong E$. Thus W is a DVR. □

If $(R, \mathbf{m})$ is a chained ring with residue field k and if X is a set of indeterminates over R, it is straightforward to see that each element of $R(X)$ is an associate of an element of R, and hence $R(X)$ is also a chained ring. Thus part (3) of (2.4) yields the following generalization of Corollary 2.7:

RESULT 2.8. *Suppose* $(R, \mathbf{m})$ *is a chained ring with residue field* k. *If* E *is an extension field of* k, *there exists a chained ring* $(W, \mathbf{n})$ *dominating* R *such that* W *has residue field* E *and each element of* W *is the associate of an element of* R.

3. Gluing of maximal ideals to obtain domination

(3.1) Doering and Lequain in [DL] introduce a "gluing process for maximal ideals" that is useful for showing certain rings are dominated by a local ring. Suppose $M_1, \ldots, M_k$ are maximal ideals of a ring T, F is a field, and $\phi_i \colon T \to F$ is a surjective ring homomorphism such that $\ker\phi_i = M_i$ for $1 \le i \le k$. Let

$$S = \{t \in T : \phi_1(t) = \phi_2(t) = \cdots = \phi_k(t)\}.$$

Then S is a subring of T containing $M = M_1 \cap \cdots \cap M_k$ as a maximal ideal, $S/M \cong F$, T is a finitely generated integral extension of S, and each of the maximal ideals M_i lies over M in S. We say that S is *a gluing of the maximal ideals* $M_1, \ldots, M_k$. If the ring T is Noetherian, then by Eakin's theorem [M2, page 18], S is Noetherian. Since S is a subring of T, each associated prime of (0) in S is the contraction to S of an associated prime of (0) in T. Moreover, if R is a subring of T such that ϕ_i and ϕ_j have the same restriction to R for every i and j, then R is a subring of S.

THEOREM 3.2. *If a zero-dimensional quasilocal ring* $(R, \mathbf{m})$ *is embeddable in a Noetherian ring, then* R *is dominated by a local Artinian ring.*

Proof. Since R is embeddable in a Noetherian ring, it is embeddable in an Artinian ring A [GH3, Prop. 2.6]. Let $M_1, \ldots, M_n$ be the maximal ideals of A. Each M_i lies over $\mathbf{m}$ in R, the canonical map of $R \to R/\mathbf{m}$ is the restriction to R of the canonical map $A \to A/M_i$, and $R/\mathbf{m}$ is a subfield of each A/M_i. Hence there exists a field E

that is an extension field of each A/M_i. We have $A \cong \prod_{i=1}^{k} A_{M_i}$ and A_{M_i} is local with residue field $A_{M_i}/M_iA_{M_i} \cong A/M_i$. By part (1) of (2.4), A_{M_i} is dominated by an Artinian local ring B_i with residue field E. Hence $B = \prod_{i=1}^{n} B_i$ is an Artinian extension ring of A with n maximal ideals and each residue field of B is R-isomorphic to E. Gluing the maximal ideals of B, we obtain an Artinian local ring C dominating R and having residue field E. □

THEOREM 3.3. *Suppose $(R, \mathbf{m})$ is a quasilocal ring. If there exists a finite family $\{I_j\}_{j=1}^{n}$ of ideals of R such that $(0) = \cap_{j=1}^{n} I_j$ and each R/I_j is dominated by a local ring, then R is dominated by a local ring.*

Proof. R is embedded in $\prod_{j=1}^{n}(R/I_j)$ and, as in the proof of (3.2), this latter ring is embedded in the direct product $S = \prod_{j=1}^{n} S_j$ of local rings S_j, where S_j dominates R/I_j and all the S_j have the same residue field. Gluing the maximal ideals of S, we obtain a local ring dominating R. □

PROPOSITION 3.4. *Let $(R, \mathbf{m})$ be a reduced quasilocal ring. Then R is dominated by a local ring if and only if R has only finitely many minimal prime ideals $P_1, \dots, P_n$ and R/P_i is dominated by a local domain for each i.*

Proof. If R is dominated by a local ring S, then (0) in R is a finite intersection of strongly primary ideals, and hence R has only finitely many minimal primes. If P is one of these minimal primes, then P is contracted from a minimal prime Q of S and R/P is dominated by the local domain S/Q.

Conversely, suppose R has only finitely many minimal prime ideals $P_1, \dots, P_n$ and that R/P_i is dominated by a local domain $(S_i, \mathbf{m}_i)$ for each i. If E is a common extension field of the fields $S_i/\mathbf{m}_i$, then part (6) of (2.4) implies the existence of a local extension domain $(T_i, \mathbf{n}_i)$ of S_i such that T_i has residue field E. Now R canonically embeds in $\prod_{i=1}^{n}(R/P_i)$ which canonically embeds in $\prod_{i=1}^{n} T_i$. Gluing the maximal ideals of T then yields a local subring of T that dominates R. □

THEOREM 3.5. *The quasilocal ring $(R, \mathbf{m})$ is dominated by a local ring if and only if there exist strongly primary ideals $Q_1, \dots, Q_n$ of R such that $(0) = \cap_{i=1}^{n} Q_i$ and R/Q_i is dominated by a local ring for each i.*

Proof. Suppose $(R, \mathbf{m})$ is dominated by the local ring $(S, \mathbf{n})$. The zero ideal of S is a finite intersection of strongly primary ideals $Q_1^*, \dots, Q_n^*$. Therefore the zero ideal of R is $\cap_{i=1}^{n}(Q_i^* \cap R)$, where each $Q_i^* \cap R$ is strongly primary in R and $R/(Q_i^* \cap R)$ is dominated by the local ring S/Q_i^*.

Conversely, if $(0) = \cap_{i=1}^{n} Q_i$ and R/Q_i is dominated by the local ring $(S_i, \mathbf{n}_i)$, then R embeds in $\prod_{i=1}^{n} S_i$, and $R/\mathbf{m}$ is a subfield of $S/\mathbf{n}_i$ for each i. Let E be a common extension field of each $S_i/\mathbf{n}_i$. By (2.1), S_i is dominated by a local ring T_i with residue field E. By gluing the maximal ideals of $\prod_{i=1}^{n} T_i$ we obtain a local subring T of $\prod_{i=1}^{n} T_i$ that dominates R. □

4. Some comments on a modification of the Kerr construction

In relation to Question 1.7, we show in this section that certain modifications of the Kerr construction described in (1.6) give zero-dimensional quasilocal rings that are dominated by an Artinian local ring.

(4.1) Let K be an infinite field of characteristic different from 2. Let X be a countably infinite set of indeterminates over K and let M denote the maximal ideal of $K[X]$ generated by X. Partition X into infinitely many nonempty subsets $S_1, S_2, \ldots$, and let I denote the ideal of $K[X]$ generated by M^3 and the set $\{uv : u, v \in S_i$ for some i and $u \neq v\}$. Let $R = K[X]/I$ and $\mathbf{m} = M/I$. It is clear that $\mathbf{m}^3 = (0)$, so R is quasilocal with maximal ideal $\mathbf{m}$. We can extend Kerr's proofs to this construction and show:

1. (1) $\mathbf{m} \supsetneq \mathbf{m}^2 \supsetneq \mathbf{m}^3 = (0)$.
2. (2) $\operatorname{Ann}(\mathbf{m}^2) = \mathbf{m} = \operatorname{Ann}(f)$ for each nonzero $f \in \mathbf{m}^2$. For $f \in \mathbf{m} - \mathbf{m}^2$, define the support of f, denoted $\operatorname{supp}(f)$, as in (1.6). Then $\operatorname{Ann}(f) \supsetneq \mathbf{m}^2$ if and only if $\operatorname{supp}(f)$ is a nonempty proper subset of some S_i, and in this case, $\operatorname{Ann}(f) = \mathbf{m}^2 + (S_i - \operatorname{supp}(f))$.
3. (3) Other than $(0), \mathbf{m}^2, \mathbf{m}$, and R, the annihilator ideals of R are of the form $(U, \mathbf{m}^2)$, where U is a nonempty proper subset of some S_i.
4. (4) R satisfies a.c.c. on annihilator ideals if and only if each S_i is finite; the length of every strictly ascending chain of annihilator ideals of R is bounded if and only if $\{|S_i|\}_{i=1}^{\infty}$ is bounded. If $n = \max\{|S_i|\}_{i=1}^{\infty}$, then $n+2$ is the maximum length of a strictly ascending chain of annihilator ideals of R (this is counting R as an annihilator ideal).
5. (5) In the case where each S_i is finite, each universally contracted ideal of R is of the form $\operatorname{Ann}(f)$ for some $f \in R$.

We use these properties to establish:

THEOREM 4.2. *With notation as in* (4.1), *assume that the positive integer n is the least upper bound of* $\{|S_i|\}_{i=1}^{\infty}$. *Then R is dominated by an Artinian local ring.*

Proof. In order to simplify notation, we first make the following reduction. Choose s so that $n \leq 2^s$, and expand each set S_i to a set T_i of 2^s indeterminates over K. Let $T = \cup_{i=1}^{\infty} T_i$. Let N denote the ideal of $K[T]$ generated by T and define J to be the ideal of $K[T]$ generated by N^3 and $\{uv : u, v \in T_i$ for some i and $u \neq v\}$. Then $K[T]/J$ is a form of the modified Kerr construction in which each set of the partition contains exactly 2^s elements. We show that $J \cap K[X] = I$, so that R is dominated by $K[T]/J$. For a proof observe that as a vector space over K, the set of pure monomials in T forms a basis H for $K[T]$. Now J is a K-subspace of $K[T]$ with basis $H_1 \subseteq H$, where $H_1 = \{uv : u, v \in T_i$ for some i and $u \neq v\} \cup \{Q :$ Q is a pure monomial in T of degree $\geq 3\}$. Moreover, $K[X]$ is a subspace of $K[T]$

with basis $H_2 \subseteq H$, where H_2 is the set of pure monomials in X. Therefore $J \cap K[X]$ is the K-subspace of $K[T]$ with basis $H_1 \cap H_2 = \{uv : u, v \in S_i$ for some i and $u \neq v\} \cup \{Q : Q$ is a pure monomial in X of degree $\geq 3\}$. Thus $J \cap K[X] = I$, as we wished to show.

In view of this reduction, we may assume that $|S_i| = 2^s = k$ for each i and that $S_i = \{X_{ij}\}_{j=1}^{k}$. We prove that for a suitable set L of indeterminates over K, there exists an embedding of the ring R into the ring

$$S = K(L)[T_1, U_1, \ldots, T_s, U_s]/(T_1^2, U_1^2, \ldots, T_s^2, U_s^2),$$

where $T_1, U_1, \ldots, T_s, U_s$ are indeterminates over the field $K(L)$. It is clear that S is a local Artinian ring. Let t_i and u_i denote the images in S of T_i and U_i, respectively. We take the set L of indeterminates over K to be the union of three sets Y, A, B, where $Y = \{Y_{ij} : 1 \leq i \leq \infty,\ 1 \leq j \leq k\}$, $A = \{A_{ij} : 1 \leq i \leq \infty,\ 1 \leq j \leq k\}$, and $B = \{B_{ij} : 1 \leq i \leq \infty,\ 1 \leq j \leq k\}$. Notice that each of the sets Y, A, B is in a natural one-to-one correspondence with the set X.

There are k binary sequences of length s; we denote them by $g_1, \ldots, g_k$, where $g_j = (g_{j1}, \ldots, g_{js})$ and each g_{ji} is either 0 or 1. Define ϕ: $K[X] \to S$ to be the K-homomorphism such that $\phi(X_{ij}) = \theta_{ij} = Y_{ij}\alpha_{ij}$, where

$$\alpha_{ij} = \prod_{h=1}^{s}(A_{ih}t_h + (-1)^{g_{jh}} B_{ih}u_h).$$

We claim that ϕ has kernel I. To see that I is contained in $\ker(\phi)$, it suffices to show that $\theta_{ia}\theta_{ib} = 0$ for $a \neq b$ and that the product of any three of the elements θ_{ij} is 0. For $a \neq b$ the sequences g_a and g_b are distinct, so $g_{am} \neq g_{bm}$ for some m. If, say, $g_{am} = 0$ and $g_{bm} = 1$, then θ_{ia} is a multiple of $A_{im}t_m + B_{im}u_m$ and θ_{ib} is a multiple of $A_{im}t_m - B_{im}u_m$, so $\theta_{ia}\theta_{ib} = 0$ since $t_m^2 = u_m^2 = 0$. We note that each θ_{ij} has a natural preimage in $K(L)[T_1, \ldots, U_s]$ that is homogeneous of degree s, so the product of three of the elements θ_{ij} has a preimage that is homogeneous of degree $3s$. Any such element of $K(L)[T_1, \ldots, U_s]$ belongs to the ideal $(T_1^2, U_1^2, \ldots, T_s^2, U_s^2)$, and hence the product of any three of the elements θ_{ij} is 0. We conclude that $I \subseteq \ker(\phi)$.

Let $f \in \ker(\phi)$. Modulo I, f is congruent to a K-linear combination of monomials X_{ij} and monomials $X_{ab}X_{cd}$, where either $a \neq c$, or $(a, b) = (c, d)$. To show that $f \in I$, it therefore suffices to show that $W = \{\theta_{ij}\} \cup \{\theta_{ab}\theta_{cd} : a = c$ implies $b = d\}$ is linearly independent over K.

Assume β is a K-linear combination of elements of W that is equal to 0. The set $\{Y_{ij}\}$ is algebraically independent over the subring $S_0 = K[\{A_{ij}, \{B_{ij}, \{t_i\}, \{u_i\}]$ of S. Now $\theta_{ij} = \alpha_{ij}Y_{ij}$ is an S_0-multiple of Y_{ij}, and $\theta_{ab}\theta_{cd}$ is an S_0-multiple of $Y_{ab}Y_{cd}$. Hence if $x\theta_{ij}$ and $y\theta_{ab}\theta_{cd}$, where $x, y \in K$, are terms occuring in this expression for β then $x\alpha_{ij} = y\alpha_{ab}\alpha_{cd} = 0$. Since $\alpha_{ij}^2 = \pm 2\prod_{h=1}^{s} A_{ih}B_{ih}t_hu_h \neq 0$, it follows that $\alpha_{ij} \neq 0$. Since nonzero elements of K are units of S, we conclude that $x = 0$.

On the other hand,

$$\begin{aligned}\alpha_{ab}\alpha_{cd} &= \prod_{h=1}^{s}(A_{ah}t_h + (-1)^{g_{bh}}B_{ah}u_h)(A_{ch}t_h + (-1)^{g_{dh}}B_{ch}u_h) \\ &= \left\{\prod_{h=1}^{s}((-1)^{g_{dh}}A_{ah}B_{ch} + (-1)^{g_{bh}}A_{ch}B_{ah})\right\} t_1u_1t_2u_2\cdots t_su_s.\end{aligned}$$

Since $y\alpha_{ab}\alpha_{cd} = 0$, it then follows that

$$y\left\{\prod_{h=1}^{s}((-1)^{g_{dh}}A_{ah}B_{ch} + (-1)^{g_{bh}}A_{ch}B_{ah})\right\},$$

which is an element of $K(L) \subseteq S$, is zero. To show that $y = 0$ it therefore suffices to show that $(-1)^{g_{dh}}A_{ah}B_{ch}+(-1)^{g_{bh}}A_{ch}B_{ah} \neq 0$ for each h. If $a \neq c$ this is clear, and if $a = c$, then $b = d$ and the element in question is $(-1)^{g_{bh}}2A_{ah}B_{ah} \neq 0$. We conclude that $y = 0$, so W is linearly independent over K and $\ker(\phi) = I$. Consequently, ϕ induces an embedding of R into S. □

Remark 4.3. The Artinian local ring S in the proof of (4.2) is also the quotient of a polynomial ring by a regular sequence and hence is a complete intersection. In particular, S is Gorenstein. Therefore each ring R as in (4.1) is dominated by a local Artinian Gorenstein ring.

5. On the structure of quasilocal extensions with a prescribed residue field extension

Discussion 5.1. Suppose $(R, \mathbf{m})$ is a quasilocal ring with residue field $R/\mathbf{m} = k$ and E is an algebraic extension field of k. Even in the case where E/k is a simple algebraic extension, the extension ring S of R provided by the construction of (2.2) and possessing the four properties listed in (2.2) is usually far from unique. However, in certain cases there are common properties possessed by every extension S constructed by means of (2.2). Some of these properties are noted in (2.4). We also have:

(1) If R is a regular local domain of dimension d, then in view of part (2) of (2.4), the extension ring S is again a regular local domain of dimension d. In particular, if R is a DVR, then S is a DVR (cf. Corollary 2.7).
(2) If R is a normal quasilocal integral domain and E/k is separable, then S is a normal quasilocal integral domain; see, for example, [AH2, (4.12), page 760].
(3) If R is a normal quasilocal integral domain and E/k is a simple extension, then S is a quasilocal integral domain but may fail to be normal (part (1) of Example 5.2). That S is a domain in this case follows from the fact that a monic polynomial irreducible over an integrally closed domain R is also irreducible

over the quotient field of R [ZS1, Thm. 4, page 260], and therefore generates a prime ideal of the polynomial ring $R[x]$.

(4) There exists a normal local integral domain $(R, \mathbf{m})$ and a purely inseparable extension $E = k(a, b)$ of k such that there exists an extension S of R as in (2.2) where S is not reduced (part (2) of Example 5.2).

(5) There exists a local integral domain $(R', \mathbf{m}')$ and a simple extension $E = k'(b)$ of the residue field k' of R' such that there are monic preimages f_1, f_2, f_3 in $R'[x]$ of the minimal polynomial of b over k', where

 (i) $S_1 = R'[x]/(f_1)$ is an integral domain,

 (ii) $S_2 = R'[x]/(f_2)$ is reduced but not an integral domain, and

 (iii) $S_3 = R'[x]/(f_3)$ is not reduced.

(See part (3) of Example 5.2.)

(6) If a quasilocal integral domain $(R, \mathbf{m})$ admits a lifting S as in (2.2) such that S is a normal integral domain, then R is normal. This follows because S meets the quotient field of R in R since S is a free R-module.

(7) There exists a quasilocal integral domain R and a simple extension E/k such that every lifting S as in (2.2) fails to be an integral domain (Example 5.3).

Example 5.2. Let $\mathbb{F}_2$ denote the prime field of characteristic 2, let a, b be indeterminates over $\mathbb{F}_2$, and let $k = \mathbb{F}_2(a^2, b^2)$. Let $D = k[[y, z]]$ be a formal power series ring over k in the variables y and z, and let F denote the quotient field of D.

(1) Let $R = D[ay + bz]$. We have $D \subsetneq R \subsetneq D[a, b] = k(a, b)[[y, z]]$. The set $\{1, a, b, ab\}$ is a module generating set for $D[a, b]$ over D and a vector space basis for the quotient field of $D[a, b]$ over F, while $\{1, ay + bz\}$ is a vector space basis for the field $F[ay + bz]$ over F. Using these bases, one sees that $R = D[ay + bz] = F[ay+bz] \cap D[a, b]$ by the following argument. If $u+v(ay+bz) = q+ra+sb+tab$, where $u, v \in F$ and $q, r, s, t \in D$, then $u = q \in D$, $vy = r$, $vz = s$, and $t = 0$. Thus $v = r/y = s/z$, and since y and z are relatively prime elements of the UFD D, it follows that $v \in D$ and $u + v(ay + bz) \in R$. Therefore R is a normal local domain with maximal ideal $\mathbf{m} = (y, z, ay + bz)$ and with residue field k. With $E = k(a) = k[x]/(x^2-a^2)$, if we take $x^2-a^2 \in R[x]$ as a preimage of $x^2-a^2 \in k[x]$, then $S = R[a] = R[x]/(x^2 - a^2)$ is an integral domain but is not integrally closed. That $R[a]$ is an integral domain follows from the fact that R is integrally closed. To see that $R[a]$ is not integrally closed, observe that $R[a] = D[a, bz]$ is properly contained in $D[a, b]$ and $R[a]$ and $D[a, b]$ have the same quotient field, so $D[a, b]$ is the integral closure of $R[a]$.

(2) Let $R = D[ay + bz]$, as in part (1), and take $E = k(a, b)$ as an extension field of the residue field k of R. If we take $x^2 - a^2 \in R[x]$ as the preimage of the minimal polynomial of a over k and then take $x^2-b^2 \in R[a][x]$ as the preimage of the minimal polynomial of b over the residue field $k(a)$ of $R[a]$, then $S = R[a][x]/(x^2 - b^2)$. Since b is in the quotient field of $R[a]$, the polynomial $x^2 - b^2$ factors as $(x - b)^2$

over the quotient field of $R[a] = D[a, bz]$. Therefore S has a localization that is not reduced, so S is not reduced.

(3) Let $R' = R[a] = D[a, bz]$. Then $k' = k(a)$ is the residue field of R'. Let $g(x) = x^2 - b^2 \in k'[x]$. Then:

(i) $f_1 = x^2 - b^2 - y \in R'[x]$ is a preimage of g that is irreducible over the quotient field of R'. Hence $S_1 = R'[x]/(f_1)$ is an integral domain.
(ii) $f_2 = (x - b)(x - b + bz) \in R'[x]$ is a preimage of g that factors over the quotient field of R' as a product of two distinct linear polynomials. Hence $S_2 = R'[x]/(f_2)$ is reduced but not an integral domain.
(iii) $f_3 = x^2 - b^2 \in R'[x]$ is a preimage of g that factors over the quotient field of R' as $(x - b)^2$. Hence $S_3 = R'[x]/(f_3)$ is not reduced.

Example 5.3. Let $E = k(c)$ be a separable algebraic extension field of k of degree $n \geq 2$. Let $V = E[[y]]$ be the formal power series ring in y over E and let $R = k + yV$. Then R is a one-dimensional complete local domain with integral closure $R[c] = V$. Let $g(x) \in k[x]$ be the minimal polynomial for c over k. Suppose $f(x) \in R[x]$ is a monic preimage of $g(x)$. To show that $R[x]/(f(x))$ is not an integral domain, it suffices to show that $f(x)$ is reducible over the quotient field of R. Since the image $g(x)$ of $f(x)$ in $(V/yV)[x] = E[x]$ is a separable polynomial of degree $n \geq 2$ that has a root in E, it follows from Hensel's Lemma that $f(x)$ is reducible in $V[x]$.

In (5.4), we give an example which shows that for an infinite algebraic extension E/k of the residue field k of a quasilocal domain $(R, \mathbf{m})$, there may exist extensions S_1 and S_2 of R with residue field E satisfying the conditions of (2.2), where S_1 is a quasilocal domain while S_2 has infinitely many minimal prime ideals.

Example 5.4. Let $p_1, p_2, \ldots$ be the sequence of positive prime integers. Let $E = \mathbb{Q}((p_1)^{1/2}, (p_2)^{1/2}, \ldots)$, let z be an indeterminate over E, and let $(V, \mathbf{m})$ denote the DVR $E[z]_{(z)}$. Let R be the one-dimensional quasilocal domain $\mathbb{Q} + \mathbf{m}$. Then $k = \mathbb{Q}$ is the residue field of R. In the notation of (2.2), we take A to be $\mathbb{Z}^+$ and y_i to be $(p_i)^{1/2}$. We show that $\{f_i = x_i^2 - p_i - z\}_{i=1}^{\infty}$ is one acceptable choice of monic polynomials for (2.2), and the resulting ring $S_1 = R[\{x_i\}_{i=1}^{\infty}]/(\{f_i\}_{i=1}^{\infty})$ is a quasilocal domain. On the other hand, we show that $\{g_i = x_i^2 - p_i\}_{i=1}^{\infty}$ is another acceptable choice, and the quasilocal ring $S_2 = R[\{x_i\}_{i=1}^{\infty}]/(\{g_i\}_{i=1}^{\infty})$ is reduced and has infinitely many minimal primes.

Proof. Let $s_i = z + p_i$ and let $t_i = (s_i)^{1/2}$ for each $i \in \mathbb{Z}^+$. For a finite subset W of $\mathbb{Z}^+$, the set of maximal ideals of $E[z]$ that ramify in the extension field $E(z)(\{t_i : i \in W\})$ of $E(z)$ is precisely the set $\{(s_i E[z] : i \in W\}$ (cf. [N, (10.18)]). It follows that the polynomial $f_n = x_n^2 - s_n$ is irreducible over the field $E(z)(\{t_i\}_{i=1}^{n-1})$ for each $n \in \mathbb{Z}^+$. For a direct argument for this assertion, see, for example, the proof given in [R]. This establishes that S_1 is a quasilocal domain.

To prove the statement concerning $S_2 = R[\{x_i\}_{i=1}^{\infty}]/(\{g_i\}_{i=1}^{\infty})$, it suffices to show that for each positive integer n, $R_n = R[\{x_i\}_{i=1}^{n}]/(\{g_i\}_{i=1}^{n})$ is reduced with 2^n minimal prime ideals. Since g_1 factors over the integral closure $V = E + \mathbf{m}$ of R as $(x_1 - y_1)(x_1 + y_1)$, a product of two distinct linear polynomials, it follows that $R_1 = R[x_1]/(g_1)$ is reduced with two minimal prime ideals each of which has associated residue class ring $R[y_1] = \mathbb{Q}(y_1) + \mathbf{m}$. Assume that R_n is reduced with 2^n minimal primes each of which has associated residue class ring isomorphic to $R[\{y_i\}_{i=1}^{n}] = \mathbb{Q}(\{y_i\}_{i=1}^{n}) + \mathbf{m}$. Since R_{n+1} is finite and free as an R_n-module, for a minimal prime P of R_n, the primes of R_{n+1} lying over P in R_n are the minimal primes of PR_{n+1} and are minimal primes of the ring R_{n+1}. Since $R_{n+1} = R_n[x_{n+1}]/(g_{n+1})$, it follows that

$$R_{n+1}/PR_{n+1} \cong (R_n/P)[x_{n+1}]/(\overline{g_{n+1}}) \cong [\mathbb{Q}(\{y_i\}_{i=1}^{n}) + \mathbf{m}][x_{n+1}]/(\overline{g_{n+1}}).$$

Therefore g_{n+1} is the product of the distinct linear polynomials $x_{n+1} - y_{n+1}$ and $x_{n+1} + y_{n+1}$ over the quotient field of R_n/P. Hence the zero ideal of R_{n+1}/PR_{n+1} is an intersection of two minimal prime ideals , each having associated residue class ring $R[\{y_i\}_{i=1}^{n+1}] = \mathbb{Q}(\{y_i\}_{i=1}^{n+1}) + \mathbf{m}$. We conclude that R_{n+1} is reduced with 2^{n+1} minimal prime ideals. This completes the proof. □

6. Embeddings into a ring with n maximal ideals

In analogy with the four questions mentioned in the introduction, we have also considered, for a positive integer n, the following four questions:

(1) Under what conditions is a ring R a subring of a ring with n maximal ideals?
(2) Under what conditions is a Noetherian ring R a subring of a ring with n maximal ideals?
(3) Under what conditions is a Noetherian ring R a subring of a Noetherian ring with n maximal ideals?
(4) Under what conditions is a ring R a subring of a Noetherian ring with n maximal ideals?

In general, a ring R is a subring of a ring with n maximal ideals if and only if the set of zero-divisors of R is contained in the union of at most n prime ideals $P \in \operatorname{Spec} R$. Thus there exists an embedding of R into a ring with at most n maximal ideals if and only if there exists a multiplicative system N of R such that $R - N$ is the union of at most n prime ideals of R and such that the canonical map of R into R_N is an injection. This gives what we regard as a satisfactory answer to Question (1).

Since the localization of a Noetherian ring at a multiplicative system is again a Noetherian ring, if a Noetherian ring R is a subring of a ring with n maximal ideals, then R is a subring of a Noetherian ring with n maximal ideals. Thus (2) and (3) are equivalent and hold precisely if the zero-divisors of R are contained in the union of at most n prime ideals $P \in \operatorname{Spec} R$. This gives what we regard as a satisfactory answer to (2) and (3). As in the local case, Question (4) is more subtle.

In analogy with Theorem 3.2, we have:

THEOREM 6.1. *If a zero-dimensional ring R with n maximal ideals is embeddable in a Noetherian ring, then R is a subring of an Artinian ring with n maximal ideals.*

Proof. Since R is zero-dimensional and has n maximal ideals, R is the direct product of n zero-dimensional quasilocal rings R_i. For $1 \leq i \leq n$, let $e_i \in R_i$ be the (idempotent) i-th component of 1 in this decomposition and let S be a Noetherian extension ring of R. Then $R_i = Re_i$ is a subring of the Noetherian ring $S_i = Se_i$. By Theorem 3.2, each R_i is dominated by an Artinian local ring C_i. Hence $R = \prod_{i=1}^n R_i$ is a subring of the Artinian ring $C = \prod_{i=1}^n C_i$ which has n maximal ideals. □

In analogy with Remark 1.1, we have:

Remark 6.2. Let n be a positive integer. It is possible for a quasilocal ring to be a subring of a Noetherian ring and yet not be a subring of a Noetherian ring having fewer than $n + 1$ maximal ideals as we show in the following example.

Example 6.3. Let $(R, \mathbf{m})$ be a one-dimensional quasilocal reduced ring with $n+1$ minimal primes and with idempotent maximal ideal. The total quotient ring of R is a product of $n + 1$ fields, a Noetherian ring with $n + 1$ maximal ideals. Suppose S is any Noetherian extension ring of R. We show that S has at least $n + 1$ maximal ideals. Since $\mathbf{m} = \mathbf{m}^2$ and since $\mathbf{m}S$ is finitely generated, it follows that $\mathbf{m}S = eS$, where $e = e^2$ is an idempotent element of S. Because eS is a homomorphic image of S, it suffices to show that the ring eS has at least $n + 1$ maximal ideals. Since $\mathbf{m}$ has annihilator (0) in R, no nonzero element of R annihilates the element e. Therefore the map $r \longrightarrow er$ is an isomorphism of R onto eR, so without loss of generality we assume that $e = 1$—that is, $\mathbf{m}S = S$. Choose $a \in \mathbf{m}$ not in the union of the $n + 1$ minimal primes of R. Then aR is $\mathbf{m}$-primary, so $S = \mathbf{m}S$ is contained in the radical of aS, and hence a is a unit of S. We conclude that the total quotient ring of R is isomorphic to a subring of S. Therefore S itself is a product of $n + 1$ nonzero ideals, so S has at least $n + 1$ maximal ideals.

To obtain a specific one-dimensional quasilocal reduced ring $(R, \mathbf{m})$ with $n + 1$ minimal prime ideals and with idempotent maximal ideal, let x, y, z be indeterminates over a field F, and define a rank-one nondiscrete valuation domain $W = F + Q$ on the field $F(x, y)$ as in (1.1). Define $R = W[z]/(z(z-x)(z-x^2)\cdots(z-x^n))$. Then R is reduced with $n + 1$ minimal primes the images of the prime ideals $(z), (z - x), \ldots, (z - x^n)$ of the polynomial ring $W[z]$, while the image $\mathbf{m}$ of the maximal ideal (Q, z) of $W[z]$ is the unique maximal ideal of R. Moreover, the fact that Q is idempotent in W implies that $\mathbf{m}$ is idempotent in R.

Remark 6.4. Using the fact that a ring R is a subring of a ring with n maximal ideals if and only if the set of zero-divisors of R is contained in the union of at most n prime ideals $P \in \operatorname{Spec} R$, it is easy to give an example of a one-dimensional Noetherian ring R such that $\operatorname{Spec} R$ is connected and R is not a subring of a ring with fewer than n maximal ideals. For example, let $p_1, \dots, p_n$ be distinct primes in the ring of integers $\mathbb{Z}$, let x be an indeterminate over $\mathbb{Z}$, and define $R = \mathbb{Z}[x]/(p_1 \cdots p_n x, x^2)$. Then R has the stated property.

In a Noetherian ring R the set of zero divisors is a finite union of prime ideals $P \in \operatorname{Spec} R$. Hence a Noetherian ring is a subring of a ring with finitely many maximal ideals. There exists, however, a non-Noetherian ring R having the property that $\operatorname{Spec} R$ is connected and every extension ring of R has infinitely many maximal ideals. For example, if R is the ring of continuous real-valued functions on the unit interval, then $\operatorname{Spec} R$ is connected and the set of zero-divisors of R is not contained in a finite union of prime ideals $P \in \operatorname{Spec} R$.

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