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A CHAIN OF CONTROLLABLE PARTITIONS OF UNITY ON THE CUBE AND THE APPROXIMATION OF HÖLDER CONTINUOUS FUNCTIONS

CHRISTIAN RICHTER

ABSTRACT. Controllable partitions of unity in $C(X)$ are partitions of unity whose supports fulfil a uniformity condition depending on the entropy numbers of the compact metric space X. We construct a chain of such partitions in $C([0,2]^m)$ such that the span of any partition is a proper subspace of the span of the following one. This chain gives rise to approximation quantities for functions from $C([0,2]^m)$ as well as for $C([0,2]^m)$-valued operators and to corresponding Jackson type inequalities. Inverse inequalities are presented for Hölder continuous functions and operators.

1. Introduction

Let (X, d) be a compact metric space. $C(X)$ is to denote the Banach space of real-valued continuous functions on X equipped with the usual norm $\|f\| = \sup\{|f(x)| : x \in X\}$. Let us recall that a finite system $\Phi = \{\varphi_1, \varphi_2, \dots, \varphi_k\} \subseteq C(X)$ is said to be a *partition of unity* if

$$0 \le \varphi_i(x) \le 1 \quad \text{for } x \in X,\ 1 \le i \le k \tag{1}$$

and

$$\sum_{i=1}^{k} \varphi_i(x) = 1 \quad \text{for } x \in X. \tag{2}$$

In her paper [Ste], I. Stephani introduced the class of *controllable* partitions of unity which are defined by a uniformity condition for the size of the *supports*

$$\operatorname{supp}(\varphi_i) = \{x \in X : \varphi_i(x) \neq 0\}$$

of the single functions. The condition refers to the concept of metric entropy in the space X. The *k-th entropy number* of a subset $S \subseteq X$ is given by

$$\varepsilon_k(S) = \inf\left\{\varepsilon > 0 : \text{ there exist points } x_1, x_2, \dots, x_k \text{ in } X \text{ such that } S \subseteq \bigcup_{i=1}^{k} B(x_i, \varepsilon)\right\},$$

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where $B(x_i, \varepsilon)$ denotes the closed ball of radius $\varepsilon \geq 0$ centered in x_i (cf. [Ca/Ste], p. 7). A partition of unity $\Phi = \{\varphi_1, \varphi_2, \dots, \varphi_k\}$ of cardinality $k \geq 2$ is called *controllable* if

$$\varepsilon_1(\operatorname{supp}(\varphi_i)) < \varepsilon_{k-1}(X) \qquad \text{for } 1 \leq i \leq k. \tag{3}$$

In [Ste] it is shown that controllable partitions of unity are *peaked*, i.e. the partition functions are of norm 1. This notion goes back to E. Michael and A. Pełczyński. In their paper [Mi/Pe1] they prove that for any infinite compact space (X, d) there exists a sequence $(\Phi_n)_{n=1}^{\infty}$ of peaked partitions of unity Φ_n, $\operatorname{card}(\Phi_n) = n$, such that the spans form an increasing sequence, i.e. $\operatorname{span}(\Phi_1) \subset \operatorname{span}(\Phi_2) \subset \operatorname{span}(\Phi_3) \subset \dots$, and the union $\bigcup_{n=1}^{\infty} \operatorname{span}(\Phi_n)$ is dense in $C(X)$. ("$\subset$" denotes the proper inclusion.) One would welcome a similar result for controllable partitions, since these can successfully be used in approximation theory (cf. [Ste], [Ri/Ste], [Ri2]). Indeed, an increasing sequence $(\operatorname{span}(\Phi_n))_{n=1}^{\infty}$ of controllable partition subspaces can be employed for a successive process of approximation of continuous functions, as will be shown in Proposition 1. In contrast with that the result of E. Michael and A. Pełczyński having far leading consequences for $C(X)$ spaces does not give quantitative results for the approximation of continuous functions comparable with the classical Jackson type theorems, which give estimates for approximation quantities of functions $f \in C(X)$ by the modulus of continuity $\omega(f, \delta) = \sup\{|f(x) - f(y)| : d(x, y) \leq \delta\}$. However, one has to pay for quantitative results. The theorem from [Mi/Pe1] can not be true in all details in the case of controllable partitions, since the property of controllability is so sharp as to imply that controllable partitions of unity exist for the cardinalities n with $\varepsilon_n(X) < \varepsilon_{n-1}(X)$ only (cf. [Ste]). These indices n are the values of Kolmogoroff's entropy function $N(X, \varepsilon)$ (cf. [Ko/Ti]). For instance, the controllable partitions on the cube $([0, 2]^m, d_{\max})$ are of cardinality k^m, for

$$\varepsilon_{k^m}([0, 2]^m) = \varepsilon_{k^m+1}([0, 2]^m) = \cdots = \varepsilon_{(k+1)^m-1}([0, 2]^m) = \frac{1}{k} \tag{4}$$

for $k = 1, 2, 3, \dots$ (cf. [Ba/Pi], [Bö/Ri]). The paper [Ste] closes with the adequate weaker question for the existence of *chains* of controllable partitions of unity in $C(X)$. A chain is meant to be a sequence of partitions Φ_n, $n = 1, 2, 3, \dots$, such that $\operatorname{span}(\Phi_1) \subset \operatorname{span}(\Phi_2) \subset \operatorname{span}(\Phi_3) \subset \cdots$.

PROPOSITION 1. *Let $(\Phi_n)_{n=1}^{\infty}$ be a chain of controllable partitions of unity in $C(X)$. Then $\bigcup_{n=1}^{\infty} \operatorname{span}(\Phi_n)$ is dense in $C(X)$. In particular: If $\operatorname{card}(\Phi_n) = k_n$ then there exist positive operators $A_n \in \mathcal{L}(C(X))$ with $A_n(C(X)) \subseteq \operatorname{span}(\Phi_n)$, $\|A_n\| = 1$, and $\|f - A_n f\| \leq \omega(f, \varepsilon_{k_n-1}(X))$ for $f \in C(X)$, such that $\lim_{n\to\infty} \|f - A_n f\| = 0$.*

Proof. We follow an idea from [Ste]. Let $\Phi_n = \{\varphi_1^{(n)}, \varphi_2^{(n)}, \dots, \varphi_{k_n}^{(n)}\}$. According to the controllability of Φ_n (cf. (3)) there exist points $x_1^{(n)}, x_2^{(n)}, \dots, x_{k_n}^{(n)} \in X$ such that

$$\operatorname{supp}\left(\varphi_i^{(n)}\right) \subseteq B\left(x_i^{(n)}, \varepsilon_{k_n-1}(X)\right) \qquad \text{for } 1 \leq i \leq k_n. \tag{5}$$

We define A_n by

$$A_n f = \sum_{i=1}^{k_n} f\left(x_i^{(n)}\right) \varphi_i^{(n)} \qquad \text{for } f \in C(X).$$

Clearly, $A_n \in \mathcal{L}(C(X))$, $A_n(C(X)) \subseteq \operatorname{span}(\Phi_n)$ and $\|A_n\| \geq 1$, for $A_n \mathbf{1}_X = \mathbf{1}_X$. A_n is positive with $\|A_n\| \leq 1$, since the values of $A_n f$ are convex combinations of $f(x_i^{(n)})$, $1 \leq i \leq k_n$. Inclusion (5) implies that

$$\begin{aligned}|(f - A_n f)(x)| &\leq \sum_{i=1}^{k_n} \left| f(x) - f\left(x_i^{(n)}\right)\right| \varphi_i^{(n)}(x) \\ &\leq \sum_{i=1}^{k_n} \omega\left(f, \varepsilon_{k_n - 1}(X)\right) \varphi_i^{(n)}(x) = \omega\left(f, \varepsilon_{k_n-1}(X)\right)\end{aligned}$$

for $x \in X$, and hence $\|f - A_n f\| \leq \omega\left(f, \varepsilon_{k_n-1}(X)\right)$.

Moreover we have $\lim_{n\to\infty} \varepsilon_{k_n-1}(X) = 0$, since $\lim_{n\to\infty} k_n = \infty$, for $\operatorname{span}(\Phi_1) \subset \operatorname{span}(\Phi_2) \subset \operatorname{span}(\Phi_3) \subset \cdots$, and $\lim_{k\to\infty} \varepsilon_k(X) = 0$ according to the compactness of X. Consequently,

$$0 \leq \lim_{n\to\infty} \|f - A_n f\| \leq \lim_{n\to\infty} \omega\left(f, \varepsilon_{k_n-1}(X)\right) = 0,$$

which completes the proof. □

In the following section we shall construct a chain of controllable partitions of unity on the cube $[0, 2]^m$ equipped with the maximum metrics $d_{\max}$, i.e.

$$d_{\max}((x_1, x_2, \ldots, x_m), (y_1, y_2, \ldots, y_m)) = \max_{1\leq j\leq m} |x_j - y_j|.$$

Corresponding approximation quantities for continuous functions on $[0, 2]^m$ as well as for $C([0, 2]^m)$-valued operators are defined and discussed in the third and fourth sections. Jackson type inequalities arise as a simple consequence of Proposition 1. The estimates will find an appropriate counterpart in inverse inequalities for Hölder continuous functions and operators.

Finally, we shall construct a related approximation scheme $E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots$ in $C([0, 2]^m)$ consisting of subspaces E_n spanned by peaked partitions of unity $\tilde{\Psi}_n$ of cardinality n such that the asymptotics of the corresponding approximation quantities of any Hölder continuous function f or operator T represents the modulus of continuity of f or T, respectively.

2. A chain of controllable partitions of unity

The essential step to a chain of controllable partitions of unity on the m-dimensional cube $[0, 2]^m$ is the construction of a chain for the one-dimensional case. Thus we begin with the consideration of the interval $I = [0, 2]$. Before defining the functions let us infer a necessary condition for the structure of the partition functions, which will justify the following rather complicated construction. According to [Mi/Pe1], Example 4.5, a chain of peaked partitions of unity in $C(I)$ can not be chosen such that all partition functions have a continuous derivative on $(0, 2)$. However, E. Michael and A. Pełczyński do not use the controllability property, which gives rise to much harder restrictions for the partition functions, as will be shown in Proposition 2. Recall that the metric space I has the entropy numbers

$$\varepsilon_k(I) = \frac{1}{k} \qquad \text{for } k = 1, 2, 3, \ldots. \tag{6}$$

Of course, the first entropy number of a subinterval of I is half of its length.

PROPOSITION 2. *Let $(\Phi_n)_{n=0}^{\infty}$ be a chain of controllable partitions of unity in $C[0,2]$. Then, for any $n_0 \geq 0$, there exists a subset $D_{n_0} \subseteq [0,2]$ of Lebesgue measure $\nu(D_{n_0}) = 2$ such that $D_{n_0} = \bigcup_{\iota \in \mathcal{I}} I_\iota$ is a countable union of intervals I_ι and any function from Φ_{n_0} is constant on any interval I_ι, $\iota \in \mathcal{I}$.*

Proof. Every partition $\Phi_n = \{\varphi_1^{(n)}, \varphi_2^{(n)}, \ldots, \varphi_{k_n}^{(n)}\}$ gives rise to an open covering $\mathcal{C}_n = \{C_1^{(n)}, C_2^{(n)}, \ldots, C_{k_n}^{(n)}\}$ by the supports $C_i^{(n)} = \operatorname{supp}(\varphi_i^{(n)})$. We consider the points $\frac{2(i-1)}{k_n-1} \in I$, $1 \leq i \leq k_n$. Clearly, any point is covered by at least one set $C_i^{(n)}$. However, each set $C_i^{(n)}$ contains at most one of the marked points, since

$$\operatorname{diam}\left(C_i^{(n)}\right) = 2\varepsilon_1\left(C_i^{(n)}\right) < \frac{2}{k_n - 1}$$

according to the controllability of Φ_n, whereas the distance of any two marked points is at least $\frac{2}{k_n-1}$. Consequently, any covering set contains exactly one of the distinguished points. We can assume that $\frac{2(i-1)}{k_n-1} \in C_i^{(n)}$ for $1 \leq i \leq k_n$. Thus,

$$\begin{aligned}
\inf\left(C_i^{(n)}\right) &\in \left(\tfrac{2(i-2)}{k_n-1}, \tfrac{2(i-1)}{k_n-1}\right) && \text{for} \quad 1 < i \leq k_n, \\
\sup\left(C_i^{(n)}\right) &\in \left(\tfrac{2(i-1)}{k_n-1}, \tfrac{2i}{k_n-1}\right) && \text{for} \quad 1 \leq i < k_n.
\end{aligned}$$

We define intervals

$$\begin{aligned}
I_1^{(n)} &= \left[0, \inf\left(C_2^{(n)}\right)\right], \\
I_i^{(n)} &= \left[\sup\left(C_{i-1}^{(n)}\right), \inf\left(C_{i+1}^{(n)}\right)\right] \qquad \text{for } 1 < i < k_n, \\
I_{k_n}^{(n)} &= \left[\sup\left(C_{k_n-1}^{(n)}\right), 2\right].
\end{aligned}$$

The above inclusions show that these are pairwise disjoint intervals of positive length and, moreover, that any interval $I_i^{(n)}$ is disjoint with $C_j^{(n)}$ for $j \neq i$, whereas $I_i^{(n)} \subseteq C_i^{(n)}$. For the partition functions this means that

$$\varphi_j^{(n)}\Big|_{I_i^{(n)}} \equiv \delta_{i,j} \qquad \text{for } i, j \in \{1, 2, \dots, k_n\}.$$

For all $n \geq n_0$, any function $\varphi_{i_0}^{(n_0)} \in \Phi_{n_0}$ is a linear combination of the functions $\varphi_j^{(n)}$ according to the chain condition. Therefore it is constant on any interval $I_i^{(n)}$, $1 \leq i \leq k_n$. We set

$$D_{n_0} = \bigcup \left\{ I_i^{(n)} \,:\, n \geq n_0, 1 \leq i \leq k_n \right\}.$$

It remains to show that $\nu(D_{n_0}) = 2$. On that account we give the following estimate for $n \geq n_0$:

$$\begin{aligned}
\nu(D_{n_0}) &\geq \nu\left(I_1^{(n)} \cup I_2^{(n)} \cup \dots \cup I_{k_n}^{(n)}\right) = \sum_{i=1}^{k_n} \nu\left(I_i^{(n)}\right) \\
&= \inf\left(C_2^{(n)}\right) + \sum_{i=2}^{k_n-1} \left(\inf\left(C_{i+1}^{(n)}\right) - \sup\left(C_{i-1}^{(n)}\right)\right) + \left(2 - \sup\left(C_{k_n-1}^{(n)}\right)\right) \\
&= 4 + \left(-\sup\left(C_1^{(n)}\right)\right) + \sum_{i=2}^{k_n-1} \left(\inf\left(C_i^{(n)}\right) - \sup\left(C_i^{(n)}\right)\right) \\
&\quad + \left(\inf\left(C_{k_n}^{(n)}\right) - 2\right) \\
&= 4 - \sum_{i=1}^{k_n} 2\varepsilon_1\left(C_i^{(n)}\right) > 4 - k_n \cdot \frac{2}{k_n - 1} = 2 - \frac{2}{k_n - 1}.
\end{aligned}$$

Letting $n \to \infty$ we obtain $\nu(D_{n_0}) = 2$, which is the desired conclusion. □

The construction of a chain $(\Phi_n)_{n=0}^{\infty}$ of controllable partitions of unity in $C(I)$ starts by defining a sequence $(\mathcal{C}_n)_{n=0}^{\infty}$ of open coverings whose covering sets $C_i^{(n)} \in \mathcal{C}_n$ serve as the supports $\operatorname{supp}(\varphi_i^{(n)})$ of the single functions $\varphi_i^{(n)} \in \Phi_n$. Accordingly, the covering sets $C_i^{(n)} \in \mathcal{C}_n$ have to fulfil the controllability condition (3). Moreover, any open set from $\mathcal{C}_n$ has to be the union of sets from $\mathcal{C}_{n+1}$, since any function from Φ_n is a linear combination of functions from Φ_{n+1} in accordance with $\operatorname{span}(\Phi_n) \subset \operatorname{span}(\Phi_{n+1})$.

Let

$$H(n) = 2 \sum_{j=n+1}^{\infty} 2^{-4^j}. \tag{7}$$

We define open coverings $\mathcal{C}_n = \{C_i^{(n)} : i = 1, 2, \dots, 2^{4^n}\}$, $n \geq 0$, by

$$C_i^{(n)} = \begin{cases} \left[0, 2 \cdot 2^{-4^n} + H(n)\right) & \text{for} \quad i = 1, \\ \left(2(i-1) \cdot 2^{-4^n} - H(n), 2i \cdot 2^{-4^n} + H(n)\right) & \text{for} \quad 1 < i < 2^{4^n}, \\ \left(2\left(2^{4^n} - 1\right) \cdot 2^{-4^n} - H(n), 2\right] & \text{for} \quad i = 2^{4^n}. \end{cases} \tag{8}$$

PROPOSITION 3. (a) *Any of the open coverings $\mathcal{C}_n$, $n \geq 0$, is controllable in the sense of formula* (3), *i.e.*,

$$\varepsilon_1\left(C_i^{(n)}\right) < \varepsilon_{2^{4^n}-1}(I) \qquad \textit{for } i = 1, 2, \dots, 2^{4^n}.$$

(b) $(\mathcal{C}_n)_{n=0}^{\infty}$ *fulfils the following chain condition*:

$$C_i^{(n)} = \begin{cases} C_1^{(n+1)} \cup C_2^{(n+1)} \cup \dots \cup C_{2^{3 \cdot 4^n}+1}^{(n+1)} & \textit{for} \quad i = 1, \\ C_{(i-1) \cdot 2^{3 \cdot 4^n}}^{(n+1)} \cup C_{(i-1) \cdot 2^{3 \cdot 4^n}+1}^{(n+1)} \cup \dots \cup C_{i \cdot 2^{3 \cdot 4^n}+1}^{(n+1)} & \textit{for} \quad 1 < i < 2^{4^n}, \\ C_{(2^{4^n}-1) \cdot 2^{3 \cdot 4^n}}^{(n+1)} \cup C_{(2^{4^n}-1) \cdot 2^{3 \cdot 4^n}+1}^{(n+1)} \cup \dots \cup C_{2^{4^{(n+1)}}}^{(n+1)} & \textit{for} \quad i = 2^{4^n}. \end{cases}$$

Proof. With the help of the fact that

$$H(n+1) < 2^{-4^{(n+1)}} \qquad \text{for } n = 0, 1, 2, \dots \tag{9}$$

and of equation (6), a simple calculation shows that

$$\varepsilon_1\left(C_i^{(n)}\right) \leq 2^{-4^n} + H(n) < 2^{-4^n} + 3 \cdot 2^{-4^{(n+1)}} < \frac{1}{2^{4^n} - 1} = \varepsilon_{2^{4^n}-1}(I).$$

Part (b) is obvious in accordance with (8). □

Now let $n \geq 0$ be fixed. The covering $\mathcal{C}_n$ gives rise to a decomposition of I for any $i \in \{1, 2, \dots, 2^{4^n}\}$:

$$I = \begin{cases} M_1^{(n)} \cup CR_1^{(n)} \cup R_1^{(n)} & \text{if} \quad i = 1, \\ L_i^{(n)} \cup CL_i^{(n)} \cup M_i^{(n)} \cup CR_i^{(n)} \cup R_i^{(n)} & \text{if} \quad 1 < i < 2^{4^n}, \\ L_{2^{4^n}}^{(n)} \cup CL_{2^{4^n}}^{(n)} \cup M_{2^{4^n}}^{(n)} & \text{if} \quad i = 2^{4^n}. \end{cases}$$

$L_i^{(n)}$ and $R_i^{(n)}$ are to denote the subintervals left beside and right beside $C_i^{(n)}$. The "critical parts" $CL_i^{(n)}$ and $CR_i^{(n)}$ are the intersections of $C_i^{(n)}$ with $C_{i-1}^{(n)}$ and $C_{i+1}^{(n)}$, respectively. The "middle part" $M_i^{(n)}$ consists of that points which are covered by

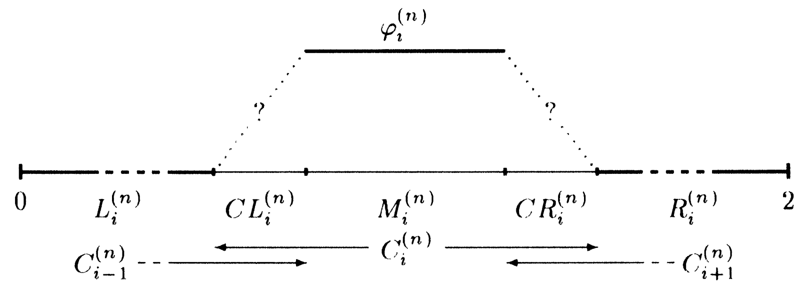

Figure 1. General structure of $\varphi_i^{(n)}$

$C_i^{(n)}$ only (cf. Fig. 1). Consequently,

$$\begin{aligned}
L_i^{(n)} &= \left[0, 2(i-1)\cdot 2^{-4^n} - H(n)\right], && 1 < i \le 2^{4^n},\\
CL_i^{(n)} &= \left(2(i-1)\cdot 2^{-4^n} - H(n), 2(i-1)\cdot 2^{-4^n} + H(n)\right), && 1 < i \le 2^{4^n},\\
M_i^{(n)} &= \begin{cases} \left[0, 2\cdot 2^{-4^n} - H(n)\right], & i = 1,\\ \left[2(i-1)\cdot 2^{-4^n} + H(n), 2i\cdot 2^{-4^n} - H(n)\right], & 1 < i < 2^{4^n},\\ \left[2\left(2^{4^n}-1\right)\cdot 2^{-4^n} + H(n), 2\right], & i = 2^{4^n},\end{cases} \qquad (10)\\
CR_i^{(n)} &= \left(2i\cdot 2^{-4^n} - H(n), 2i\cdot 2^{-4^n} + H(n)\right), && 1 \le i < 2^{4^n},\\
R_i^{(n)} &= \left[2i\cdot 2^{-4^n} + H(n), 2\right], && 1 \le i < 2^{4^n}.
\end{aligned}$$

We want to construct a partition of unity $\Phi_n = \{\varphi_i^{(n)} : i = 1, 2, \ldots, 2^{4^n}\}$ such that $\mathrm{supp}(\varphi_i^{(n)}) = C_i^{(n)}$. Hence we have to define

$$\varphi_i^{(n)}\Big|_{L_i^{(n)}\cup R_i^{(n)}} \equiv 0 \qquad \text{and} \qquad \varphi_i^{(n)}\Big|_{M_i^{(n)}} \equiv 1, \tag{11}$$

where $L_i^{(n)} \cup R_i^{(n)}$ has to be replaced by $R_1^{(n)}$ if $i = 1$ and by $L_{2^{4^n}}^{(n)}$ if $i = 2^{4^n}$. The crucial point is the construction in the critical regions $CL_i^{(n)}$ and $CR_i^{(n)}$. We first shall give the definition on dense subsets $D_{i-1}^{(n)} \subseteq CL_i^{(n)}$ and $D_i^{(n)} \subseteq CR_i^{(n)}$. This requires some additional concepts.

$\{-1, 1\}^*$ is meant to be the set of all finite words over the alphabet $\{-1, 1\}$, i.e. $\{-1, 1\}^* = \{e, -1, 1, (-1, -1), (-1, 1), (1, -1), (1, 1), \ldots\}$ where e stands for the empty word. The length k of a word $w = (l_1, l_2, \ldots, l_k)$ is denoted by $|w|$; in particular, $|e| = 0$. Let

$$h(w, n) = h((l_1, l_2, \ldots, l_k), n) = \sum_{j=1}^{k} 2l_j \cdot 2^{-4^{(n+j)}} \tag{12}$$

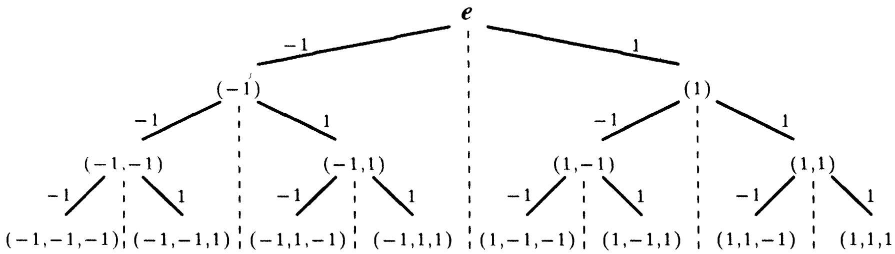

Figure 2. Inorder of words w with $|w| \leq 3$

for $w = (l_1, l_2, \dots, l_k) \in \{-1, 1\}^*$, in particular $h(e, n) = 0$. Now let

$$\begin{aligned} I_i^{(n)}(w) = \Big[& 2i \cdot 2^{-4^n} + h(w,n) - 2 \cdot 2^{-4^{(n+|w|+1)}} + H(n + |w| + 1), \\ & 2i \cdot 2^{-4^n} + h(w,n) + 2 \cdot 2^{-4^{(n+|w|+1)}} - H(n + |w| + 1) \Big] \end{aligned} \tag{13}$$

for $1 \leq i < 2^{4^n}$. Inequality (9) ensures that $I_i^{(n)}(w)$ is an interval of positive length.

For different words w and w' there is a natural ordering between the intervals $I_i^{(n)}(w)$ and $I_i^{(n)}(w')$. It can be described by the so-called *inorder* in $\{-1, 1\}^*$, which is an irreflexive ordering defined as follows (cf. e.g. [Ge], p. 446): $(l_1, l_2, \dots, l_k) \prec (l'_1, l'_2, \dots, l'_{k'})$ *if and only if one of the following three statements is true*:

$$\begin{array}{ll} (i) & k < k' \textit{ and } l_1 = l'_1, l_2 = l'_2, \dots, l_k = l'_k \textit{ and } l'_{k+1} = 1. \\ (ii) & k > k' \textit{ and } l_1 = l'_1, l_2 = l'_2, \dots, l_{k'} = l'_{k'} \textit{ and } l_{k'+1} = -1. \\ (iii) & \textit{There exists } m \in \{1, 2, \dots, \min\{k, k'\}\} \textit{ such that } l_1 = l'_1, \\ & l_2 = l'_2, \dots, l_{m-1} = l'_{m-1} \textit{ and } l_m = -1, l'_m = 1. \end{array} \tag{14}$$

Figure 2 shows the binary tree consisting of the words w of length $|w| \leq 3$. The word "inorder" is due to the fact that any word w is greater than the words from the subtree left below w and less than the words from the subtree right below w. This means for example for the empty word e that e is greater than any word with first letter -1 and less than any word with first letter 1. Consequently, the words from Figure 2 are ordered "from the left to the right", that is $(-1, -1, -1) \prec (-1, -1) \prec (-1, -1, 1) \prec -1 \prec (-1, 1, -1) \prec (-1, 1) \prec (-1, 1, 1) \prec e \prec (1, -1, -1) \prec (1, -1) \prec (1, -1, 1) \prec 1 \prec (1, 1, -1) \prec (1, 1) \prec (1, 1, 1)$. An equivalent definition can easily be verified:

$$(l_1, l_2, \dots, l_k) \prec (l'_1, l'_2, \dots, l'_{k'}) \qquad \text{iff} \qquad \sum_{j=1}^{k} l_j \cdot 2^{-j} < \sum_{j=1}^{k'} l'_j \cdot 2^{-j}. \tag{15}$$

The proof of the following proposition on the ordering of the intervals $I_i^{(n)}(w)$ can be carried out by discussing the three cases from (14) with the help of (7), (9), (12) and (13).

PROPOSITION 4. *Let $w, w' \in \{-1, 1\}^*$ such that $w \prec w'$. Then* $\sup(I_i^{(n)}(w)) < \inf(I_i^{(n)}(w'))$. *In particular, the intervals $I_i^{(n)}(w)$, $w \in \{-1, 1\}^*$, are pairwise disjoint.*

Now let

$$D_i^{(n)} = \bigcup_{w \in \{-1,1\}^*} I_i^{(n)}(w) \qquad \text{for} \qquad 1 \le i < 2^{4^n}. \tag{16}$$

PROPOSITION 5. *The set $D_i^{(n)}$ is a dense subset of the interval $CR_i^{(n)} = CL_{i+1}^{(n)}$.*

Proof. It can easily be checked that any interval $I_i^{(n)}(w)$, $w \in \{-1, 1\}^*$, is contained in $CR_i^{(n)}$. Consequently, $D_i^{(n)} \subseteq CR_i^{(n)}$.

We prove the density of $D_i^{(n)}$ in $CR_i^{(n)}$ by showing the equality of the Lebesgue measures of $D_i^{(n)}$ and $CR_i^{(n)}$. In fact, Proposition 4 and formulas (7), (10) and (13) yield

$$\begin{aligned}
\nu\left(D_i^{(n)}\right) &= \sum_{k=0}^{\infty} \sum_{\substack{w \in \{-1,1\}^* \\ |w|=k}} \nu\left(I_i^{(n)}(w)\right) \\
&= \sum_{k=0}^{\infty} \operatorname{card}(\{w : |w| = k\}) \cdot 2\left(2 \cdot 2^{-4^{(n+k+1)}} - H(n+k+1)\right) \\
&= 2\sum_{k=0}^{\infty} 2^k \left(H(n+k) - 2H(n+k+1)\right) \\
&= 2\left(\sum_{k=0}^{\infty} 2^k H(n+k) - \sum_{k=1}^{\infty} 2^k H(n+k)\right) \\
&= 2H(n) = \nu\left(CR_i^{(n)}\right).
\end{aligned}$$

□

We define the function $\varphi_i^{(n)}$ on $D_{i-1}^{(n)} \subseteq CL_i^{(n)}$ and $D_i^{(n)} \subseteq CR_i^{(n)}$ to be constant on each interval $I_{i-1}^{(n)}(w)$ and $I_i^{(n)}(w)$, $w = (l_1, l_2, \ldots, l_k) \in \{-1, 1\}^*$; namely

$$\varphi_i^{(n)}\Big|_{I_{i-1}^{(n)}(l_1,l_2,\ldots,l_k)} \equiv \frac{1}{2} + \frac{1}{2}\sum_{j=1}^{k} l_j \cdot 2^{-j} \tag{17}$$

(cf. Fig. 3) and

$$\varphi_i^{(n)}\Big|_{I_i^{(n)}(l_1,l_2,\ldots,l_k)} \equiv \frac{1}{2} - \frac{1}{2}\sum_{j=1}^{k} l_j \cdot 2^{-j}, \tag{18}$$

respectively. (Note that $\varphi_i^{(n)}$ has a kind of self-similar structure as the well-known *Lebesgue singular function* (cf. e.g. [Se], p. 23).)

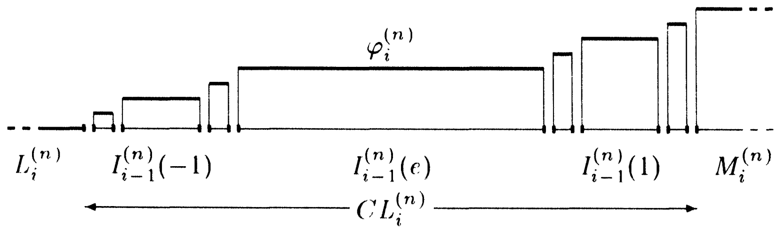

Figure 3. Behaviour of $\varphi_i^{(n)}$ on $CL_i^{(n)}$

PROPOSITION 6. *For any $n \geq 0$ and any $i \in \{1, 2, \ldots, 2^{4^n}\}$ there exists a uniquely determined continuous function $\varphi_i^{(n)} \in C(I)$ subject to the definitions* (11), (17) *and* (18).

Proof. The above definitions fix $\varphi_i^{(n)}$ on a dense subset of I. Thus the uniqueness is obvious.

By (11), $\varphi_i^{(n)}$ is already defined on $L_i^{(n)}$, $M_i^{(n)}$ and $R_i^{(n)}$. Hence is suffices to show that (17) admits a continuous extension of $\varphi_i^{(n)}$ to $CL_i^{(n)}$ which fits continuously together with $\varphi_i^{(n)}|_{L_i^{(n)}}$ and $\varphi_i^{(n)}|_{M_i^{(n)}}$. Of course, $\varphi_i^{(n)}$ can similarly be treated on $CR_i^{(n)}$.

Observe that $\varphi_i^{(n)}$ is monotonically increasing on $D_{i-1}^{(n)}$ according to (17). Indeed, let $x, y \in D_{i-1}^{(n)}$ such that $x < y$. Then there exist $w = (l_1, l_2, \ldots, l_k)$, $w' = (l'_1, l'_2, \ldots, l'_{k'}) \in \{-1, 1\}^*$ with $x \in I_{i-1}^{(n)}(w)$ and $y \in I_{i-1}^{(n)}(w')$. If $w = w'$ then we have $\varphi_i^{(n)}(x) = \varphi_i^{(n)}(y)$. If $w \neq w'$ then $w \prec w'$ in accordance with Proposition 4. The equivalence (15) yields $\sum_{j=1}^{k} l_j \cdot 2^{-j} < \sum_{j=1}^{k'} l'_j \cdot 2^{-j}$. Thus we obtain $\varphi_i^{(n)}(x) < \varphi_i^{(n)}(y)$ by (17).

Moreover, the image of $\varphi_i^{(n)}|_{D_{i-1}^{(n)}}$ is a dense subset of $[0, 1]$. Consequently, there exists a continuous extension $\varphi_i^{(n)}|_{CL_i^{(n)}}$ which fits continuously together with $\varphi_i^{(n)}|_{L_i^{(n)}} \equiv 0$, since

$$\lim_{x \to \inf\left(CL_i^{(n)}\right)} \varphi_i^{(n)}|_{CL_i^{(n)}}(x) = \inf\left(\varphi_i^{(n)}|_{D_{i-1}^{(n)}}\left(D_{i-1}^{(n)}\right)\right) = 0,$$

and with $\varphi_i^{(n)}|_{M_i^{(n)}} \equiv 1$, since

$$\lim_{x \to \sup\left(CL_i^{(n)}\right)} \varphi_i^{(n)}|_{CL_i^{(n)}}(x) = \sup\left(\varphi_i^{(n)}|_{D_{i-1}^{(n)}}\left(D_{i-1}^{(n)}\right)\right) = 1,$$

This is our claim. $\square$

PROPOSITION 7. *For any $n \geq 0$ the system $\Phi_n = \{\varphi_i^{(n)} : i = 1, 2, \dots, 2^{4^n}\}$ forms a controllable partition of unity in $C(I)$.*

Proof. We have

$$\varphi_i^{(n)}(I) = [0, 1] \quad \text{and} \quad \operatorname{supp}\left(\varphi_i^{(n)}\right) = C_i^{(n)}$$

according to the above definitions. This proves (1) and the controllability (3) by Proposition 3 (a). The second partition property (2) can be verified by considering points x from the dense subset

$$M_1^{(n)} \cup D_1^{(n)} \cup M_2^{(n)} \cup D_2^{(n)} \cup \dots \cup M_{2^{4^n}-1}^{(n)} \cup D_{2^{4^n}-1}^{(n)} \cup M_{2^{4^n}}^{(n)} \subseteq I.$$

If $x \in M_{i_0}^{(n)}$ then $\{i : x \in \operatorname{supp}(\varphi_i^{(n)})\} = \{i_0\}$ and, by (11),

$$\sum_{i=1}^{2^{4^n}} \varphi_i^{(n)}(x) = \varphi_{i_0}^{(n)}(x) = 1.$$

Otherwise we have $x \in D_{i_0}^{(n)}$, i.e. $x \in I_{i_0}^{(n)}(w)$ with a suitable $w \in \{-1, 1\}^*$. Then $\{i : x \in \operatorname{supp}(\varphi_i^{(n)})\} = \{i_0, i_0 + 1\}$ and, by (17) and (18),

$$\sum_{i=1}^{2^{4^n}} \varphi_i^{(n)}(x) = \varphi_{i_0}^{(n)}(x) + \varphi_{i_0+1}^{(n)}(x) = 1.$$

This completes the proof. $\square$

Next we have to verify the chain condition $\operatorname{span}(\Phi_n) \subset \operatorname{span}(\Phi_{n+1})$. Figure 4 illustrates how a function $\varphi_i^{(n)}$ can be represented as a linear combination of functions from Φ_{n+1} on the critical region $CR_i^{(n)}$.

PROPOSITION 8. *For any $n \geq 0$ we have* $\operatorname{span}(\Phi_n) \subset \operatorname{span}(\Phi_{n+1})$. *In particular,*

$$\begin{aligned}
\varphi_1^{(n)} &= \left(\varphi_1^{(n+1)} + \varphi_2^{(n+1)} + \dots + \varphi_{2^{3\cdot 4^n}-1}^{(n+1)}\right) + \frac{1}{2}\left(\varphi_{2^{3\cdot 4^n}}^{(n+1)} + \varphi_{2^{3\cdot 4^n}+1}^{(n+1)}\right), \\
\varphi_i^{(n)} &= \frac{1}{2}\left(\varphi_{(i-1)\cdot 2^{3\cdot 4^n}}^{(n+1)} + \varphi_{(i-1)\cdot 2^{3\cdot 4^n}+1}^{(n+1)}\right) + \left(\varphi_{(i-1)\cdot 2^{3\cdot 4^n}+2}^{(n+1)} + \varphi_{(i-1)\cdot 2^{3\cdot 4^n}+3}^{(n+1)} + \dots\right. \\
&\quad \left.\dots + \varphi_{i\cdot 2^{3\cdot 4^n}-2}^{(n+1)} + \varphi_{i\cdot 2^{3\cdot 4^n}-1}^{(n+1)}\right) + \frac{1}{2}\left(\varphi_{i\cdot 2^{3\cdot 4^n}}^{(n+1)} + \varphi_{i\cdot 2^{3\cdot 4^n}+1}^{(n+1)}\right), \ 1 < i < 2^{4^n}, \\
\varphi_{2^{4^n}}^{(n)} &= \frac{1}{2}\left(\varphi_{(2^{4^n}-1)\cdot 2^{3\cdot 4^n}}^{(n+1)} + \varphi_{(2^{4^n}-1)\cdot 2^{3\cdot 4^n}+1}^{(n+1)}\right) + \\
&\quad + \left(\varphi_{(2^{4^n}-1)\cdot 2^{3\cdot 4^n}+2}^{(n+1)} + \varphi_{(2^{4^n}-1)\cdot 2^{3\cdot 4^n}+3}^{(n+1)} + \dots + \varphi_{2^{4^{(n+1)}}}^{(n+1)}\right).
\end{aligned}$$

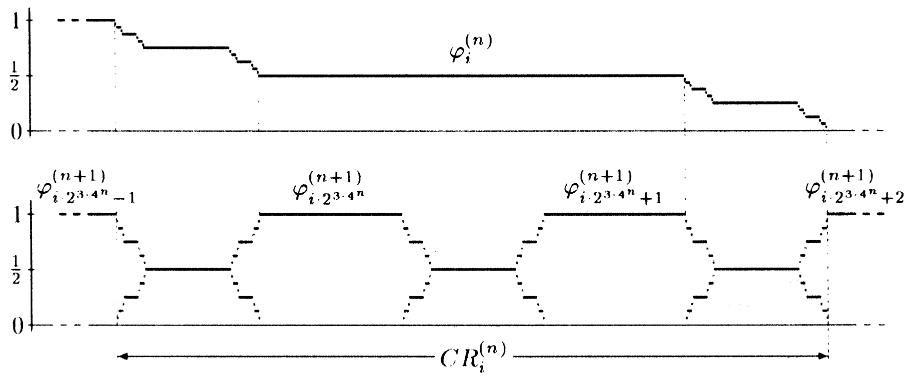

Figure 4. $\varphi_i^{(n)}$ on $CR_i^{(n)}$ as a linear combination

Proof. Let $i_0 \in \{2, 3, \dots, 2^{4^n} - 1\}$ be fixed. (The cases $i_0 = 1$ and $i_0 = 2^{4^n}$ admit a similar treatment.) Proposition 3 (b) shows that the supports of $\varphi_{i_0}^{(n)}$ and of the function from the right-hand side (r.h.s.) of the asserted equation coincide. Hence we can restict our considerations to points x from the set $D_{i_0-1}^{(n)} \cup M_{i_0}^{(n)} \cup D_{i_0}^{(n)}$, which is dense in $C_{i_0}^{(n)} = \operatorname{supp}(\varphi_{i_0}^{(n)})$.

We start with $x \in M_{i_0}^{(n)}$. According to equations (7), (8) and (10) we obtain

$$M_{i_0}^{(n)} = I \setminus \left(\bigcup_{i=1}^{(i_0-1)\cdot 2^{3\cdot 4^n}+1} C_i^{(n+1)} \cup \bigcup_{i=i_0\cdot 2^{3\cdot 4^n}}^{2^{4^{(n+1)}}} C_i^{(n+1)} \right)$$

and thus

$$\left\{ i : x \in \operatorname{supp}\left(\varphi_i^{(n+1)}\right)\right\} \subseteq \left\{ (i_0-1)\cdot 2^{3\cdot 4^n}+2,\, (i_0-1)\cdot 2^{3\cdot 4^n}+3, \dots, i_0\cdot 2^{3\cdot 4^n}-1 \right\}.$$

Consequently,

$$\text{r.h.s.}(x) = \sum_{i=(i_0-1)\cdot 2^{3\cdot 4^n}+2}^{i_0\cdot 2^{3\cdot 4^n}-1} \varphi_i^{(n+1)}(x) = \sum_{i=1}^{2^{4^{(n+1)}}} \varphi_i^{(n+1)}(x) = 1 = \varphi_{i_0}^{(n)}(x)$$

by Proposition 7 and by (11).

Now let $x \in D_{i_0-1}^{(n)}$. Then there exists a word $w = (l_1, l_2, \dots, l_k) \in \{-1, 1\}^*$ such that $x \in I_{i_0-1}^{(n)}(w)$. We treat the three cases $w = e$, $w \prec e$ and $e \prec w$ separately.

If $w = e$, i.e., $x \in I_{i_0-1}^{(n)}(e)$, then we observe that

$$I_{i_0-1}^{(n)}(e) = I \setminus \left(\bigcup_{i=1}^{(i_0-1)\cdot 2^{3\cdot 4^n}-1} C_i^{(n+1)} \cup \bigcup_{i=(i_0-1)\cdot 2^{3\cdot 4^n}+2}^{2^{4^{(n+1)}}} C_i^{(n+1)} \right)$$

(cf. (13) and (8)) and hence

$$\left\{i : x \in \operatorname{supp}\left(\varphi_i^{(n+1)}\right)\right\} \subseteq \left\{(i_0 - 1) \cdot 2^{3 \cdot 4^n}, (i_0 - 1) \cdot 2^{3 \cdot 4^n} + 1\right\}.$$

This shows that

$$\begin{aligned} \text{r.h.s.}(x) &= \frac{1}{2}\left(\varphi^{(n+1)}_{(i_0-1)\cdot 2^{3\cdot 4^n}}(x) + \varphi^{(n+1)}_{(i_0-1)\cdot 2^{3\cdot 4^n}+1}(x)\right) \\ &= \frac{1}{2}\sum_{i=1}^{2^{4(n+1)}} \varphi_i^{(n+1)}(x) = \frac{1}{2} = \varphi_{i_0}^{(n)}(x) \end{aligned}$$

in accordance with Proposition 7 and with definition (17).

Now let us assume that $w \prec e$, i.e., $l_1 = -1$ and $w = (-1, l_2, l_3, \ldots, l_k)$. In the same manner as above we can see that

$$I_{i_0-1}^{(n)}(w) \subseteq I \setminus \left(\bigcup_{i=1}^{(i_0-1)\cdot 2^{3\cdot 4^n}-2} C_i^{(n+1)} \cup \bigcup_{i=(i_0-1)\cdot 2^{3\cdot 4^n}+1}^{2^{4(n+1)}} C_i^{(n+1)} \right)$$

and

$$\left\{i : x \in \operatorname{supp}\left(\varphi_i^{(n+1)}\right)\right\} \subseteq \left\{(i_0 - 1) \cdot 2^{3 \cdot 4^n} - 1, (i_0 - 1) \cdot 2^{3 \cdot 4^n}\right\}.$$

Thus the right-hand side of the asserted equation reduces to

$$\text{r.h.s.}(x) = \frac{1}{2}\,\varphi^{(n+1)}_{(i_0-1)\cdot 2^{3\cdot 4^n}}(x). \tag{19}$$

Moreover, combining (13) with (12) yields

$$x \in I_{i_0-1}^{(n)}(w) = I_{i_0-1}^{(n)}(-1, l_2, l_3, \ldots, l_k) = I^{(n+1)}_{(i_0-1)\cdot 2^{3\cdot 4^n}-1}(l_2, l_3, \ldots, l_k).$$

Hence, by (17), equation (19) can be continued to

$$\text{r.h.s.}(x) = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2}\sum_{j=1}^{k-1} l_{j+1} \cdot 2^{-j}\right) = \frac{1}{2} + \frac{1}{2}\sum_{j=1}^{k} l_j \cdot 2^{-j} = \varphi_{i_0}^{(n)}(x).$$

Similarly, if $e \prec w$, i.e., $l_1 = 1$, then we obtain

$$I_{i_0-1}^{(n)}(w) \subseteq I \setminus \left(\bigcup_{i=1}^{(i_0-1)\cdot 2^{3\cdot 4^n}} C_i^{(n+1)} \cup \bigcup_{i=(i_0-1)\cdot 2^{3\cdot 4^n}+3}^{2^{4(n+1)}} C_i^{(n+1)} \right),$$

hence

$$\left\{i : x \in \operatorname{supp}\left(\varphi_i^{(n+1)}\right)\right\} \subseteq \left\{(i_0 - 1) \cdot 2^{3 \cdot 4^n} + 1, (i_0 - 1) \cdot 2^{3 \cdot 4^n} + 2\right\}$$

and thus

$$\text{r.h.s.}(x) = \frac{1}{2}\,\varphi^{(n+1)}_{(i_0-1)\cdot 2^{3\cdot 4^n}+1}(x) + \varphi^{(n+1)}_{(i_0-1)\cdot 2^{3\cdot 4^n}+2}(x). \tag{20}$$

Moreover, we get

$$x \in I^{(n)}_{i_0-1}(w) = I^{(n)}_{i_0-1}(1, l_2, l_3, \dots, l_k) = I^{(n+1)}_{(i_0-1)\cdot 2^{3\cdot 4^n}+1}(l_2, l_3, \dots, l_k).$$

Applying (18) to $\varphi^{(n+1)}_{(i_0-1)\cdot 2^{3\cdot 4^n}+1}$ and (17) to $\varphi^{(n+1)}_{(i_0-1)\cdot 2^{3\cdot 4^n}+2}$ we can rewrite (20) as

$$\begin{aligned}\text{r.h.s.}(x) &= \frac{1}{2}\left(\frac{1}{2} - \frac{1}{2}\sum_{j=1}^{k-1} l_{j+1}\cdot 2^{-j}\right) + \left(\frac{1}{2} + \frac{1}{2}\sum_{j=1}^{k-1} l_{j+1}\cdot 2^{-j}\right)\\ &= \frac{1}{2} + \frac{1}{2}\sum_{j=1}^{k} l_j \cdot 2^{-j} = \varphi^{(n)}_{i_0}(x).\end{aligned}$$

This completes the considerations for points $x \in D^{(n)}_{i_0-1}$.

The remaining proof for $x \in D^{(n)}_{i_0}$ runs as for $x \in D^{(n)}_{i_0-1}$. □

Propositions 7 and 8 show that $(\Phi_n)_{n=0}^{\infty}$ is a chain of controllable partitions of unity in $C(I)$. Thus the consideration of the one-dimensional case is complete. Finally, we obtain a chain $(\Psi_n)_{n=0}^{\infty}$ of partitions of unity in $C([0,2]^m)$ by putting

$$\begin{aligned}&\Psi_n = \left\{\psi^{(n)}_{(i_1,i_2,\dots,i_m)} : i_1, i_2, \dots, i_m \in \left\{1, 2, \dots, 2^{4^n}\right\}\right\} \quad \text{with}\\ &\psi^{(n)}_{(i_1,i_2,\dots,i_m)}(x_1, x_2, \dots, x_m) = \varphi^{(n)}_{i_1}(x_1)\varphi^{(n)}_{i_2}(x_2)\dots\varphi^{(n)}_{i_m}(x_m).\end{aligned} \tag{21}$$

The typical graph of a function $\psi^{(n)}_{(i_1,i_2)}$ on $[0,2]^2$ is displayed in Figure 5.

THEOREM 1. *$(\Psi_n)_{n=0}^{\infty}$ is a chain of controllable partitions of unity on* $([0,2]^m, d_{\max})$.

Proof. It remains to verify the controllability of Ψ_n, which means that

$$\varepsilon_1\left(\operatorname{supp}\left(\psi^{(n)}_{(i_1,i_2,\dots,i_m)}\right)\right) < \varepsilon_{(2^{4^n})^m-1}([0,2]^m) = \frac{1}{2^{4^n}-1}$$

according to (3), since $\operatorname{card}(\Psi_n) = \left(2^{4^n}\right)^m$, and to (4). We obtain the required estimate by

$$\begin{aligned}\varepsilon_1\left(\operatorname{supp}\left(\psi^{(n)}_{(i_1,i_2,\dots,i_m)}\right)\right) &= \varepsilon_1\left(\operatorname{supp}\left(\varphi^{(n)}_{i_1}\right) \times \dots \times \operatorname{supp}\left(\varphi^{(n)}_{i_m}\right)\right)\\ &= \max_{1\le j\le m} \varepsilon_1\left(\operatorname{supp}\left(\varphi^{(n)}_{i_j}\right)\right) < \varepsilon_{2^{4^n}-1}(I) = \frac{1}{2^{4^n}-1}.\end{aligned}$$

(Note that we use the same symbol $\varepsilon_k(\cdot)$ for entropy numbers with respect to the different metric spaces $[0,2]^m$ and I.) This completes the proof. □

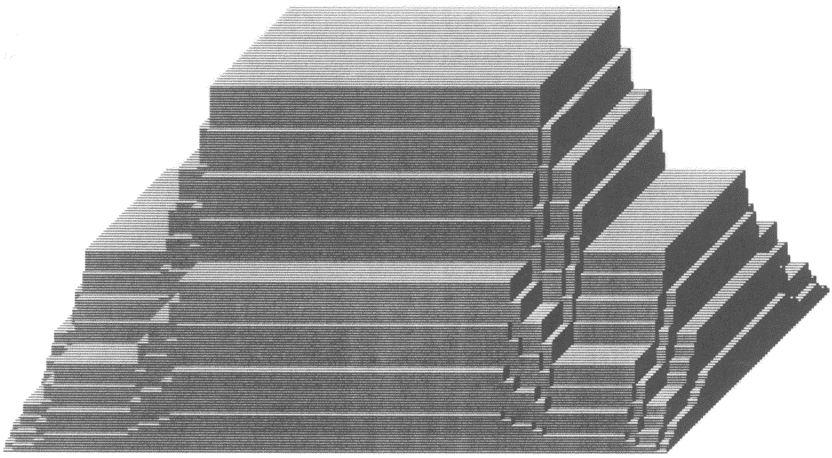

Figure 5. Graph of a function $\psi^{(n)}_{(i_1,i_2)}$

3. Approximation properties

Obviously, the linear spaces $\operatorname{span}(\Psi_n)$, $n \geq 0$, can serve as an *approximation scheme* on the Banach space $C([0,2]^m)$. The corresponding approximation numbers of a function $f \in C([0,2]^m)$ are given by

$$E(f, \operatorname{span}(\Psi_n)) = \inf\{\|f - \psi\| : \psi \in \operatorname{span}(\Psi_n)\} \qquad \text{for } n \geq 0$$

(cf. e.g. [Bu/Sch], p. 50). We apply Proposition 1 to the chain $(\Psi_n)_{n=0}^{\infty}$ and obtain $\|f - A_n f\| \leq \omega(f, \frac{1}{2^{4^n}-1})$, where $A_n f \in \operatorname{span}(\Psi_n)$. This gives a theorem of Jackson type.

THEOREM 2. *Let $f \in C([0,2]^m)$ and $n \geq 0$. Then*

$$E(f, \operatorname{span}(\Psi_n)) \leq \omega\left(f, \frac{1}{2^{4^n}-1}\right).$$

A related concept for approximating linear operators $T \in \mathcal{L}(E, C([0,2]^m))$ from a Banach space E into $C([0,2]^m)$ is based on the approximation quantities

$$E(T, \operatorname{span}(\Psi_n)) = \inf\{\, \|T - A\| : A \in \mathcal{L}(E, C([0,2]^m)),\ A(E) \subseteq \operatorname{span}(\Psi_n)\,\}$$

for $n \geq 0$. Let us recall that the *modulus of continuity* $\omega(T, \cdot)$ *of the operator* T is given by

$$\omega(T, \delta) = \sup_{\|z\| \leq 1} \omega(Tz, \delta)$$

(cf. [Ca/Ste], p. 174). Similarly, as above, we obtain $\|T - A_n T\| \leq \omega(T, \frac{1}{2^{4^n}-1})$ by Proposition 1. This in particular shows that $T = \lim_{n\to\infty} A_n T$ if T is compact, since compactness of T is equivalent to $\lim_{\delta\to+0}\omega(T,\delta) = 0$.

THEOREM 3. *Let $T \in \mathcal{L}(E, C([0,2]^m))$ be an operator from a Banach space E into $C([0,2]^m)$ and let $n \geq 0$. Then*

$$E(T, \operatorname{span}(\Psi_n)) \leq \omega\left(T, \frac{1}{2^{4^n}-1}\right).$$

Inverse theorems refer to the concept of Hölder continuity. A function $f \in C([0,2]^m)$ is called *Hölder continuous of type α*, $0 < \alpha \leq 1$, if

$$|f|_\alpha = \sup_{\delta>0} \frac{\omega(f,\delta)}{\delta^\alpha} < \infty.$$

Analogously, one can define the value $|T|_\alpha$ and the concept of *Hölder continuity of an operator* $T \in \mathcal{L}(E, C([0,2]^m))$ (cf. [Ca/Ste], p. 196). The goal of this section is the proof of the following results.

THEOREM 4. *Let $f \in C([0,2]^m)$ be non-constant and Hölder continuous of type α, $0 < \alpha \leq 1$. Then*

$$\liminf_{n\to\infty} \frac{E(f, \operatorname{span}(\Psi_n))}{\omega\left(f, \frac{1}{2^{4^n}-1}\right)} \geq \frac{1}{2\left(\alpha^{-\frac{1}{2}}+1\right)}.$$

THEOREM 5. *Let E be a Banach space and let $T \in \mathcal{L}(E, C([0,2]^m))$ be Hölder continuous of type α, $0 < \alpha \leq 1$, such that the image $T(E)$ does not consist of constant functions only. Then*

$$\liminf_{n\to\infty} \frac{E(T, \operatorname{span}(\Psi_n))}{\omega\left(T, \frac{1}{2^{4^n}-1}\right)} \geq \frac{1}{2\left(\alpha^{-\frac{1}{2}}+1\right)}.$$

The above estimates depend essentially on the definition of the partition functions from Φ_n, $n \geq 0$, in the regions $CR_i^{(n)}$ and $CL_i^{(n)}$. We start the considerations with a proposition on the distance between intervals $I_i^{(n)}(w)$ which form the dense subsets $D_i^{(n)} \subseteq CR_i^{(n)} = CL_{i+1}^{(n)}$. Later on, when approximating a Hölder continuous function f, the numbers k appering in Proposition 9 will be chosen in dependence on the coefficient α of Hölder continuity.

PROPOSITION 9. *Let $k \geq 1$ and let $\{w_1, w_2, \ldots, w_{2^k-1}\} = \{w \in \{-1, 1\}^* : |w| < k\}$ such that $w_1 \prec w_2 \prec \cdots \prec w_{2^k-1}$. Then, for any $n \geq 0$ and any $1 \leq i < 2^{4^n}$,*

$$\begin{aligned}
\min\left(I_i^{(n)}(w_1)\right) - \max\left(M_i^{(n)}\right) &= 2H(n+k), \\
\min\left(I_i^{(n)}(w_t)\right) - \max\left(I_i^{(n)}(w_{t-1})\right) &= 2H(n+k) \quad \textit{for} \quad 1 < t < 2^k, \\
\min\left(M_{i+1}^{(n)}\right) - \max\left(I_i^{(n)}(w_{2^k-1})\right) &= 2H(n+k).
\end{aligned}$$

Proof. Clearly, $w_1 = (-1, -1, \ldots, -1)$ and $w_{2^k-1} = (1, 1, \ldots, 1)$ with $|w_1| = |w_{2^k-1}| = k - 1$. (In particular, $w_1 = w_{2^k-1} = e$ if $k = 1$.) One easily obtains the first and the third equation from (10) and (13).

So let $1 < t < 2^k$. We consider the word $w_{t-1} = (l_1, l_2, \ldots, l_{|w_{t-1}|})$. The ordering of the words with respect to $\prec$ (cf. (14)) implies the following: If $|w_{t-1}| < k - 1$ then $w_t = (l_1, l_2, \ldots, l_{|w_{t-1}|}, 1, -1, -1, \ldots, -1)$ with $|w_t| = k - 1$. Otherwise, if $|w_{t-1}| = k - 1$ then w_t must contain at least one letter -1, since $w_{t-1} \prec w_{2^k-1}$ and $|w_{t-1}| = |w_{2^k-1}|$. Thus we have $w_{t-1} = (l_1, l_2, \ldots, l_{s-1}, -1, 1, 1, \ldots, 1)$ where $s = \max\{1 \leq r \leq |w_{t-1}| : l_r = -1\}$. This gives $w_t = (l_1, l_2, \ldots, l_{s-1})$.

Now it is a simple calculation to infer the asserted equation from (13). □

The following two propositions are the central statements for the proof of Theorems 4 and 5. Let us remark that we do not demand that the right-hand sides of the considered estimates be non-negative for all f, n, k or T, n, k, respectively.

PROPOSITION 10. *Let $f \in C([0, 2]^m)$ and $n, k \in \{0, 1, 2, \ldots\}$. Then*

$$E(f, \operatorname{span}(\Psi_n)) \geq \frac{1}{2(2^k+1)}\,\omega\left(f, 2\left(2^{-4^n} - H(n)\right)\right) - \frac{2^k}{2(2^k+1)}\,\omega\left(f, 2H(n+k)\right).$$

Proof. There exist points $x_0 = (x_1^{(0)}, x_2^{(0)}, \ldots, x_m^{(0)}), x_1 = (x_1^{(1)}, x_2^{(1)}, \ldots, x_m^{(1)}) \in [0, 2]^m$ such that

$$\begin{aligned}
\omega\left(f, 2\left(2^{-4^n} - H(n)\right)\right) &= |f(x_0) - f(x_1)|, \\
d_{\max}(x_0, x_1) &\leq 2\left(2^{-4^n} - H(n)\right).
\end{aligned} \tag{22}$$

We have $[0, 2]^m = (M_1^{(n)} \cup CR_1^{(n)} \cup M_2^{(n)} \cup CR_2^{(n)} \cup \cdots \cup M_{2^{4^n}}^{(n)})^m$ and, by (10), $\operatorname{diam}(M_i^{(n)}) \geq 2(2^{-4^n} - H(n))$. Hence, for any $1 \leq j \leq m$, there exists an index $r_j \in \{1, 2, \ldots, 2^{4^n} - 1\}$ such that $x_j^{(0)}, x_j^{(1)} \in M_{r_j}^{(n)} \cup CR_{r_j}^{(n)} \cup M_{r_j+1}^{(n)}$. Consequently, x_0 and x_1 belong to the following subspace $Y \subseteq X$:

$$x_0, x_1 \in \left(M_{r_1}^{(n)} \cup CR_{r_1}^{(n)} \cup M_{r_1+1}^{(n)}\right) \times \cdots \times \left(M_{r_m}^{(n)} \cup CR_{r_m}^{(n)} \cup M_{r_m+1}^{(n)}\right) = Y. \tag{23}$$

Figure 6 illustrates the situation for $m = 2$.

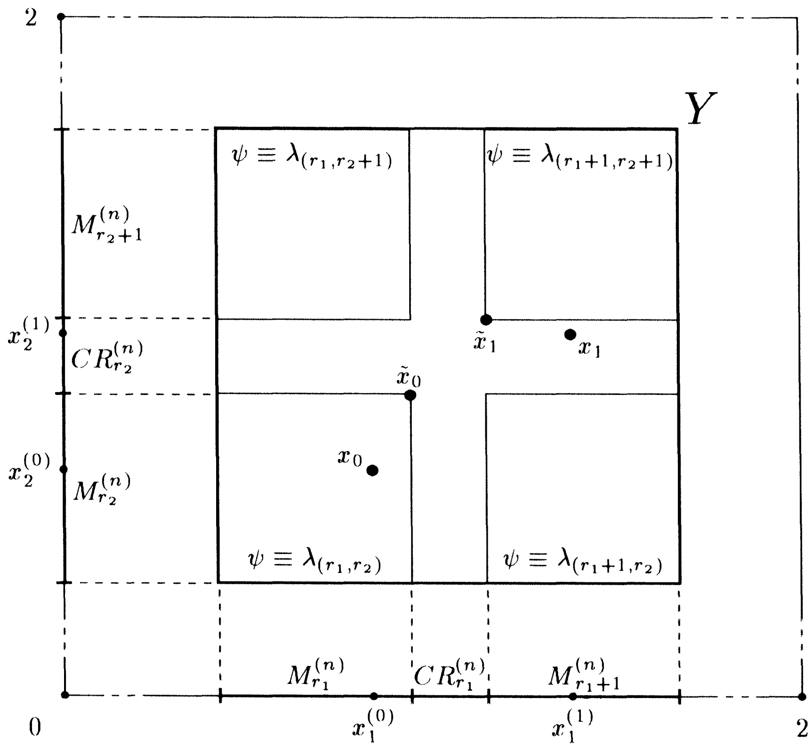

Figure 6. The subspace Y

Let $\psi \in \operatorname{span}(\Psi_n)$ be fixed arbitrarily. We shall show that

$$\|f-\psi\| \geq \frac{1}{2(2^k+1)}\,\omega\left(f, 2\left(2^{-4^n} - H(n)\right)\right) - \frac{2^k}{2(2^k+1)}\,\omega\left(f, 2H(n+k)\right). \quad (24)$$

We have

$$\psi = \sum_{(i_1,i_2,\ldots,i_m)\in\{1,2,\ldots,2^{4^n}\}^m} \lambda_{(i_1,i_2,\ldots,i_m)} \cdot \psi^{(n)}_{(i_1,i_2,\ldots,i_m)} \quad (25)$$

where $\operatorname{supp}(\psi^{(n)}_{(i_1,i_2,\ldots,i_m)}) = \operatorname{supp}(\varphi^{(n)}_{i_1}) \times \operatorname{supp}(\varphi^{(n)}_{i_2}) \times \cdots \times \operatorname{supp}(\varphi^{(n)}_{i_m})$ according to (21). We consider the following set I_Y of indices:

$$\begin{aligned} I_Y &= \left\{(i_1, i_2, \ldots, i_m) \in \left\{1, 2, \ldots, 2^{4^n}\right\}^m : \operatorname{supp}\left(\psi^{(n)}_{(i_1,i_2,\ldots,i_m)}\right) \cap Y \neq \emptyset\right\} \\ &= \{(r_1+s_1, r_2+s_2, \ldots, r_m+s_m) : s_1, s_2, \ldots, s_m \in \{0,1\}\}. \end{aligned}$$

Accordingly, $\{\psi^{(n)}_{(i_1,i_2,\ldots,i_m)}|_Y : (i_1, i_2, \ldots, i_m) \in I_Y\}$ is a partition of unity on Y and $\psi|_Y$ is a linear combination of this partition with the coefficients $\lambda_{(i_1,i_2,\ldots,i_m)}$,

$(i_1, i_2, \ldots, i_m) \in I_Y$. Hence $\psi(x)$ is a convex combination of these numbers for any $x \in Y$. We choose extremal coefficients such that

$$\psi(x) \in \left[\lambda_{(r_1+u_1, r_2+u_2, \ldots, r_m+u_m)}, \lambda_{(r_1+v_1, r_2+v_2, \ldots, r_m+v_m)}\right] \qquad \text{for} \quad x \in Y \tag{26}$$

with fixed $u_j, v_j \in \{0, 1\}$. Without loss of generality we assume $u_j \leq v_j$ for $1 \leq j \leq m$. (If $u_j > v_j$ then the j-th coordinate x_j has to be replaced by $2 - x_j$, which is nothing but a reflection of the cube $[0, 2]^m$.) We define $\tilde{x}_0 = (\tilde{x}_1^{(0)}, \tilde{x}_2^{(0)}, \ldots, \tilde{x}_m^{(0)})$ and $\tilde{x}_1 = (\tilde{x}_1^{(1)}, \tilde{x}_2^{(1)}, \ldots, \tilde{x}_m^{(1)})$ by

$$\tilde{x}_j^{(0)} = \max\left(M_{r_j+u_j}^{(n)}\right), \tag{27}$$

$$\tilde{x}_j^{(1)} = \begin{cases} \max\left(M_{r_j+v_j}^{(n)}\right) = \tilde{x}_j^{(0)} & \text{if} \quad u_j = v_j, \\ \min\left(M_{r_j+v_j}^{(n)}\right) & \text{if} \quad u_j \neq v_j \ \text{(i.e. } u_j = 0,\ v_j = 1). \end{cases} \tag{28}$$

Thus we obtain $\tilde{x}_0 \in M_{r_1+u_1}^{(n)} \times M_{r_2+u_2}^{(n)} \times \cdots \times M_{r_m+u_m}^{(n)}$ and $\tilde{x}_1 \in M_{r_1+v_1}^{(n)} \times M_{r_2+v_2}^{(n)} \times \cdots \times M_{r_m+v_m}^{(n)}$ and, therefore, $\psi(\tilde{x}_0) = \lambda_{(r_1+u_1, r_2+u_2, \ldots, r_m+u_m)}$ and $\psi(\tilde{x}_1) = \lambda_{(r_1+v_1, r_2+v_2, \ldots, r_m+v_m)}$ by (25). With the help of (22), (23) and (26), this yields

$$\begin{aligned} |\psi(\tilde{x}_0) - \psi(\tilde{x}_1)| &= \left|\lambda_{(r_1+u_1, r_2+u_2, \ldots, r_m+u_m)} - \lambda_{(r_1+v_1, r_2+v_2, \ldots, r_m+v_m)}\right| \\ &= \max_{x \in Y} \psi(x) - \min_{x \in Y} \psi(x) \\ &\geq |\psi(x_0) - \psi(x_1)| \\ &\geq |f(x_0) - f(x_1)| - |f(x_0) - \psi(x_0)| - |\psi(x_1) - f(x_1)| \\ &\geq \omega\left(f, 2\left(2^{-4^n} - H(n)\right)\right) - 2\|f - \psi\|. \end{aligned} \tag{29}$$

Moreover, (27) and (28) together with (10) give

$$\begin{aligned} d_{\max}(\tilde{x}_0, \tilde{x}_1) &= \max_{1 \leq j \leq m} \left|\tilde{x}_j^{(0)} - \tilde{x}_j^{(1)}\right| \\ &\leq \max_{1 \leq j \leq m} \left(\min\left(M_{r_j+1}^{(n)}\right) - \max\left(M_{r_j}^{(n)}\right)\right) = 2H(n). \end{aligned}$$

Accordingly, by (29),

$$\begin{aligned} \omega(f, 2H(n)) &\geq |f(\tilde{x}_0) - f(\tilde{x}_1)| \\ &\geq |\psi(\tilde{x}_0) - \psi(\tilde{x}_1)| - |\psi(\tilde{x}_0) - f(\tilde{x}_0)| - |f(\tilde{x}_1) - \psi(\tilde{x}_1)| \\ &\geq \omega\left(f, 2\left(2^{-4^n} - H(n)\right)\right) - 4\|f - \psi\|. \end{aligned}$$

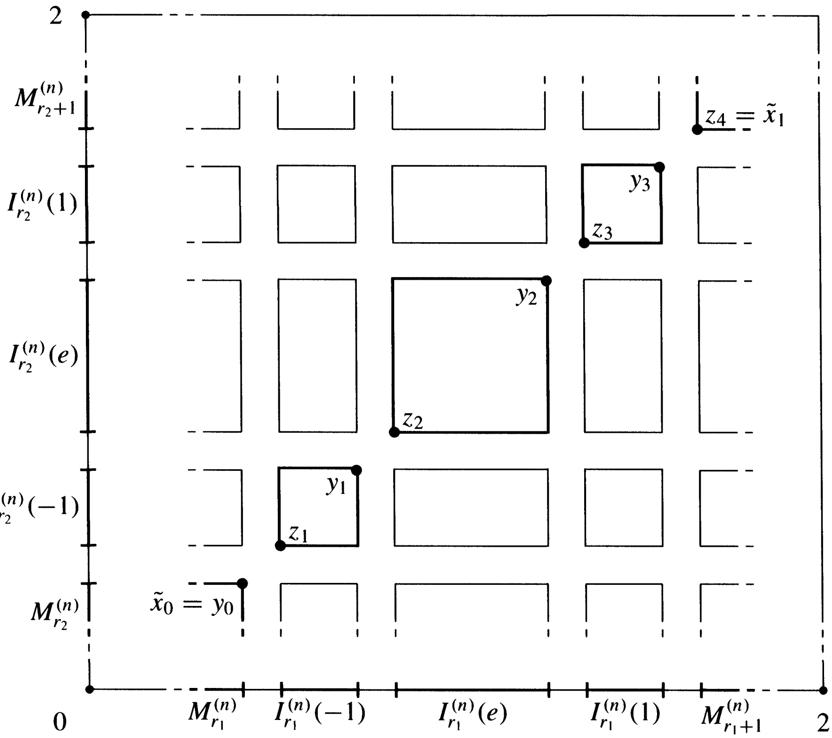

Figure 7. The points $y_0, y_1, \dots, y_{2^k-1}$ and $z_1, z_2, \dots, z_{2^k}$

This yields

$$\|f - \psi\| \geq \frac{1}{4}\,\omega\left(f, 2\left(2^{-4^n} - H(n)\right)\right) - \frac{1}{4}\,\omega\left(f, 2H(n)\right),$$

which is our claim (24) if $k = 0$.

Now let $k \geq 1$. We consider the set $\{w_1, w_2, \dots, w_{2^k-1}\} = \{w \in \{-1, 1\}^* : |w| < k\}$ with $w_1 \prec w_2 \prec \cdots \prec w_{2^k-1}$ as in Proposition 9. We define points $y_0, y_1, \dots, y_{2^k-1}$ and $z_1, z_2, \dots, z_{2^k}$ where

$$y_0 = \tilde{x}_0 \qquad \text{and} \qquad z_{2^k} = \tilde{x}_1. \tag{30}$$

The j-th coordinates of the remaining points $y_t = (y^{(t)}_1, y^{(t)}_2, \dots, y^{(t)}_m)$ and $z_t = (z^{(t)}_1, z^{(t)}_2, \dots, z^{(t)}_m)$, $1 \leq t < 2^k$, are given as follows (cf. Figure 7):

If $u_j = v_j$ then we put $y_j^{(t)} = z_j^{(t)} = \tilde{x}_j^{(0)}$ for all $1 \le t < 2^k$. We have $y_j^{(0)} = \tilde{x}_j^{(0)} = \tilde{x}_j^{(1)} = z_j^{(2^k)}$ by (28). Thus we obtain

$$\left|z_j^{(t)} - y_j^{(t-1)}\right| = 0 \le 2H(n+k) \qquad \text{for } 1 \le t \le 2^k, \tag{31}$$

$$\varphi\left(z_j^{(t)}\right) = \varphi\left(y_j^{(t)}\right) \qquad \text{for } 1 \le t < 2^k,\ \varphi \in \operatorname{span}(\Phi_n). \tag{32}$$

Otherwise, if $u_j \ne v_j$, i.e., $u_j = 0$ and $v_j = 1$, then we define

$$z_j^{(t)} = \min\left(I_{r_t}^{(n)}(w_t)\right), \quad y_j^{(t)} = \max\left(I_{r_t}^{(n)}(w_t)\right) \qquad \text{for } 1 \le t < 2^k.$$

Any function from $\operatorname{span}(\Phi_n)$ is constant on the intervals $I_i^{(n)}(w)$. Thus we obtain

$$\varphi\left(z_j^{(t)}\right) = \varphi\left(y_j^{(t)}\right) \qquad \text{for } 1 \le t < 2^k,\ \varphi \in \operatorname{span}(\Phi_n). \tag{33}$$

Definitions (30), (27) and (28) give

$$y_j^{(0)} = \tilde{x}_j^{(0)} = \max\left(M_{r_j}^{(n)}\right), \quad z_j^{(2^k)} = \tilde{x}_j^{(1)} = \min\left(M_{r_j+1}^{(n)}\right).$$

Thus we can apply Proposition 9 and get

$$\left|z_j^{(t)} - y_j^{(t-1)}\right| = 2H(n+k) \qquad \text{for } 1 \le t \le 2^k. \tag{34}$$

From (31) and (34) it follows that

$$d_{\max}(z_t, y_{t-1}) \le 2H(n+k) \qquad \text{for } 1 \le t \le 2^k. \tag{35}$$

Formulas (32) and (33) show that $\psi^{(n)}(z_t) = \psi^{(n)}(y_t)$ for any $1 \le t < 2^k$ and any function $\psi^{(n)} \in \Psi_n$ by (21). Thus, by (25),

$$\psi(z_t) = \psi(y_t) \qquad \text{for } 1 \le t < 2^k.$$

On account of (30) we conclude that

$$\begin{aligned} |\psi(\tilde{x}_0) - \psi(\tilde{x}_1)| &= |\psi(y_0) - \psi(z_{2^k})| \\ &\le \sum_{t=1}^{2^k-1} \left(|\psi(y_{t-1}) - \psi(z_t)| + |\psi(z_t) - \psi(y_t)|\right) + |\psi(y_{2^k-1}) - \psi(z_{2^k})| \\ &= \sum_{t=1}^{2^k} |\psi(y_{t-1}) - \psi(z_t)|. \end{aligned}$$

Hence there exists an index $t_0 \in \{1, 2, \dots, 2^k\}$ such that

$$|\psi(y_{t_0-1}) - \psi(z_{t_0})| \ge \frac{1}{2^k} |\psi(\tilde{x}_0) - \psi(\tilde{x}_1)|.$$

From this and formulas (35) and (29) we obtain

$$\begin{aligned}\omega(f, 2H(n+k)) &\geq |f(y_{t_0-1}) - f(z_{t_0})| \\ &\geq |\psi(y_{t_0-1}) - \psi(z_{t_0})| - |\psi(y_{t_0-1}) - f(y_{t_0-1})| - |f(z_{t_0}) - \psi(z_{t_0})| \\ &\geq \frac{1}{2^k} |\psi(\tilde{x}_0) - \psi(\tilde{x}_1)| - 2\|f - \psi\| \\ &\geq \frac{1}{2^k} \omega\left(f, 2\left(2^{-4^n} - H(n)\right)\right) - \frac{2}{2^k} \|f - \psi\| - 2\|f - \psi\| \\ &= \frac{1}{2^k} \omega\left(f, 2\left(2^{-4^n} - H(n)\right)\right) - \frac{2(2^k+1)}{2^k} \|f - \psi\|.\end{aligned}$$

Consequently,

$$\|f - \psi\| \geq \frac{1}{2(2^k+1)} \omega\left(f, 2\left(2^{-4^n} - H(n)\right)\right) - \frac{2^k}{2(2^k+1)} \omega(f, 2H(n+k)).$$

This is our claim (24). □

PROPOSITION 11. *Let E be a Banach space, $T \in \mathcal{L}(E, C([0,2]^m))$ and $n, k \in \{0, 1, 2, \ldots\}$. Then*

$$E(T, \operatorname{span}(\Psi_n)) \geq \frac{1}{2(2^k+1)} \omega\left(T, 2\left(2^{-4^n} - H(n)\right)\right) - \frac{2^k}{2(2^k+1)} \omega(T, 2H(n+k)).$$

Proof. According to Proposition 10 we can estimate the norm $\|T - A\|$ for any $A \in \mathcal{L}(E, C([0,2]^m))$ with $A(E) \subseteq \operatorname{span}(\Psi_n)$ as follows:

$$\begin{aligned}\|T - A\| &= \sup_{\|z\|\leq 1} \|Tz - Az\| \geq \sup_{\|z\|\leq 1} E(Tz, \operatorname{span}(\Psi_n)) \\ &\geq \sup_{\|z\|\leq 1} \left(\tfrac{1}{2(2^k+1)} \omega\left(Tz, 2\left(2^{-4^n} - H(n)\right)\right) - \tfrac{2^k}{2(2^k+1)} \omega(Tz, 2H(n+k))\right) \\ &\geq \tfrac{1}{2(2^k+1)} \sup_{\|z_1\|\leq 1} \omega\left(Tz_1, 2\left(2^{-4^n} - H(n)\right)\right) - \tfrac{2^k}{2(2^k+1)} \sup_{\|z_2\|\leq 1} \omega(Tz_2, 2H(n+k)) \\ &= \tfrac{1}{2(2^k+1)} \omega\left(T, 2\left(2^{-4^n} - H(n)\right)\right) - \tfrac{2^k}{2(2^k+1)} \omega(T, 2H(n+k)).\end{aligned}$$

Passing to the infimum over all operators A completes the proof. □

Next we need a statement on the behaviour of the modulus of continuity of Hölder continuous functions and operators, respectively. We shall prove this fact for the wider class of *metrically convex* compact metric spaces X instead of $X = [0,2]^m$. A space (X, d) is called metrically convex if for any two points $x, y \in X$ and any $\lambda \in (0, 1)$ there exists a point $z \in X$ such that $d(x, z) = \lambda d(x, y)$ and $d(z, y) = (1 - \lambda) d(x, y)$. This concept is due to K. Menger (cf. [Me]). The modulus of continuity $\omega(f, \cdot)$ of a function $f \in C(X)$ has the property of subadditivity if X is metrically convex, i.e., $\omega(f, \delta_1 + \delta_2) \leq \omega(f, \delta_1) + \omega(f, \delta_2)$ for $\delta_1, \delta_2 \geq 0$ (cf. [Go]). In particular, $\omega(f, k\delta) \leq k\,\omega(f, \delta)$ for $k \in \{1, 2, 3, \ldots\}$, $\delta \geq 0$. Clearly, if T is an operator from a Banach space into $C(X)$ then $\omega(T, \cdot)$ has the same property.

PROPOSITION 12. *Let (X, d) be a metrically convex compact metric space, $0 < \alpha \leq 1$, and let $\omega(\cdot)$ be the modulus of continuity of a function $f \in C(X)$ or of an operator $T \in \mathcal{L}(E, C(X))$ mapping a Banach space E into $C(X)$, such that f or T is Hölder continuous of type α, respectively. Suppose that f is non-constant and that $T(E)$ does not consist of constant functions only, respectively.*

Then, for any functions $g_1, g_2 : (0, \infty) \to (0, \infty)$ with $g_1^\alpha = o(g_2)$ as $\delta \to +0$ and $\lim_{\delta\to+0} g_2(\delta) = 0$, we have

$$\lim_{\delta\to+0} \frac{\omega(g_1(\delta))}{\omega(g_2(\delta))} = 0.$$

Proof. We assume $\omega(\cdot) = \omega(f, \cdot)$. (The same proof works if f is replaced by T.) Note that $\omega(g_2(\delta)) > 0$, since f is non-constant. According to the Hölder continuity there exists $\delta_0 > 0$ such that

$$\frac{\omega(\delta_0)}{\delta_0^\alpha} \geq \frac{1}{2} \sup_{\delta>0} \frac{\omega(\delta)}{\delta^\alpha} = \frac{1}{2} |f|_\alpha.$$

Let $\delta > 0$ be fixed. We can assume that δ is sufficiently small so that $g_2(\delta) \leq \delta_0$. Then there exists $l \geq 0$ such that

$$2^l g_2(\delta) \leq \delta_0 < 2^{l+1} g_2(\delta).$$

This implies that

$$\begin{aligned}\frac{\omega(g_1(\delta))}{\omega(g_2(\delta))} &\leq \frac{\omega(g_1(\delta))}{\omega(g_2(\delta))} \frac{2}{|f|_\alpha} \frac{\omega(\delta_0)}{\delta_0^\alpha} \leq \frac{|f|_\alpha\,(g_1(\delta))^\alpha}{\omega(g_2(\delta))} \frac{2}{|f|_\alpha} \frac{\omega(2^{l+1} g_2(\delta))}{\delta_0^\alpha} \\ &\leq 2 \frac{(g_1(\delta))^\alpha}{\omega(g_2(\delta))} \frac{2^{l+1}\,\omega(g_2(\delta))}{\delta_0^\alpha} = \frac{4}{\delta_0^\alpha} 2^l\,(g_1(\delta))^\alpha \leq \frac{4}{\delta_0^\alpha} \frac{\delta_0}{g_2(\delta)} (g_1(\delta))^\alpha.\end{aligned}$$

Passing to the limit as $\delta \to +0$ gives our claim, for $g_1^\alpha = o(g_2)$. □

Now a lower estimate for the approximation quantities, even slightly sharper than Theorems 4 and 5, can be given.

PROPOSITION 13. *Let $f \in C([0, 2]^m)$ be non-constant and Hölder continuous of type α, $0 < \alpha \leq 1$, and let $k \geq 0$ be chosen such that $4^{-(k+1)} < \alpha \leq 1$. Then*

$$\liminf_{n\to\infty} \frac{E(f, \operatorname{span}(\Psi_n))}{\omega\left(f, 2\left(2^{-4^n} - H(n)\right)\right)} \geq \frac{1}{2(2^k + 1)}.$$

Moreover, if E is a Banach space and the operator $T \in \mathcal{L}(E, C([0,2]^m))$ is Hölder continuous of type α such that the image $T(E)$ does not consist of constant functions only, then

$$\liminf_{n\to\infty} \frac{E(T, \operatorname{span}(\Psi_n))}{\omega\left(T, 2\left(2^{-4^n} - H(n)\right)\right)} \geq \frac{1}{2(2^k+1)}.$$

Proof. Proposition 10 gives

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{E(f, \operatorname{span}(\Psi_n))}{\omega\left(f, 2\left(2^{-4^n} - H(n)\right)\right)} \\
&\geq \frac{1}{2(2^k+1)} - \frac{2^k}{2(2^k+1)} \limsup_{n\to\infty} \frac{\omega(f, 2H(n+k))}{\omega\left(f, 2\left(2^{-4^n} - H(n)\right)\right)}.
\end{aligned} \tag{36}$$

Now we consider the functions $G_1(n) = 2H(n+k)$ and $G_2(n) = 2(2^{-4^n} - H(n))$. Clearly, $\lim_{n\to\infty} G_2(n) = 0$. Moreover, $G_1^\alpha = o(G_2)$ as $n \to \infty$. Indeed, definition (7) and estimate (9) yield

$$\begin{aligned}
\lim_{n\to\infty} \frac{(G_1(n))^\alpha}{G_2(n)} &= \lim_{n\to\infty} \frac{\left(2\left(2\cdot 2^{-4^{(n+k+1)}} + H(n+k+1)\right)\right)^\alpha}{2\left(2^{-4^n} - 2\cdot 2^{-4^{(n+1)}} - H(n+1)\right)} \\
&\leq \lim_{n\to\infty} \frac{\left(2\left(3\cdot 2^{-4^{(n+k+1)}}\right)\right)^\alpha}{2\left(2^{-4^n} - 3\cdot 2^{-4^{(n+1)}}\right)} \\
&\leq \lim_{n\to\infty} \frac{6^\alpha \cdot 2^{-\alpha 4^{(n+k+1)}}}{2^{-4^n}} \\
&= \lim_{n\to\infty} 6^\alpha \cdot 2^{-4^n(\alpha 4^{(k+1)} - 1)} = 0,
\end{aligned}$$

for $\alpha\, 4^{(k+1)} - 1 > 0$. Hence we can apply Proposition 12 and obtain

$$\lim_{n\to\infty} \frac{\omega(f, 2H(n+k))}{\omega\left(f, 2\left(2^{-4^n} - H(n)\right)\right)} = \lim_{n\to\infty} \frac{\omega(f, G_1(n))}{\omega(f, G_2(n))} = 0.$$

Combining this with (36) gives the desired estimate.

The estimate for the operator T can similarly be obtained from Proposition 11. □

Proof of Theorems 4 *and* 5. Let α and f be as in Theorem 4. We choose $k \geq 0$ such that $4^{-(k+1)} < \alpha \leq 4^{-k}$. Then Proposition 13 yields

$$\liminf_{n\to\infty} \frac{E(f, \operatorname{span}(\Psi_n))}{\omega\left(f, 2\left(2^{-4^n} - H(n)\right)\right)} \geq \frac{1}{2(2^k+1)} \geq \frac{1}{2\left(\alpha^{-\frac{1}{2}} + 1\right)}. \tag{37}$$

On the other hand, $2\left(2^{-4^n} - H(n)\right) > \frac{1}{2^{4^n}-1}$ for $n \geq 1$, since

$$\begin{aligned} 2\left(2^{-4^n} - H(n)\right) &= 2\left(2^{-4^n} - 2 \cdot 2^{-4^{(n+1)}} - H(n+1)\right) \\ &> 2\left(2^{-4^n} - 3 \cdot 2^{-4^{(n+1)}}\right) = \frac{2\left(1 - 3 \cdot 2^{-3 \cdot 4^n}\right)}{2^{4^n}} \\ &> \frac{2\left(1 - 3 \cdot 2^{-3 \cdot 4^0}\right)}{2^{4^n}} = \frac{5}{4} \cdot \frac{1}{2^{4^n}} > \frac{1}{2^{4^n} - 1} \end{aligned}$$

by (7) and (9). Consequently,

$$\liminf_{n\to\infty} \frac{E(f, \operatorname{span}(\Psi_n))}{\omega\left(f, \frac{1}{2^{4^n}-1}\right)} \geq \liminf_{n\to\infty} \frac{E(f, \operatorname{span}(\Psi_n))}{\omega\left(f, 2\left(2^{-4^n} - H(n)\right)\right)}.$$

Combining this with (37) completes the proof of Theorem 4.

Obviously, the same proof works for Theorem 5. □

4. A completed approximation scheme

Although Theorems 2–5 give pleasant error estimates, the above approximation scheme has the essential disadvantage that the dimensions $\dim(\operatorname{span}(\Psi_n)) = 2^{m \cdot 4^n}$ increase rapidly. In the sequel we shall complete the scheme such that we obtain a chain $(\tilde{\Psi}_n)_{n=1}^{\infty}$ of peaked partitions of unity $\tilde{\Psi}_n$ on $[0,2]^m$ of cardinality $\operatorname{card}(\tilde{\Psi}_n) = n$ with similar approximation properties. Of course, the partitions $\tilde{\Psi}_n$ will not be controllable in general.

We begin with the definition of a chain $(\tilde{\Phi}_{4^l})_{l=0}^{\infty}$ of partitions of unity $\tilde{\Phi}_{4^l} = \{\tilde{\varphi}_1^{(4^l)}, \tilde{\varphi}_2^{(4^l)}, \ldots, \tilde{\varphi}_{4^l}^{(4^l)}\}$ on the interval $I = [0,2]$. We shall use the partitions Φ_n, $n \geq 1$, constructed in the second section in order to preserve the approximation properties for the new scheme. Besides that, we try to choose the additional partitions as simple as possible, such that a function $\tilde{\varphi}_i^{(4^l)}$ is closely related to the characteristic function of the interval $[2(i-1)\cdot 4^{-l}, 2i \cdot 4^{-l}] \subseteq I$. Clearly, $\tilde{\Phi}_{4^0} = \{\tilde{\varphi}_1^{(1)}\} = \{\mathbf{1}_{[0,2]}\}$. For any $l \geq 1$ we introduce the corresponding exponent $e(l) \in \{0, 1, 2, \ldots\}$ by

$$4^l \in \left\{2^{4^{e(l)}}, 2^{4^{e(l)}} + 1, \ldots, 2^{4^{e(l)+1}} - 1\right\}. \tag{38}$$

If $4^l = 2^{4^{e(l)}}$ then we let $\tilde{\Phi}_{4^l} = \Phi_{e(l)}$, i.e., we use the partitions introduced in the second section. In particular,

$$\tilde{\varphi}_i^{(4^l)} = \varphi_i^{(e(l))} \qquad \text{for } 1 \leq i \leq 4^l.$$

If $4^l > 2^{4^{e(l)}}$ then we define $\tilde{\Phi}_{4^l} = \{\tilde{\varphi}_1^{(4^l)}, \tilde{\varphi}_2^{(4^l)}, \ldots, \tilde{\varphi}_{4^l}^{(4^l)}\}$ by

$$\tilde{\varphi}_1^{(4^l)} = \left(\varphi_1^{(e(l)+1)} + \cdots + \varphi_{2^{4^{e(l)+1}-2l}-1}^{(e(l)+1)}\right) + \frac{1}{2}\left(\varphi_{2^{4^{e(l)+1}-2l}}^{(e(l)+1)} + \varphi_{2^{4^{e(l)+1}-2l}+1}^{(e(l)+1)}\right),$$

$$\tilde{\varphi}_i^{(4^l)} = \frac{1}{2}\left(\varphi^{(e(l)+1)}_{(i-1)\cdot 2^{4^{e(l)+1}-2l}} + \varphi^{(e(l)+1)}_{(i-1)\cdot 2^{4^{e(l)+1}-2l}+1}\right) + \left(\varphi^{(e(l)+1)}_{(i-1)\cdot 2^{4^{e(l)+1}-2l}+2} + \cdots \right.$$
$$\left. \cdots + \varphi^{(e(l)+1)}_{i\cdot 2^{4^{e(l)+1}-2l}-1}\right) + \frac{1}{2}\left(\varphi^{(e(l)+1)}_{i\cdot 2^{4^{e(l)+1}-2l}} + \varphi^{(e(l)+1)}_{i\cdot 2^{4^{e(l)+1}-2l}+1}\right) \quad , 1 < i < 4^l,$$

$$\tilde{\varphi}_{4^l}^{(4^l)} = \frac{1}{2}\left(\varphi^{(e(l)+1)}_{(4^l-1)\cdot 2^{4^{e(l)+1}-2l}} + \varphi^{(e(l)+1)}_{(4^l-1)\cdot 2^{4^{e(l)+1}-2l}+1}\right)$$
$$+ \left(\varphi^{(e(l)+1)}_{(4^l-1)\cdot 2^{4^{e(l)+1}-2l}+2} + \cdots + \varphi^{(e(l)+1)}_{2^{4^{e(l)+1}}}\right).$$

Proposition 8 implies that these formulas remain valid if $4^l = 2^{4^{e(l)}}$.

PROPOSITION 14. *$(\tilde{\Phi}_{4^l})_{l=0}^{\infty}$ is a chain of peaked partitions of unity on I with* $\operatorname{card}(\tilde{\Phi}_{4^l}) = 4^l$, *such that $\tilde{\Phi}_{2^{4^n}} = \Phi_n$ for $n \geq 1$. Moreover, the partition functions fulfil the uniformity condition*

$$\varepsilon_1\left(\operatorname{supp}\left(\tilde{\varphi}_i^{(4^l)}\right)\right) \leq \frac{7}{4} \cdot 4^{-l}$$

for $1 \leq i \leq 4^l$.

Proof. We have to show that $\tilde{\Phi}_{4^l} \subseteq \operatorname{span}(\tilde{\Phi}_{4^{l+1}})$ for $l \geq 0$. Clearly, this is true for $l = 0$. So let $l \geq 1$. If $e(l) = e(l+1)$ then the above definition gives

$$\tilde{\varphi}_i^{(4^l)} = \tilde{\varphi}_{4i-3}^{(4^{l+1})} + \tilde{\varphi}_{4i-2}^{(4^{l+1})} + \tilde{\varphi}_{4i-1}^{(4^{l+1})} + \tilde{\varphi}_{4i}^{(4^{l+1})}.$$

If $e(l) < e(l+1)$, i.e. $e(l+1) = e(l)+1$, then we have $4^{l+1} = 2^{4^{e(l+1)}}$, since 4^{l+1} must be the smallest integral power of 4 in $\{2^{4^{e(l)+1}}, 2^{4^{e(l)+1}}+1, \ldots, 2^{4^{e(l)+2}}-1\}$ by (38). Consequently, $\tilde{\Phi}_{4^{l+1}} = \Phi_{e(l)+1}$ and therefore $\tilde{\Phi}_{4^l} \subseteq \operatorname{span}(\tilde{\Phi}_{4^{l+1}})$ again by the above formulas.

The second part of Proposition 14 is trivial if $l = 0$. If $l \geq 1$ then we observe that $\operatorname{supp}(\tilde{\varphi}_i^{(4^l)})$ is the union of the supports $C_j^{(e(l)+1)}$ of certain functions $\varphi_j^{(e(l)+1)}$. We obtain

$$\operatorname{supp}\left(\tilde{\varphi}_i^{(4^l)}\right) = \begin{cases} \left[0,\ 2\cdot 4^{-l} + H(e(l))\right), & i = 1, \\ \left(2(i-1)\cdot 4^{-l} - H(e(l)),\ 2i\cdot 4^{-l} + H(e(l))\right), & 1 < i < 4^l, \\ \left(2(4^l-1)\cdot 4^{-l} - H(e(l)),\ 2\right], & i = 4^l, \end{cases}$$

in accordance with (8). This yields

$$\varepsilon_1\left(\operatorname{supp}\left(\tilde{\varphi}_i^{(4^l)}\right)\right) \leq 4^{-l} + H(e(l)). \tag{39}$$

On the other hand we can give the estimate

$$4^{-l} \geq 4 \cdot 2^{-4^{e(l)+1}} \qquad \text{for } l \geq 1. \tag{40}$$

Indeed, (38) implies that $2^{2l} = 4^l < 2^{4^{e(l)+1}}$, hence $l < \frac{1}{2} \cdot 4^{e(l)+1}$, $l \le \frac{1}{2} \cdot 4^{e(l)+1} - 1$ and therefore $4^{-l} \ge 4^{-\frac{1}{2} \cdot 4^{e(l)+1}+1} = 4 \cdot 2^{-4^{e(l)+1}}$.

Finally, (39), (7), (9) and (40) amount to

$$\varepsilon_1\left(\operatorname{supp}\left(\tilde{\varphi}_i^{(4^l)}\right)\right) < 4^{-l} + 3 \cdot 2^{-4^{e(l)+1}} \le 4^{-l} + \frac{3}{4} \cdot 4^{-l} = \frac{7}{4} \cdot 4^{-l},$$

which is our claim. □

Now we can introduce the chain $(\tilde{\Psi}_n)_{n=1}^{\infty}$ of peaked partitions of unity on the cube $[0,2]^m$. If $n = 4^{ml}$ for some $l \ge 0$ then we define $\tilde{\Psi}_{4^{ml}}$ by the aid of the one-dimensional functions from $\tilde{\Phi}_{4^l}$ as in the second section, that is,

$$\begin{aligned} &\tilde{\Psi}_{4^{ml}} = \left\{\tilde{\psi}^{(4^{ml})}_{(i_1,i_2,\dots,i_m)} : i_1, i_2, \dots, i_m \in \left\{1, 2, \dots, 4^l\right\}\right\} \quad \text{with} \\ &\tilde{\psi}^{(4^{ml})}_{(i_1,i_2,\dots,i_m)}(x_1, x_2, \dots, x_m) = \tilde{\varphi}^{(4^l)}_{i_1}(x_1)\tilde{\varphi}^{(4^l)}_{i_2}(x_2)\dots\tilde{\varphi}^{(4^l)}_{i_m}(x_m). \end{aligned} \tag{41}$$

(Note that $\tilde{\Psi}_{4^{ml}} = \Psi_{e(l)}$ if $4^l = 2^{4^{e(l)}}$.) The remaining peaked partitions $\tilde{\Psi}_n$ of cardinality $\operatorname{card}(\tilde{\Psi}_n) = n \ne 4^{ml}$ have to be chosen such that the chain condition $\operatorname{span}(\tilde{\Psi}_1) \subseteq \operatorname{span}(\tilde{\Psi}_2) \subseteq \operatorname{span}(\tilde{\Psi}_3) \subseteq \cdots$ remains valid. The existence of such partitions (or, in other words, the existence of suitable intermediate spaces being isometrically isomorphic to l_∞^n) is based on Lemma 3.2 from [Mi/Pe2]: *If the subspace E of l_∞^s is isometrically isomorphic to l_∞^r ($r < s$), then there is a subspace $F \supset E$ of l_∞^s which is isometrically isomorphic to l_∞^{r+1}.*

Corresponding approximation quantities can be defined as in the last section. We obtain the following estimates from above.

THEOREM 6. *There exist positive operators $\tilde{A}_n \in \mathcal{L}(C([0,2]^m))$, $n \ge 1$, mapping $C([0,2]^m)$ into $\operatorname{span}(\tilde{\Psi}_n)$ such that:*
(a) For any $f \in C([0,2]^m)$,

$$E\left(f, \operatorname{span}\left(\tilde{\Psi}_n\right)\right) \le \left\|f - \tilde{A}_n f\right\| \le 7\,\omega(f, \varepsilon_n([0,2]^m)).$$

(b) For any Banach space E and any operator $T \in \mathcal{L}(E, C([0,2]^m))$,

$$E\left(T, \operatorname{span}\left(\tilde{\Psi}_n\right)\right) \le \left\|T - \tilde{A}_n T\right\| \le 7\,\omega(T, \varepsilon_n([0,2]^m)).$$

Proof. Let n be fixed. We choose $l \ge 0$ such that $4^{ml} \le n < 4^{m(l+1)}$. Proposition 14 and definition (41) show that

$$\varepsilon_1\left(\operatorname{supp}\left(\tilde{\psi}^{(4^{ml})}_{(i_1,i_2,\dots,i_m)}\right)\right) = \max_{1 \le j \le m} \varepsilon_1\left(\operatorname{supp}\left(\tilde{\varphi}^{(4^l)}_{i_j}\right)\right) \le 7 \cdot 4^{-(l+1)}$$

for $(i_1, i_2, \dots, i_m) \in \{1, 2, \dots, 4^l\}^m$. Hence there exist points $x_{(i_1,i_2,\dots,i_m)}$ such that

$$\operatorname{supp}\left(\tilde{\psi}^{(4^{ml})}_{(i_1,i_2,\dots,i_m)}\right) \subseteq B\left(x_{(i_1,i_2,\dots,i_m)}, 7 \cdot 4^{-(l+1)}\right). \tag{42}$$

Let

$$\tilde{A}_n f = \sum_{(i_1,i_2,\ldots,i_m)\in\{1,2,\ldots,4^l\}^m} f\left(x_{(i_1,i_2,\ldots,i_m)}\right)\tilde{\psi}^{(4^{ml})}_{(i_1,i_2,\ldots,i_m)}$$

for $f \in C([0,2]^m)$. Clearly, $\tilde{A}_n$ is a positive operator with $\tilde{A}_n(C([0,2]^m)) = \operatorname{span}(\tilde{\Psi}_{4^{ml}}) \subseteq \operatorname{span}(\tilde{\Psi}_n)$. As in Proposition 1 we obtain

$$\left\| f - \tilde{A}_n f \right\| \leq \omega\left(f, 7\cdot 4^{-(l+1)}\right)$$

by (42). The subadditivity of $\omega(f,\cdot)$ and formula (4) give

$$\left\| f - \tilde{A}_n f \right\| \leq 7\omega\left(f, 4^{-(l+1)}\right) = 7\omega(f, \varepsilon_{4^{m(l+1)}}([0,2]^m)) \leq 7\omega(f, \varepsilon_n([0,2]^m)).$$

This is the desired estimate. Part (b) is a simple conclusion. □

Again we can prove inverse estimates for Hölder continuous functions and operators, respectively.

THEOREM 7. (a) *Let $f \in C([0,2]^m)$ be non-constant and Hölder continuous of type α, $0 < \alpha \leq 1$. Then*

$$\liminf_{n\to\infty} \frac{E\left(f, \operatorname{span}\left(\tilde{\Psi}_n\right)\right)}{\omega\left(f, \varepsilon_n([0,2]^m)\right)} \geq \frac{1}{16\left(2\alpha^{-\frac{1}{2}}+1\right)}.$$

(b) *Let E be a Banach space and let $T \in \mathcal{L}(E, C([0,2]^m))$ be Hölder continuous of type α, $0 < \alpha \leq 1$, such that the image $T(E)$ does not consist of constant functions only. Then*

$$\liminf_{n\to\infty} \frac{E\left(T, \operatorname{span}\left(\tilde{\Psi}_n\right)\right)}{\omega\left(T, \varepsilon_n([0,2]^m)\right)} \geq \frac{1}{16\left(2\alpha^{-\frac{1}{2}}+1\right)}.$$

Although the main ideas coincide with those of the last section, we present the main steps of the proof of Theorem 7. As the analogue of Propositions 10 and 11 we obtain the following.

PROPOSITION 15. (a) *Let $f \in C([0,2]^m)$, $l \geq 1$ and $k \geq 0$. Then*

$$\begin{aligned} E\left(f, \operatorname{span}\left(\tilde{\Psi}_{4^{ml}}\right)\right) \geq\ & \frac{1}{2(2^k+1)}\,\omega\left(f, 2\left(4^{-l} - H(e(l))\right)\right) \\ & - \frac{2^k}{2(2^k+1)}\,\omega\left(f, 2H(e(l)+k)\right). \end{aligned}$$

(b) *Let E be a Banach space, $T \in \mathcal{L}(E, C([0,2]^m))$, $l \geq 1$ and $k \geq 0$. Then*

$$E\left(T, \operatorname{span}\left(\tilde{\Psi}_{4^{ml}}\right)\right) \geq \frac{1}{2(2^k+1)}\,\omega\left(T, 2\left(4^{-l} - H(e(l))\right)\right) - \frac{2^k}{2(2^k+1)}\,\omega\left(T, 2H(e(l)+k)\right).$$

Proof. We decompose the support of the function $\tilde{\varphi}_i^{(4^l)} \in \Phi_{4^l}$ into the middle part $\tilde{M}_i^{(4^l)}$ and the critical parts $\tilde{CL}_i^{(4^l)}$ and $\tilde{CR}_i^{(4^l)}$ as we did with the functions $\varphi_i^{(n)} \in \Phi_n$ in the second section. According to the definition of $\tilde{\varphi}_i^{(4^l)}$ we obtain

$$\tilde{CL}_i^{(4^l)} = \left(\inf\left(\operatorname{supp}\left(\varphi^{(e(l)+1)}_{(i-1)\cdot 2^{4^{e(l)+1}-2l}}\right)\right), \sup\left(\operatorname{supp}\left(\varphi^{(e(l)+1)}_{(i-1)\cdot 2^{4^{e(l)+1}-2l}+1}\right)\right)\right)$$

for $1 < i \leq 4^l$. We observe that $\tilde{CL}_i^{(4^l)}$ is nothing but a translate of $CL_j^{(e(l))}$, $1 < j \leq 2^{4^{e(l)}}$, since $CL_j^{(e(l))}$ admits a similar representation with $\varphi^{(e(l)+1)}_{(j-1)\cdot 2^{3\cdot 4^{e(l)}}}$ and $\varphi^{(e(l)+1)}_{(j-1)\cdot 2^{3\cdot 4^{e(l)}}+1}$ because of Proposition 8. Moreover, *the graph of the restriction of $\tilde{\varphi}_i^{(4^l)}$ to $\tilde{CL}_i^{(4^l)}$ coincides with the graph of the restriction of $\varphi_j^{(e(l))}$ to $CL_j^{(e(l))}$*, since

$$\tilde{\varphi}_i^{(4^l)}\Big|_{\tilde{CL}_i^{(4^l)}} = \left(\frac{1}{2}\left(\varphi^{(e(l)+1)}_{(i-1)\cdot 2^{4^{e(l)+1}-2l}} + \varphi^{(e(l)+1)}_{(i-1)\cdot 2^{4^{e(l)+1}-2l}+1}\right) + \varphi^{(e(l)+1)}_{(i-1)\cdot 2^{4^{e(l)+1}-2l}+2}\right)\Big|_{\tilde{CL}_i^{(4^l)}}$$

and

$$\varphi_j^{(e(l))}\Big|_{CL_j^{(e(l))}} = \left(\frac{1}{2}\left(\varphi^{(e(l)+1)}_{(j-1)\cdot 2^{3\cdot 4^{e(l)}}} + \varphi^{(e(l)+1)}_{(j-1)\cdot 2^{3\cdot 4^{e(l)}}+1}\right) + \varphi^{(e(l)+1)}_{(j-1)\cdot 2^{3\cdot 4^{e(l)}}+2}\right)\Big|_{CL_j^{(e(l))}}.$$

Of course, the situation is analogous for $\tilde{CR}_i^{(4^l)}$, $1 \leq i < 4^l$. Hence we can define intervals $\tilde{I}_i^{(4^l)}(w) \subseteq \tilde{CR}_i^{(4^l)} = \tilde{CL}_{i+1}^{(4^l)}$, $w \in \{-1,1\}^*$, such that $\tilde{\varphi}_i^{(4^l)}$ *behaves on $\tilde{I}_{i-1}^{(4^l)}(w)$ and $\tilde{I}_i^{(4^l)}(w)$ as $\varphi_j^{(e(l))}$ on $I_{j-1}^{(e(l))}(w)$ and $I_j^{(e(l))}(w)$, respectively* (cf. (17), (18)). In particular, *Proposition* 9 *applies to the intervals $\tilde{I}_i^{(4^l)}(w)$ as far as $H(n+k)$ is replaced by $H(e(l)+k)$.*

The intervals $\tilde{M}_i^{(4^l)}$, on which the value of $\tilde{\varphi}_i^{(4^l)}$ is 1, are

$$\tilde{M}_i^{(4^l)} = \begin{cases} \left[0,\ 2\cdot 4^{-l} - H(e(l))\right], & i = 1, \\ \left[2(i-1)\cdot 4^{-l} + H(e(l)),\ 2i\cdot 4^{-l} - H(e(l))\right], & 1 < i < 4^l, \\ \left[2(4^l-1)\cdot 4^{-l} + H(e(l)),\ 2\right], & i = 4^l. \end{cases}$$

We note that

$$\operatorname{diam}\left(\tilde{M}_i^{(4^l)}\right) \geq 2\left(4^{-l} - H(e(l))\right)$$

for $1 \leq i \leq 4^l$.

Using these observations we can follow the proof of Proposition 10 when replacing n, $M_i^{(n)}$, $CL_i^{(n)}$, $CR_i^{(n)}$ and $I_i^{(n)}(\cdot)$ by $e(l)$, $\tilde{M}_i^{(4^l)}$, $\tilde{CL}_i^{(4^l)}$, $\tilde{CR}_i^{(4^l)}$ and $\tilde{I}_i^{(4^l)}(\cdot)$, respectively.

Part (b) of Proposition 15 can be inferred from part (a) as Proposition 11 from Proposition 10. □

PROPOSITION 16. *Let $f \in C([0,2]^m)$ be non-constant and Hölder continuous of type α, $0 < \alpha \leq 1$, and let $k \geq 1$ be chosen such that $4^{-k} < \alpha \leq 1$. Then*

$$\liminf_{l\to\infty} \frac{E\left(f, \operatorname{span}\left(\tilde{\Psi}_{4^{ml}}\right)\right)}{\omega\left(f, 2\left(4^{-l} - H(e(l))\right)\right)} \geq \frac{1}{2(2^k+1)}.$$

Moreover, if E is a Banach space and the operator $T \in \mathcal{L}(E, C([0,2]^m))$ is Hölder continuous of type α such that the image $T(E)$ does not consist of constant functions only, then

$$\liminf_{l\to\infty} \frac{E\left(T, \operatorname{span}\left(\tilde{\Psi}_{4^{ml}}\right)\right)}{\omega\left(T, 2\left(4^{-l} - H(e(l))\right)\right)} \geq \frac{1}{2(2^k+1)}.$$

Proof. We consider the functions $G_1(l) = 2H(e(l)+k)$ and $G_2(l) = 2(4^{-l} - H(e(l)))$. One can show that $G_1^\alpha = o(G_2)$ as $l \to \infty$ by formulas (7), (9) and (40). Then the proof of Proposition 13 applies. □

Proof of Theorem 7. Let n be fixed. We choose $l \geq 0$ such that $4^{ml} \leq n < 4^{m(l+1)}$. Obviously,

$$E\left(f, \operatorname{span}\left(\tilde{\Psi}_n\right)\right) \geq E\left(f, \operatorname{span}\left(\tilde{\Psi}_{4^{m(l+1)}}\right)\right).$$

Using formulas (7), (9) and (40) we get the estimate $2(4^{-(l+1)} - H(e(l+1))) > 2(4^{-(l+1)} - 3 \cdot 2^{-4^{e(l+1)+1}}) \geq 2(4^{-(l+1)} - \frac{3}{4} \cdot 4^{-(l+1)}) = \frac{1}{8} \cdot 4^{-l}$ and thus obtain

$$\begin{aligned}\omega(f, \varepsilon_n([0,2]^m)) &\leq \omega\left(f, \varepsilon_{4^{ml}}([0,2]^m)\right) = \omega\left(f, 4^{-l}\right) \\ &\leq 8\,\omega\left(f, \tfrac{1}{8} \cdot 4^{-l}\right) \leq 8\,\omega\left(f, 2\left(4^{-(l+1)} - H(e(l+1))\right)\right).\end{aligned}$$

Now we choose $k \geq 1$ such that $4^{-k} < \alpha \leq 4^{-k+1}$. Proposition 16 yields

$$\liminf_{n\to\infty} \frac{E\left(f, \operatorname{span}\left(\tilde{\Psi}_n\right)\right)}{\omega\left(f, \varepsilon_n([0,2]^m)\right)} \geq \frac{1}{8} \liminf_{l\to\infty} \frac{E\left(f, \operatorname{span}\left(\tilde{\Psi}_{4^{m(l+1)}}\right)\right)}{\omega\left(f, 2\left(4^{-(l+1)} - H(e(l+1))\right)\right)}$$
$$\geq \frac{1}{16\left(2^k+1\right)} \geq \frac{1}{16\left(2\alpha^{-\frac{1}{2}}+1\right)},$$

which is our claim.

Obviously, the same proof works for T instead of f. $\square$

Let us point out two simple conclusions from Theorems 6 and 7. The first one summarizes the fact that the modulus of continuity $\omega(\cdot, \varepsilon_n([0,2]^m))$ of any Hölder continuous function $f \in C([0,2]^m)$ or operator $T \in \mathcal{L}(E, C([0,2]^m))$ has the same asymptotics as the approximation quantity $E(\cdot, \operatorname{span}(\tilde{\Psi}_n))$.

COROLLARY 1. (a) *Let $f \in C([0,2]^m)$ be a Hölder continuous function. Then there exists a constant $c_f > 0$ such that*

$$c_f \cdot \omega(f, \varepsilon_n([0,2]^m)) \leq E\left(f, \operatorname{span}\left(\tilde{\Psi}_n\right)\right) \leq 7\,\omega(f, \varepsilon_n([0,2]^m))$$

for all $n \geq 1$.
(b) *Let E be a Banach space and $T \in \mathcal{L}(E, C([0,2]^m))$ a Hölder continuous operator. Then there exists a constant $c_T > 0$ such that*

$$c_T \cdot \omega(T, \varepsilon_n([0,2]^m)) \leq E\left(T, \operatorname{span}\left(\tilde{\Psi}_n\right)\right) \leq 7\,\omega(T, \varepsilon_n([0,2]^m))$$

for all $n \geq 1$.

COROLLARY 2. (a) *None of the functions $f \in \operatorname{span}(\tilde{\Psi}_n)$, $n \geq 0$, is Hölder continuous so far as it is non-constant.*
(b) *Let E be a Banach space and let $T \in \mathcal{L}(E, C([0,2]^m))$ be an operator with $T(E) \subseteq \operatorname{span}(\tilde{\Psi}_n)$ such that the image $T(E)$ does not consist of constant functions only. Then T is not Hölder continuous.*

Corollary 2 shows in particular that the partitions of unity $\tilde{\Psi}_n$ do not contain Hölder continuous functions and that the operators given by Theorem 6 are not Hölder continuous.

Finally, we note that the chain $(\tilde{\Psi}_n)_{n=1}^{\infty}$ gives similar results on any metric space $([0,2]^m, d)$ apart from controllability if the metrics d is equivalent to $d_{\max}$, so that Theorems 6 and 7 and Corollary 1 admit obvious generalizations. Indeed, the property of Hölder continuity of functions and operators is not influenced by the change from $d_{\max}$ to d and, moreover, the entropy numbers of $([0,2]^m, d)$ have the same asymptotics as the entropy numbers of $([0,2]^m, d_{\max})$. For instance, the above approximation scheme can be used on the cube $[0,2]^m$ equipped with the Euclidean metrics d_2 or on the m-dimensional Euclidean ball with the Euclidean metrics, since this metric space is isometric to a suitable space $([0,2]^m, d)$.

5. Concluding remarks

Let us conclude with a remark on chains of controllable partitions of unity on arbitrary compact metric spaces (X, d). In [Ri/Ste] the related concept of a *controllable partition of* X is introduced. This is a finite partition $\mathcal{P}$ *of the set* X into pairwise disjoint subsets which fulfil the same uniformity condition (3) as the supports of controllable partitions of unity. A chain $(\mathcal{P}_n)_{n=0}^{\infty}$ of partitions of the space X is meant to be a sequence of partitions such that any member $\mathcal{P}_n$ is a proper refinement of the preceding one. It is proved in [Ri1] that any compact metric space (X, d) possesses a chain of controllable partitions of X. For certain totally disconnected compact metric spaces X, the characteristic functions of the partition sets from the controllable partitions $\mathcal{P}_n$ are continuous and, accordingly, form controllable partitions of unity. Obviously, a chain of controllable partitions $\mathcal{P}_n$ of that type gives rise to a chain of controllable partitions of unity. In particular, I. Stephani has shown in this way that there exist chains of controllable partitions of unity on the Cantor set 2^{ω} (private communication).

As a first step to the general case, the author was able to prove the following fact by methods from [Ri1]: *Any infinite compact metric space* (X, d) *possesses a sequence* $(\mathcal{C}_n)_{n=1}^{\infty}$ *of finite open coverings such that any covering* $\mathcal{C}_n$ *is controllable in the sense of* (3) *and any open set from* $\mathcal{C}_n$ *is a union of sets from* $\mathcal{C}_{n+1}$. Is it possible to use these coverings as the systems of supports of a chain $(\Phi_n)_{n=1}^{\infty}$ of contollable partitions of unity in $C(X)$?

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Friedrich-Schiller-Universität Jena, Mathematisches Institut, Ernst-Abbe-Platz 1-4, D-07743 Jena, Germany

richterc@minet.uni-jena.de